Cosmic acceleration and higher-dimensional gravity

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Abstract

The self-accelerating branch of the Dvali-Gabadadze-Porrati (DGP) five-dimensional braneworld has provided a compelling model for the current cosmic acceleration. Recent observations, however, have not favored it so much. We discuss the solutions which contain a de Sitter 3-brane in the cascading DGP braneworld model, which is a kind of higher-dimensional generalizations of the DGP model, where a $p$-dimensional brane is placed on a $(p+1)$-dimensional one and the $p$-brane action contains the $(p+1)$-dimensional induced scalar curvature term. In the simplest six-dimensional model, we derive the solutions. Our solutions can be classified into two branches, which reduce to the self-accelerating and normal solutions in the limit of the original five-dimensional DGP model. In the presence of the six-dimensional bulk gravity, the ‘normal’ branch provides a new self-accelerating solution. The expansion rate of this new branch is generically lower than that of the original one, which may alleviate the fine-tuning problem.

Recent observational data with high precision suggest that our Universe is currently in an accelerating phase [1, 2]. They are consistent with the presence of a nonzero cosmological constant or quantum vacuum energy, but its value must be extremely tiny. In the context of the braneworld, the Dvali-Gabadadze-Porrati (DGP) five-dimensional model has been a compelling model for the cosmic acceleration [3–5]. The DGP model contains a mechanism to modify the gravitational law just on cosmological scales by the effects of the four-dimensional Einstein-Hilbert term put into the action of our 3-brane Universe. Such an intrinsic curvature term would be induced due to quantum loops of the matter fields which are localized on the 3-brane. The effect of the four-dimensional intrinsic curvature term on the 3-brane recovers the Einstein gravity on small scales but on large distance scales gravitational law becomes five-dimensional. The DGP model realizes the so-called self-accelerating Universe that features a four-dimensional de Sitter phase even though our 3-brane Universe is completely empty. Recent studies, however, have indicated that the observational data have not favored the self-accelerating branch of DGP (see e.g., Ref. [6] of [6]). The self-accelerating solutions have also faced the disastrous issue of ghost excitations (see e.g., Ref. [7] of [6]): The energy is not bounded from below and therefore the theory is already pathological even at the classical level.

There are possibilities that the realistic cosmological model may be obtained by generalizing the five-dimensional DGP model to a higher-dimensional spacetime. An interesting model is so-called the cascading DGP model [7]. The model is constructed by a set of branes of the different dimensionality, where a $p$-brane is placed on a $(p+1)$-dimensional one and the $p$-brane action contains the $(p+1)$-dimensional induced scalar curvature term. For instance, in the simplest six-dimensional model, a 3-brane Universe whose action contains an induced four-dimensional scalar curvature term is placed on a 4-brane whose action contains an induced five-dimensional scalar curvature term, embedded into a (possibly infinitely extended) six-dimensional spacetime. An extension to the case of an arbitrary number of spacetime dimensions is straightforward in principle. It is expected that in such kind of model, in the infrared region the gravitational force falls off sufficiently fast to exhibit ‘degravitation’ [7]. The linearized analysis has confirmed this idea in part at the level of the linearized theory [7].

One of crucial questions is the viability of the cascading DGP model. To answer to this question, of course, one should go beyond the linearized analysis and in particular investigate the cosmology. Nonlinearities may detect effects which may not appear in the linearized treatment. In addition, cosmology can help to have a better understanding of the model and of the idea of gravity localized through
intrinsic curvature terms on the 3-brane and 4-brane. As the first step to this direction, we will look for
the solutions which contain a de Sitter 3-brane. They may give rise to the self-accelerating cosmological
solutions in the simplest six-dimensional cascading DGP model.

The system of our interest is that our 3-brane Universe \( \Sigma_4 \) is placed on a 4-brane \( \Sigma_5 \), embedded into
the six-dimensional bulk \( \mathcal{M}_6 \). For simplicity, we suppress the matter terms in the bulk and on the branes.
The total action is given by

\[
S = \frac{M_6}{2} \int_{\Sigma_5} d^4 x \sqrt{-q^{(4)}(r)} R + \frac{M_5^2}{2} \int_{\Sigma_5} d^5 y \sqrt{-q^{(5)}(r)} R + \frac{M_4^2}{\sqrt{4}} \int_{\Sigma_4} d^4 x \sqrt{-g^{(4)}} R,
\]

where \( G_{AB} \), \( q_{ab} \), and \( g_{\mu\nu} \) represent metrics in \( \mathcal{M}_6 \), on \( \Sigma_5 \) and \( \Sigma_4 \), respectively. \(^i R \) \((i = 6, 5, 4)\) are
Ricci scalar curvature terms associated with respect to \( G_{AB} \), \( q_{ab} \) and \( g_{\mu\nu} \). For the later discussion, it is
useful to introduce the crossover mass scales \( m_5 := M_5^2/M_6^2 \) and \( m_6 := M_6^6/M_6^2 \), which determines the
energy scale where the five-dimensional and six-dimensional physics appear, respectively. We assume that
\( m_5 > m_6 \). Then, it is natural to expect that the effective gravitational theory becomes four-dimensional
for \( H > m_5 \), five-dimensional for \( m_5 > H > m_6 \), and finally six-dimensional for \( H < m_6 \), where \( H \) is the
cosmic expansion rate.

We consider the six-dimensional Minkowski spacetime, which is covered by the following choice of the
coordinates

\[
ds_6^2 = G_{AB} dX^A dX^B = dr^2 + d\theta^2 + H^2 r^2 \gamma_{\mu\nu} dx^\mu dx^\nu,
\]

where \( \gamma_{\mu\nu} \) is the metric of the four-dimensional de Sitter spacetime with the expansion rate \( H \). The \( r \)
and \( \theta \) coordinates represent two extra dimensions and \( x^\mu \) \((\mu = 0, 1, 2, 3)\) do the ordinary four-dimensional
spacetime. The surface of \( r = 0 \) corresponds to a (Rindler-like) horizon and only the region of \( r \geq 0 \) is
considered. Note that the boundary surface of \( r = 0 \) does not cause any pathological effect because it is
not a singularity. We consider a 4-brane located along the trajectory \( (r(\xi), \theta(\xi)) \), where the affine
parameter \( \xi \) gives the proper coordinate along the 4-brane. The 3-brane is placed at \( \xi = 0 \), and for
decreasing value of \( |\xi| \) one approaches the 3-brane. We assume the \( Z_2 \)-symmetry across the 4-brane
and hence an identical copy is glued to the opposite side. Along the trajectory of the 4-brane \( r^2 + \theta^2 = 1 \),
where the dot represents the derivative with respect to \( \xi \). The induced metric on the 4-brane is given by

\[
ds_4^2 = q_{ab} dy^a dy^b = d\xi^2 + H^2 r(\xi)^2 \gamma_{\mu\nu} dx^\mu dx^\nu.
\]

The point where \( r(\xi) = 0 \) on the 4-brane corresponds to a horizon and the 4-brane is not extended beyond
it. The 3-brane geometry is exactly de Sitter spacetime with the normalization condition \( H r(0) = 1 \),

\[
ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = \gamma_{\mu\nu} dx^\mu dx^\nu.
\]

The nonvanishing components of the tangential and normal vectors to the 4-brane are given by \( u^r = \dot{r}, u^\theta = \dot{\theta}, u^\phi = \dot{\phi} \), and \( n^r = -\dot{r} \). We restrict that the region to be considered is to be \( r > 0 \) and the
3-brane is sitting on \( r \)-axis \((\theta = 0)\). The 4-brane trajectory is \( Z_2 \)-symmetric across the 3-brane. \( \dot{\theta} > 0 \)
for increasing \( \xi \). In the case of \( \epsilon = 1 \), the bulk space is in the side of increasing \( r \), while in the case of
\( \epsilon = -1 \), the bulk space is the side of decreasing \( r \).

The extrinsic curvature tensor is defined by \( K_{ab} := \nabla_a n_b \). The junction condition is given by

\[
M_6^4 \left[ K_{ab} - q_{ab} K \right] = \left( M_5^3 G_{ab} + M_4^2 G_{\mu\nu} \delta^\mu_\theta \delta^\nu_\phi \delta(\xi) \right),
\]

where the square bracket denotes the jump of a bulk quantity across the 4-brane. By taking the \( Z_2 \)
-symmetry across the 4-brane into consideration, the matching condition becomes

\[
-M_6^4 \frac{4}{r}(1 - r^2)^{1/2} = -3M_5^3 \frac{1 - r^2}{r^2}, \quad M_6^4 \epsilon \left( -\frac{3}{r} (1 - r^2)^{1/2} + \frac{\ddot{r}}{(1 - r^2)^{1/2}} \right) = \frac{3M_5^3}{2} \frac{\dot{r}^2 + r\ddot{r} - 1}{r^2}.
\]

The way to construct the solution is essentially the same as the case of a tensional 3-brane on a tensional
4-brane (See Appendix).
In our case, it is suitable to take $\epsilon = +1$ branch. Then, the junction condition tells that the trajectory of the 4-brane is given by $r(\xi) = a^{-1} \cos(a|\xi| - a\xi_0)$ with
\begin{equation}
   a = \frac{4m_6}{3},
\end{equation}
where we assume $0 < a\xi_0 < \pi/2$. $r(\xi)$ vanishes at $|\xi| = |\xi_{\text{max}}| = \pi/(2a) + \xi_0$. Note that, as mentioned before, the surface of $r = 0$ corresponds to a horizon and on the 4-brane there are horizons at $|\xi| = |\xi_{\text{max}}|$, namely at a finite proper distance from the 3-brane. The 4-brane is not extended beyond them [8]. Now an identical copy is attached across the 4-brane. The normalization of the overall factor of the metric function at the 3-brane place requires $\cos(a\xi_0) = a/H \leq 1$. Note that
\begin{equation}
   H \geq \frac{4m_6}{3}.
\end{equation}
The $\ddot{r}$ term gives rise to the contribution proportional to $\delta(\xi)$. Here, by noting that
\begin{equation}
   \frac{d}{d\xi} \arctan \left( \frac{\dot{r}}{\sqrt{1 - r^2}} \right) = \frac{\ddot{r}}{\sqrt{1 - r^2}},
\end{equation}
and integrating the $(\mu, \nu)$-component of the junction equation Eq. (6) across $\xi = 0$, one finds
\begin{equation}
   M_4^2(4a\xi_0) = 6M_3^2a\tan(a\xi_0) - 3H^2M_5^2,
\end{equation}
which with Eq. (7) leads to
\begin{equation}
   \frac{H}{2m_5} = \left( \sqrt{1 - \frac{16m_5^2}{9H^2}} - \frac{2m_6}{3H} \arctan \left( \sqrt{\frac{9H^2}{16m_5^2} - 1} \right) \right) = 0.
\end{equation}
The solution of Eq. (11) determines the value of the expansion rate $H$. The 3-brane induces the deficit angle $4a\xi_0$ in the bulk. The configuration of the bulk space is shown in Fig. 1. The bulk space is outside the curve of the 4-brane and has an infinite volume. As mentioned before, the surface of $r = 0$ corresponds to a horizon and, in particular, on the 4-brane there are horizons at a finite proper distance from the 3-brane. The 4-brane is not extended beyond them. Note that this surface does not cause any pathological effect (see [6] for the detailed configuration of the bulk space).

For generic values of $m_6$, in Fig 1, the left-hand-side of Eq. (11) is shown as a function of $H/m_5$ for each fixed ratio $m_6/m_5$. It is found that below the critical ratio $m_6/m_5 < (m_6/m_5)_{\text{crit}} \approx 0.46978$, there are two branches of solutions, which are here denoted by $H_+ > H_-$. On the other hand, for $m_6/m_5 > (m_6/m_5)_{\text{crit}}$, there is no solution of Eq. (11). In the marginal case of $m_6/m_5 = (m_6/m_5)_{\text{crit}}$, there is the degenerate solution given by $H \approx m_5$. For generic values of $m_6/m_5$, in Fig. 2 and 3, the...
solutions $H_+$ and $H_-$ are shown as functions of $m_6/m_5(< (m_6/m_5)_{\text{crit}})$, respectively. In the limit of $m_6 \ll m_5$, another solution is given approximately given by

$$H_+ \approx 2m_5, \quad H_- \approx \frac{4m_6}{3}.\quad (12)$$

In the absence of the bulk gravity, $m_6 \to 0$, the (+) and (−)-branches coincide with the ‘self-accelerating’ and ‘normal’ solutions in the DGP model, with $H_+ = 2m_5$ and $H_- = 0$, respectively. By taking the presence of the six-dimensional bulk into consideration, the self-accelerating branch essentially remains the same. But the normal branch solution provides a new self-accelerating solution if $H_-$, which could be much smaller than $H_+$ for $m_6 \ll m_5$. Note that the existence of both of these new solutions relies on the presence of the 4-brane, since in the limit of $M_5 \to 0$ none of these solutions can exist.

As we mentioned, the self-accelerating branch of the original DGP model is not favored by recent observations and also suffers a ghost instability. What we found is that in the six-dimensional cascading DGP model, one of two branches, which corresponds to the ‘normal’ branch in the original DGP model, provides a new self-accelerating solution whose expansion rate could be much smaller than that in the other branch, which corresponds to the original ‘self-accelerating’ branch. Thus, the fine-tuning would be relaxed in some degrees. In the self-accelerating solution of the DGP model, the bulk spacetime is infinitely extended and a mode which satisfies the background solution is not normalizable. Thus, the scalar mode is hence different from the zero mode, which already implies the potential pathology about the ghost instability. In our new solutions the 4-brane where the 3-brane resides can never reach the infinity and has a finite volume. Therefore, in analogy with the case of the standard DGP, it implies that the bending mode of the 3-brane would be normalized and hence solutions could be healthy, although the detailed investigations about the stability are left for a future work.

References