Cosmological Influence on Gravitationally Bound Local System: Case of Lemaître–Tolman–Bondi Spacetime and its Application to Secular Increase of Astronomical Unit

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Abstract

We investigated the influence of inhomogeneity of the Universe on the gravitationally bound local system such as the solar system based on the Lemaître–Tolman–Bondi (LTB) solution. In this study, we concentrated on the dynamical perturbation to the planetary motion and derived the leading order effect arisen from LTB model in the straightforward way; first we expressed the perturbation attributed to LTB model in the standard comoving coordinates \((t, r, \theta, \phi)\), then transformed it into the curvature or proper coordinates \((t, R, \theta, \phi)\), imposing the approximate relation between \(r\) and \(R\). It was shown that not only the familiar cosmological contribution arisen from the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) Universe but also the correction due to the radial inhomogeneity of LTB model are considerably weak and currently undetectable. We also applied the results to the problem of secular increase in the astronomical unit reported by Krasinsky and Brumberg (2004) and found that the inhomogeneity of the Universe does not cause the significant effect for explaining the observed \(\frac{d\text{AU}}{dt} = 15 \pm 4 \text{ [m/century]}.\)

1 Dynamical Perturbation in LTB Model

We will obtain the cosmological perturbation attributed to the LTB model, which is analogous to \(F_R^{(\text{FLRW})}, F_\phi^{(\text{FLRW})}\) in previous section. The metric of LTB spacetime in the standard comoving form is given by \([3, 12, 16, 19]\),

\[
ds^2 = -c^2 dt^2 + \frac{1}{1 + 2\mathcal{E}(r)} \left( \frac{dR}{dr} \right)^2 dr^2 + R^2 d\Omega^2, \tag{1}
\]

here \(R\) is the function of \(t\) and \(r\), and

\[
\mathcal{E}(r) = \frac{1}{2c^2} \left( \frac{dR}{dt} \right)^2 - \frac{M(r)}{R} - \frac{1}{6} \Lambda R^2, \tag{2}
\]

\[
M(r) = \frac{4\pi G}{c^2} \int \rho(t,r)R^2 \frac{dR}{dr} dr, \tag{3}
\]

in which \(\Lambda\) is the cosmological constant, \(\rho(t,r)\) is the density of the cosmological pressureless particles, and \(\mathcal{E}(r)\) and \(M(r)\) are the arbitrary functions of \(r\); \(\mathcal{E}(r)\) can be considered as the generalization of curvature parameter \(k\) in the FLRW model and \(M(r)\) is the active gravitational mass that generates the gravitational field. \(R\) has the dimension of physical length, namely, the source area distance or the luminosity distance, while \(r\) is the coordinate value then dimensionless, see Plebański and Krasiński [16] for more details.

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Making use of (1), the equations of motion for $r, \phi$ become,

$$\frac{d^2 r}{dt^2} = - \left[ 2 \frac{\partial^2 R}{\partial t \partial r} \frac{dr}{dt} + \frac{\partial^2 R}{\partial r^2} \left( \frac{dr}{dt} \right)^2 \right]$$

$$- (1 + 2\mathcal{E}) \mathcal{R} \left( \frac{d\phi}{dt} \right)^2 \frac{1}{\partial \mathcal{R} / \partial t}$$

$$+ \frac{1}{1 + 2\mathcal{E}} \frac{d\mathcal{E}}{dr} \left( \frac{dr}{dt} \right)^2,$$

(4)

$$\frac{d^2 \phi}{dt^2} = - \frac{2}{\mathcal{R}} \left[ \frac{\partial \mathcal{R}}{\partial t} + \frac{\partial \mathcal{R}}{\partial r} \frac{dr}{dt} \right] \frac{d\phi}{dt} + \frac{1}{1 + 2\mathcal{E}} \frac{d\mathcal{E}}{dr} \left( \frac{dr}{dt} \right)^3,$$

(5)

where we dropped the $\mathcal{O}(c^{-2})$ and higher order terms. When the flat FLRW limit, $\mathcal{R} \rightarrow R = ra(t), \mathcal{E} \rightarrow k = 0$.

In order to relate between $r$ and $\mathcal{R}$ explicitly, we suppose that the background LTB spacetime is the regular at the origin $r = 0$ where the central body is located, and that the test particle, such as a planet, moves around $r = 0$ then the redshift $z$ in this area is sufficiently small, $z < 1$. So according to Mashhoon et al. [13], we adopt the following expansion forms for $\mathcal{R}, \mathcal{E}, \mathcal{M}$ around $r = 0$ as,

$$\mathcal{R}(t, r) = ra(t) \left[ 1 + \frac{1}{2} \frac{\Delta(t)}{a(t)} \right] + O(r^2),$$

$$\Delta(t) \equiv \left. \frac{\partial^2 \mathcal{R}}{\partial r^2} \right|_{r=0} \ll 1,$$

(6)

$$\mathcal{E}(r) = \frac{1}{2} \epsilon r^2 + O(r^3), \quad \epsilon \equiv \left. \frac{d^2 \mathcal{E}}{dr^2} \right|_{r=0} \ll 1,$$

(7)

$$\mathcal{M}(r) = \frac{1}{6} mr^3 + O(r^4), \quad m \equiv \left. \frac{d^3 \mathcal{M}}{dr^3} \right|_{r=0} \ll 1,$$

(8)

in which the scale factor $a(t)$ is defined as,

$$a(t) \equiv \left. \frac{\partial \mathcal{R}}{\partial r} \right|_{r=0}.$$

(9)

Using these relations, (4) and (5) are rewritten as,

$$\frac{d^2 \mathcal{R}}{dt^2} - \mathcal{R} \left( \frac{d\phi}{dt} \right)^2 = \mathcal{F}_{\mathcal{R}}^{(LTB)},$$

(10)

$$\frac{d}{dt} \left( \mathcal{R}^2 \frac{d\phi}{dt} \right) = \mathcal{F}_{\phi}^{(LTB)},$$

(11)

where the leading order dynamical perturbations, $\mathcal{F}_{\mathcal{R}}^{(LTB)}$ and $\mathcal{F}_{\phi}^{(LTB)}$ are expressed as,

$$\mathcal{F}_{\mathcal{R}}^{(LTB)} = \left[ \frac{\dot{a}}{a} + \left( \frac{1}{\Delta} \frac{d\Delta}{dt} \right) \right] \mathcal{R} - \frac{2\epsilon}{\Delta} \left[ \frac{\mathcal{R} \dot{a}^2}{a^2} - \frac{\dot{a}}{a} \dot{\mathcal{R}} \right]$$

$$- q \left( \frac{\dot{a}}{a} \right)^2 \left( \frac{1}{\Delta} \frac{d\Delta}{dt} \right)^2 \mathcal{R}$$

$$- 2 \epsilon \left[ \frac{\mathcal{R} \dot{a}^2}{a^2} - \frac{\dot{a}}{a} \dot{\mathcal{R}} \right],$$

(12)

$$\mathcal{F}_{\phi}^{(LTB)} = 0.$$

(13)

In (12), we used the standard relation in the FLRW model,

$$\frac{\dot{a}}{a} = - q \left( \frac{\dot{a}}{a} \right)^2,$$

(14)
in which q is the deceleration parameter. The first term in (12) is $F_{(FLRW)}$, the second to forth terms are corrections arisen from the LTB model. It may be possible to consider that the second term in (12) is analogous to $F_{(FLRW)} = -q(\dot{a}/a)^2$.

In (12), we need to evaluate not only $\epsilon$ determining the geometry or curvature of the Universe but also $\Delta(t)$ characterizing the radial inhomogeneity. Since current observations indicate the flat Universe then we let $\epsilon \rightarrow 0$ here. While, $\Delta(t)$ may be in principle obtained from the modified luminosity-redshift relation [15],

$$d_L = c \left[ \frac{z}{H} + \frac{z^2}{2H}(1 - q - C) \right], \quad C = \frac{1}{aH^2} \frac{d\Delta}{dt}.$$  

(15)

Finally, the Newtonian equation of motion may be given by,

$$\frac{d^2 R}{dt^2} - R \left( \frac{d\phi}{dt} \right)^2 = -\frac{GM}{R} + F_{(LTB)}^R,$$

(16)

$$\frac{d}{dt} \left( R^2 \frac{d\phi}{dt} \right) = F_{\phi}^{(LTB)}.$$  

(17)

2 Application to Secular Increase in Astronomical Unit

In this section, as the application of (16) and (17), we consider the secular increase in the astronomical unit reported by Krasinsky and Brumberg [11] (see also Standish [18] and section 5 of Arakida [1]). They found that from the analysis of planetary radar and martian orbiters/landers, the astronomical unit (AU) increase with respect to meters as $dAU/dt = 15 \pm 4$ [m/century]. This secular trend currently cannot be related to any theoretical predictions and so far the origin of this secular increase is far from clear.

Krasinsky and Brumberg suggested one possibility as the future work, namely, the inhomogeneity of the Universe may induce the observable effect and explain $dAU/dt$. Then, in terms of LTB model, let us give consideration to this possibility. Since current cosmological observations assist the flat geometry of the Universe, then we adopt $\epsilon = 0$. In this case, the cosmological contribution is governed by,

$$F_{R}^{(LTB)} = \left[ -q \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{1}{\Delta} \frac{d\Delta}{dt} \right)^2 \right] R.$$  

(18)

In our approximation, $(\dot{a}/a)R$ or $-q(\dot{a}/a)R$ is a dominant cosmological effect, nevertheless, this contribution is considerably small, see Arakida [1], Adkins and McDonnell [2], Carrera and Giulini [4], Cooperstock et al. [5], Faraoni and Jacques [6], Järnefelt [7,8,9], Kliioner and Soffel [10], Noerdlinger and Ptroesian [14], Sereno and Jetzer [17]. From the assumption (6), $\Delta$ may be regarded as the correction to scale factor, $a(t)$. And its time variation $d\Delta/dt$ may be also smaller than $da/dt$. Further, it is known that the deviation in the observed Cosmic Microwave Background (CMB) radiation is of the order of $10^{-5}$, then the inhomogeneity of the Universe also causes only negligibly weak contribution.

Therefore, it is currently difficult to detect the cosmological contribution attributed to not only the FLRW model but also the inhomogeneity of the Universe, and then an inhomogeneity of the background cosmological matter distribution cannot causes the detectable effect and explain the observed $dAU/dt$.

References