All Near-Horizon Geometries of Extremal Vacuum Black Holes

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Abstract

This talk is aimed at introducing the paper of [arXiv:0909.3462] by S. Hollands and the present author, which addresses the classification problem of all vacuum near-horizon geometries in $D$-dimensions with $(D-3)$ commuting rotational symmetries. Here we present some of the key formulas and main results of the paper.

1 Key formulas and main results

We are concerned with the classification problem of higher dimensional black hole solutions: Given a higher dimensional gravity theory, we wish to first (i) identify all (physical) parameters that uniquely determine black hole solutions of the theory, and then (ii) construct explicitly all possible black hole solutions characterized by the parameters. In this generality, however, the black hole classification problem appears to be difficult to address, so we restrict attention to some interesting subclass of solutions.

Many known families of black hole solutions possess a limit wherein the black hole horizon becomes degenerate; such black holes are called extremal. Due to the limiting procedure, extremal black holes are in some sense at the fringe of the space of all black holes, and therefore possess special properties which make them easier to study in various respects. It is known that any extremal black hole with a degenerate Killing horizon admits the near-horizon geometry, which is obtained by taking a suitable scaling process to the metric in an immediate vicinity of the degenerate horizon. More precisely, for any spacetime with a degenerate Killing horizon, one can introduce, in a neighborhood of the horizon, Gaussian-null coordinates, $(v, u, y^a)$, such that the metric takes the form

$$\begin{align*}
\text{d}s^2 &= 2\text{d}v(u^2\alpha\text{d}u + u\beta_a\text{d}y^a) + \gamma_{ab}\text{d}y^a\text{d}y^b,
\end{align*}$$

where the horizon is located at $u = 0$ and where $v$ is the Killing parameter and at the same time, an affine parameter along the null generators of the horizon. The metric functions $\alpha, \beta_a, \gamma_{ab}$, which are independent of $v$, can be seen as functions on horizon cross section, $H$, at $u = 0 = v$. We consider diffeomorphism $v \mapsto v/\epsilon, \ u \mapsto \epsilon u$, leaving $y^a$ unchanged, and then take $\epsilon \to 0$. The obtained near-horizon metric looks exactly like the original one, but now the new metric functions $\alpha, \beta_a, \gamma_{ab}$ depend on neither $v$ nor $u$. The near horizon metric satisfies the same dynamics as the original black hole solution. It is, in fact, this near-horizon geometry that enters many interesting applications, such as the arguments pertaining to the derivation of the Bekenstein-Hawking entropy.

The purpose of the paper [1] is to classify all near-horizon geometries which can arise from $D$-dimensional extremal, stationary vacuum black holes. We assume further that the geometries admit $D-3$ commuting rotational symmetries, generated by axial Killing vector fields, $\psi_i = \partial/\partial \varphi^i, \ i = 1, \ldots, D-3$. Then, the metric functions for such a near-horizon geometry depend only on a single coordinate, say $x$, which may correspond to a polar coordinate and can be chosen $x \in [-1, 1]$. Furthermore, it can be shown that a near horizon geometry possesses more symmetries, $O(2,1) \times U(1)^{D-3}$, than the original solution does [2]. Previously, Kunduri-Lucietti [3] gave a classification of such near-horizon geometries in $D = 4, 5$. However, their method does not appear to be generalized to higher dimensional case $D \geq 6$. We use a different method based on a matrix (sigma-model) formulation of the vacuum Einstein equations that works in arbitrary dimensions. In the following we briefly explain our method. First, as a consequence of

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our symmetry assumption and the dynamics (part of the vacuum Einstein equations) we can find a new coordinate system, \((v, r, x, \varphi')\), in which the near-horizon metric, \((1)\), are rewritten as

\[
\begin{align*}
    ds^2 &= \frac{1 - x^2}{\det f} (2dvd^r - C_{ij}^2dv^j) + \frac{dx^2}{C_{ij}^2\det f} + f_{ij}(d\varphi^i + rC_k^j dv^j)(d\varphi^j + rC_k^j dv^j),
\end{align*}
\]

where \(k^i\) and \(C\) are some constants and \(f_{ij} := ds^2(\psi_i, \psi_j)\) depends only on \(x\). The horizon is now at \(r = 0\) and \(v\) is the Killing parameter as before. In obtaining the above form of the metric, we have used up all but the \(ij\)-components of the Einstein equations. We use the remaining Einstein equations to determine the matrix elements, \(f_{ij}\). For this purpose, we introduce twist potentials, \(\chi_i\), defined up to a constant by

\[
    d\chi_i = \ast (\psi_1 \wedge \cdots \wedge \psi_{D-3} \wedge d\psi_i),
\]

and then set the matrix

\[
    \Phi = \begin{pmatrix}
        (\det f)^{-1} & -(\det f)^{-1}\chi_i \\
        -(\det f)^{-1}\chi_i & f_{ij} + (\det f)^{-1}\chi_i\chi_{ij}
    \end{pmatrix}.
\]

The matrix elements are functions of only \(x\). Then, the content of the remaining vacuum Einstein equations is expressed as the ordinary differential equations

\[
    \partial_x[(1 - x^2)\Phi^{-1}\partial_x\Phi] = 0.
\]

This matrix equation is easily integrated to

\[
    \Phi(x) = Q \left(\frac{1 + x}{1 - x}\right)^L,
\]

where \(Q = \Phi(0)\), \(L = (1/2)(1 - x^2)\Phi(x)\^{-1}\partial_x\Phi(x)\) are both constant real \((D - 2) \times (D - 2)\) matrices. Since the matrix, \(\Phi\), is symmetric, unimodular, and positive definite, one can show that \(\det Q = 1\), \(\text{Tr} L = 0\), \(Q = Q^T > 0\), \(L^T Q = Q L\), and further that \(Q = S^T S\) for some real invertible matrix \(S = (s_{IJ})\) of \(\det S = \pm 1\). Using these properties, the matrix \(\Phi\) is given by

\[
    \Phi_{IJ}(x) = \sum_{K=0}^{D-3} \left(\frac{1 + x}{1 - x}\right)^{K}\sigma_k s_K i s_{KJ}.
\]

This is the most general solutions to eq. (4) with the real parameters \(s_{IJ}, \sigma_I\), subject to the constraints \(\det S = \pm 1, \sum_{I=0}^{D-3} \sigma_I = 0\). This solution determines \(f_{ij}\) and \(\chi_i\) and in turn fix the constants \(k^i\) and \(C\), thus completely fixing the near-horizon geometry. It turns out that the smoothness of the near-horizon metric implies further constraints on \(s_{IJ}\) and \(\sigma_I\). The results are summarized as follows:

\[
    \sigma_I = \begin{cases} 
        0 & \text{if } I \leq D - 5, \\
        -1 & \text{if } I = D - 4, \\
        1 & \text{if } I = D - 3,
    \end{cases}
\]

and

\[
    k^i = \frac{2e_+ e_-}{e_+ - e_-} \left( \frac{a^i_{+}}{\mu_i a^i_{+}} + \frac{a^i_{-}}{\mu_i a^i_{-}} \right), \quad C = \frac{4e^2_+}{(e_+ - e_-)\mu_i a^i_+} = \frac{4e^2_-}{(e_+ - e_-)\mu_i a^i_-},
\]

where

\[
    \mu_i = s_{(D-3)i} = s_{(D-4)i}, \quad e_+ = s_{(D-3)0}, \quad e_- = s_{(D-4)0},
\]

and \(a^i_{\pm} \in \mathbb{Z}\) are real integer parameters taken so that the linear combination \(\sum a^i_{\pm} \psi_i\) vanishes at the boundary points \(x = \pm 1\). It turns out that \(a^i_{\pm}\) determine the horizon topology as commented later on.

Thus, we have determined all quantities \(C, k^i, f_{ij}\) in the near-horizon metric (2). Making the final coordinate change \(x = \cos \theta\), \(0 \leq \theta \leq \pi\), and performing some algebraic manipulations, we get the following result:
Theorem 1  All vacuum, non-static, near horizon metrics (except topology type $H \cong T^{D-2}$) with assumed symmetry are parametrized by the real parameters $c_{\pm}, \mu, s_i$, and the integers $a'_\pm$ where $I = 0, \ldots, D - 5$ and $i = 1, \ldots, D - 3$, and $\text{g.c.d.}(a'_\pm) = 1$. The explicit form of the near horizon metric in terms of these parameters is

$$g = e^{-\lambda}(2d\varphi^2 - C^2 d\vartheta^2 + C^{-2} d\theta^2) + e^+ \left\{ (c_+ - c_-)^2 (\sin \theta) \Omega^2 \right.$$  
$$+ (1 + \cos \theta)^2 c_+^2 \sum_i \left( \omega_i - \frac{s_I \cdot a_+}{\mu \cdot a_+} \Omega \right)^2 + (1 - \cos \theta)^2 c_-^2 \sum_i \left( \omega_i - \frac{s_J \cdot a_-}{\mu \cdot a_-} \Omega \right)^2$$  
$$+ \frac{c_+^2 \sin^2 \theta}{(\mu \cdot a_+)^2} \sum_{i < j} \left( (s_I \cdot a_+) \omega_i - (s_J \cdot a_+) \omega_j \right)^2 \right\} \Omega^2. \tag{10}$$

Here, the sums run over $I, J$ from $0, \ldots, D - 5$, the function $\lambda(\theta)$ is given by

$$\exp[-\lambda(\theta)] = c_+^2 (1 + \cos \theta)^2 + c_-^2 (1 - \cos \theta)^2 + \frac{c_+^2 \sin^2 \theta}{(\mu \cdot a_+)^2} \sum_I (s_I \cdot a_+)^2, \tag{11}$$

$C$ is given by $C = 4c_\pm^2 [(c_+ - c_-)(\mu \cdot a_\pm)]^{-1}$, and we have defined the 1-forms

$$\Omega(r) = \mu \cdot d\varphi + 4Ct \frac{c_+ - c_-}{c_+ - c_-} dv$$  
$$\omega_I(r) = s_I \cdot d\varphi + \frac{r}{2} C^2 (s_I \cdot a_+ + s_I \cdot a_-) dv. \tag{12}$$

We are also using the shorthand notations such as $s_I a'_\pm = s_I a_\pm$, or $\mu \cdot d\varphi = \mu_i d\varphi_i$, etc. The parameters are subject to the constraints $\mu \cdot a_\pm \neq 0$ and

$$\frac{c_+^2}{\mu \cdot a_+} = \frac{c_-^2}{\mu \cdot a_-}, \quad \frac{c_+(s_I \cdot a_+)}{\mu \cdot a_+} = \frac{c_-(s_I \cdot a_-)}{\mu \cdot a_-}, \quad \pm 1 = (c_+ - c_-) \epsilon^{ijk \ldots m} s_{0i}s_{1j}s_{2k} \cdots \mu_m \tag{14}$$

but they are otherwise free. The coordinates $\varphi^i$ are $2\pi$-periodic, $0 \leq \theta \leq \pi$, and $v, r$ are arbitrary. When writing “$\pm$”, we mean that the formulae hold for both signs.

Remarks: (i) The meaning of the parameters are as follows. The parameters $a'_\pm \in \mathbb{Z}$ are related to the horizon topology. Up to a globally defined coordinate transformation of the form $\varphi^i \mapsto \sum A^i_j \varphi^j$ mod $2\pi$, $A \in SL(\mathbb{Z}, D - 3)$, we have

$$a_+ = (1, 0, 0, \ldots, 0), \quad a_- = (q, p, 0, \ldots, 0), \quad p, q \in \mathbb{Z}, \quad \text{g.c.d.}(p, q) = 1. \tag{15}$$

A general analysis of compact manifolds with a cohomogeneity-one torus action implies that the topology of $H$ is

$$H \cong \begin{cases} 
S^1 \times T^{D-5} & \text{if } p = \pm 1, q = 0, \\
S^2 \times T^{D-4} & \text{if } p = 0, q = 1, \\
L(p, q) \times T^{D-5} & \text{otherwise.} 
\end{cases} \tag{16}$$

The constants $\mu_i, c_\pm, a'_\pm$ are directly related to the horizon area by

$$A_H = \frac{(2\pi)^{D-3}(c_+ - c_-)^2 (\mu \cdot a_\pm)^2}{8c_\pm^2}, \tag{17}$$

and we also have

$$J_i := \frac{1}{2} \int_H \star (d\psi_i) = (2\pi)^{D-3} \frac{e_+ - e_-}{2c_+ c_-} \mu_i. \tag{18}$$

In an asymptotically flat or Kaluza-Klein black hole spacetime with a single horizon $H$, $J_i$ would be equal to the Komar expressions for the angular momentum. The near horizon limits that we consider do not of
course satisfy any such asymptotic conditions, and hence this cannot be done. Nevertheless, if the near horizon metric under consideration arises from an asymptotically flat or asymptotically Kaluza-Klein spacetime, then the $J_i$ are the angular momenta of that spacetime. Hence, we see that the parameters $c_{\pm}, \mu_i, a_{\pm}^i$ are directly related to geometrical/topological properties of the metric. This seems to be less clear for the remaining parameters $s_{Ii}$.

(ii) The number of continuous parameters on which our metric depend can be counted as follows. First, the matrix $s_{Ii}$ has $(D - 3)(D - 4)$ independent components, $\mu_i$ has $(D - 3)$ and $c_{\pm}$ has 2 components. These parameters are subject to the $(D - 2)$ constrains, eqs. (14). However, changing $s_{Ii}$ to $\sum_{J=0}^{D-5} R^J i s_{Ji}$, with $R^J$ an orthogonal matrix in $O(D - 4)$, does not change the metric. Since such a matrix depends on $(D - 4)(D - 5)/2$ parameters, our metrics depend only on $(D - 3)(D - 4) + (D - 3) + 2 - (D - 2) - (D - 4)(D - 5)/2 = (D - 2)(D - 3)/2$ real continuous parameters.

(iii) By contrast to the case $D \leq 5$ given in [3], not all near horizon metrics that we have found can be obtained as the near horizon limits of known black hole solutions in dimensions $D \geq 6$. It is conceivable that there are further extremal black hole solutions—to be found—which give our metrics in the near horizon limit, but it is also possible that some of our metrics in $D \geq 6$ simply do not arise in this way.

References

