

Dissertation

G/G Theory and the Bethe Ansatz
for the Integrable System

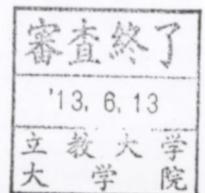
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Abstract

This thesis is devoted to the study of the Gauge/Bethe correspondence between a topological gauge theory and an integrable system.

In the first part of this thesis, we show the correspondence between the G/G gauged Wess-Zumino-Witten model on a Riemann surface and the phase model. When we apply equivariant localization methods to the G/G gauged Wess-Zumino-Witten model, the diagonal components of a group element satisfy Bethe Ansatz equations for the phase model. We show that the partition function of the G/G gauged Wess-Zumino-Witten model coincides with a summation of norms with respect to all the eigenstates of the Hamiltonian with the fixed number of particles in the phase model.

In the second part of this thesis, we generalize the G/G gauged Wess-Zumino-Witten model to the one with additional matters. We show that this model corresponds to the q -boson model as with the first part. Also, we consider the Gauge/Bethe correspondence from a general point of view.

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CHAPTER 1

Introduction

It is well known that there exists various connections between topological or supersymmetric gauge theories and integrable systems. In this thesis, we consider “Gauge/Bethe correspondence” between a topological gauge theory and a quantum integrable system. A simplest example in this correspondence is the equivalence between the BF theory with the $U(N)$ gauge group on a Riemann surface and the system of N free non-relativistic fermions on a circle. As a generalization, Moore, Nekrasov and Shatashvili discovered a correspondence between the topological Yang-Mills-Higgs theory with the gauge group $U(N)$ and the non-linear Schrödinger model [1]. The topological Yang-Mills theory is the BF theory coupled to a one-form valued adjoint Higgs field and describes the $U(1)$ -equivariant intersection theory on the Hitchin’s moduli space [2], [3], just as the BF theory describes the intersection theory on a moduli space of flat connections on a Riemann surface [4].

In [1], they applied the equivariant localization to the topological Yang-Mills-Higgs theory and found that the localization configurations lead to Bethe Ansatz equations for the non-linear Schrödinger model. Later, Gerasimov and Shatashvili discovered that the partition function of this model is related to norms of wave functions in the non-linear Schrödinger model [5], [6]. Therefore, we expect that the partition functions of other topological gauge theories are also related to norms of wave functions for the corresponding integrable systems.

The Gauge/Bethe correspondence is realized for not only a topological gauge theory but also vacua in a supersymmetric gauge theory. This is natural because the vacua of the supersymmetric gauge theory transfers to physical states in the topological field theory by the topological twist. Nekrasov and Shatashvili discovered that coulomb branches in a supersymmetric gauge theory corresponds to some integrable system. Especially, they found that a twisted superpotential in $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in two dimensions coincides with a Yang-Yang function for XXX model [7], [8]. Further, this correspondence is not restricted to two-dimensional topological gauge theory. Three dimensional $\mathcal{N} = 2$ gauge theory compactified on a circle and four dimensional $\mathcal{N} = 2$ gauge theory compactified on a torus also correspond to the XXZ model and the XYZ model, respectively.

In this way, it expects that the Gauge/Bethe correspondence works well for various models. However, this correspondence is not investigated in detail yet. An underlying

mathematical principle for this correspondence also is not known. In this thesis, we take the G/G gauged Wess-Zumino-Witten (WZW) model which is a topological field theory as an example and study the Gauge/Bethe correspondence. The G/G gauged WZW model is constructed from G WZW model which is a two-dimensional conformal field theory. The G WZW model has rich structures and various applications in mathematics and also in physics. For example, the Hilbert space of the Chern-Simons (CS) theory with a gauge group G on $\mathbb{R} \times \Sigma_h$ is equivalent to the space of the conformal block for the G/G gauged WZW model on a Riemann surface Σ_h . The partition function of the CS theory on a three manifold can be obtained by sewing the boundary Riemann surfaces which is implemented by an inner product of states on Σ . One can also calculate Wilson loop expectation values which give knot invariants in terms of fusion coefficients and modular matrices [9].

In the WZW model, one can construct the G/H gauged WZW model by gauging an anomaly free subgroup H of the global symmetry group G . The G/H gauged WZW model is an explicit lagrangian realization of the coset construction in the CFT. When $H = G$, the G/G gauged WZW model becomes a topological field theory [10], [11], [12]. There exists a method for calculating the partition function and correlation functions without relying on the CFT techniques nor the representation theory of the affine Lie algebra. Actually, these were derived by a field theoretic approach in [13], [14]. In this approach, it is important for the G/G gauged WZW model to possess a certain BRST-type symmetry whose square generates a G -gauge transformation. This symmetry makes it possible to work out the path integrals with insertions of BRST closed operators via equivariant localization procedure. In higher rank of the gauge group, the localization configurations for the diagonal components of G -elements are complicated. However, the final expression for the partition function is simply expressed by modular matrices.

In this thesis, we will firstly show that the integrable system corresponding to the $U(N)/U(N)$ gauged WZW model is the phase model [15]. The phase model is a quantum integrable field theory on one-dimensional lattice [16]. We can apply the algebraic Bethe Ansatz. For example, see [17], [18] for the Algebraic Bethe Ansatz method. It is known that the phase model appears in the $SU(N)$ WZW model. Recently, Korff and Stroppel established the $\widehat{\mathfrak{su}}(N)_k$ Verlinde algebra [19] in terms of the algebraic Bethe Ansatz for the phase model and derived an efficient recursion relation for calculating fusion coefficients [20], [21]. See also a short review [22]. We will consider relations between the Gauge/Bethe correspondence and [20]. We also point out that the partition function of the CS theory on $S^1 \times \Sigma_h$ is related to norms of eigenstates of the Hamiltonian in the phase model.

We further consider a generalization of the Gauge/Bethe correspondence for the G/G gauged WZW model and the phase model. The phase model is realized by a strong coupling limit of the q -boson model [16]. Therefore, it is natural that G/G theory corresponding to the q -boson model also exists. We call a model like this as the G/G gauged Wess-Zumino-Witten-Higgs (WZW-Higgs) model. The G/G gauged WZW-Higgs is the G/G gauged WZW model coupled to additional scalar matters and is regarded as a non-linear deformation of the BF theory with the gauge group G coupled to additional scalar matters. Then, we will show that an integrable system corresponding to this model in

fact is the q-boson model. Further, we consider relations with the commutative Frobenius algebra constructed by Korff [23]. See [24] for the content in this chapter.

This thesis is organized as follows. In chapter 2, we study the integrable system, especially the q-boson model and apply the Algebraic Bethe Ansatz to this model. We also study the phase model which is a strong coupling limit of the q-boson model. In chapter 3, we study relations between the G/G gauged WZW model and the phase model. To investigate this, we calculate the partition function by applying the localization method to the $U(N)/U(N)$ gauged WZW model. We then find relations between the partition function of the $U(N)/U(N)$ gauged WZW model and the Bethe norms in the phase model. In chapter 4, we consider a generalization of Chapter 3. We introduce the G/G gauged WZW-Higgs model. We apply the localization method to this model and study relations between the G/G gauged WZW-Higgs model and the q-boson model. The chapter 5 is devoted to conclusion. In appendix A, we summarize a convention for G/G gauged WZW model and its generalization in Chapter 3 and 4. In appendix B, we prove an expression for a Bethe norm in detail.

CHAPTER 2

Integrable System

In this chapter, we study “Bethe” part of the Gauge/Bethe correspondence, that is, integrable systems. The integrable system is usually defined as a system which possesses as many commuting conserved charges as a degree of freedom of the system. Therefore, the system has a number of symmetries and becomes exactly solvable. Further, there exists characteristic methods to calculate various observables, the energy eigenvalues, the eigenvectors and the correlation functions and so on in the integrable system.

In this thesis, we especially study the q -boson model and the phase model. The q -boson model is a quantum integrable field theory on one-dimensional lattice and a strongly correlated boson system. The phase model is defined as a strong coupling limit of the q -boson model. These models are firstly introduced by [16] to study a strongly correlated system. See also [17], [18], [20] and [23]. These models are the quantum integrable systems which can apply the Bethe Ansatz methods [25]. The Bethe Ansatz is a general term for methods to calculate the observable of the quantum integrable system. In this thesis, we consider the Algebraic Bethe Ansatz based on the algebraic commutation relations [26]. See e.g. [27], [28] and [29] for the Algebraic Bethe Ansatz.

We study the q -boson model in the section 2.1 and the phase model in the section 2.2. See the Appendix B for the derivation of a norm between eigenvectors in the q -boson model.

2.1 The q -boson model

In this section, we define the q -boson model and apply the Algebraic Bethe Ansatz to this model. The q -boson model is considered as the quantum group deformation of the ordinary boson because the q -boson model becomes the free boson model by taking a limit $q \rightarrow 1$.

2.1.1 The q-boson model

Firstly, we define the q-boson model. Consider the operators $\{q^{\pm\hat{N}}, \beta, \beta^\dagger\}$ satisfying an algebra called by the q-boson algebra or the q-oscillator algebra \mathcal{H}_q

$$\begin{aligned} q^{\hat{N}} q^{-\hat{N}} &= q^{-\hat{N}} q^{\hat{N}} = 1, & q^{\hat{N}} \beta &= \beta q^{\hat{N}-1}, & q^{\hat{N}} \beta^\dagger &= \beta^\dagger q^{\hat{N}+1}, \\ \beta \beta^\dagger - \beta^\dagger \beta &= (1 - q^2) q^{2\hat{N}}, & \beta \beta^\dagger - q^2 \beta^\dagger \beta &= 1 - q^2, \end{aligned} \quad (2.1)$$

where $q^{\pm\hat{N}}$ denote generators and $q^{\pm p\hat{N}+x}$ is shorthand for $(q^{\pm\hat{N}})^p q^x$. The parameter q is a generic c-number and $0 \leq q < 1$. We see that the operators \hat{N} , β and β^\dagger also serve as the number operator, the annihilation operator and creation operator, respectively.

We change the operators obeying the q-boson algebra to the operators $\{a, a^\dagger, \hat{N}\}$ obeying the free boson algebra or the harmonic oscillator algebra

$$[\hat{N}, a] = -a, \quad [\hat{N}, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = 1, \quad (2.2)$$

as follows

$$\beta = \sqrt{\frac{1 - q^{2(\hat{N}+1)}}{1 + \hat{N}}} a, \quad \beta^\dagger = a^\dagger \sqrt{\frac{1 - q^{2(\hat{N}+1)}}{1 + \hat{N}}} \quad (2.3)$$

where these are defined as formal power series. Thus, this algebra is the q -deformation of the harmonic oscillator algebra. Therefore, we see that the q-boson is regarded as the q -deformation of the usual free boson.

Next, we construct a Fock space \mathcal{F} for the q-boson algebra given by (2.1). A set $\{|m\rangle := (\beta^\dagger)^m / (q^2)_m |0\rangle \mid m \in \mathbb{Z}_{\geq 0}\}$ forms the basis of the Fock space. Here, $(x)_m$ is $(x)_m = \prod_{i=0}^{m-1} (1 - x^{i+1})$. It also holds the following relation,

$$q^{\hat{N}} |m\rangle = q^m |m\rangle, \quad \beta^\dagger |m\rangle = (1 - q^{2m+2}) |m+1\rangle, \quad \beta |m\rangle = |m-1\rangle. \quad (2.4)$$

The Hamiltonian of the q-boson model on the lattice size L is defined by

$$H = -\frac{1}{2} \sum_{j=1}^L \left(\beta_i \beta_{j+1}^\dagger + \beta_i^\dagger \beta_{j+1} \right) \quad (2.5)$$

where we impose the periodic boundary condition $L+1 \equiv 1$ and set the lattice spacing $\Delta = 1$. The operators $\{\beta_i, \beta_i^\dagger, q^{\hat{N}_i}\}_{i=1, \dots, L}$ obey the L -fold tensor product $\mathcal{H}^{\otimes L}$ of the q-boson algebra (2.1). $\mathcal{H}^{\otimes L}$ is defined by

$$\begin{aligned} \beta_i \beta_j - \beta_j \beta_i &= \beta_i^\dagger \beta_j^\dagger - \beta_j^\dagger \beta_i^\dagger = q^{\hat{N}_i} q^{\hat{N}_j} - q^{\hat{N}_j} q^{\hat{N}_i} = 0 \\ q^{\hat{N}_i} \beta_j &= \beta_j q^{\hat{N}_i - \delta_{ij}}, & q^{\hat{N}_i} \beta_j^\dagger &= \beta_j^\dagger q^{\hat{N}_i + \delta_{ij}}, \\ \beta_i \beta_j^\dagger - \beta_j^\dagger \beta_i &= \delta_{ij} (1 - q^2) q^{2\hat{N}_i}, & \beta_i \beta_i^\dagger - q^2 \beta_i^\dagger \beta_i &= (1 - q^2). \end{aligned} \quad (2.6)$$

where the index i labels the sites of a lattice. Therefore, the Hamiltonian belongs to $\mathcal{H}^{\otimes L}$ and acts on the L -fold tensor product of the Fock space $\mathcal{F}^{\otimes L}$. The basis of $\mathcal{F}^{\otimes L}$ are the sets $\{|m_1, \dots, m_L\rangle := |m_1\rangle \otimes \dots \otimes |m_L\rangle \mid m_i \in \mathbb{Z}_{\geq 0}\}$.

We rewrite the Hamiltonian (2.5) by using the substitution (2.3) as

$$H = \sum_{j=1}^L \left(\sqrt{\frac{1 - q^{2(\hat{N}_j+1)}}{1 + \hat{N}_j}} a_j a_{j+1}^\dagger \sqrt{\frac{1 - q^{2(\hat{N}_{j+1}+1)}}{1 + \hat{N}_{j+1}}} + a_j^\dagger \sqrt{\frac{1 - q^{2(\hat{N}_j+1)}}{1 + \hat{N}_j}} \sqrt{\frac{1 - q^{2(\hat{N}_{j+1}+1)}}{1 + \hat{N}_{j+1}}} a_{j+1} \right). \quad (2.7)$$

Thus, we see that the Hamiltonian has infinite interaction terms in front of the hopping term. Therefore we find that the q-boson model is the strongly interacting system and the field theory with non-local interactions on the lattice.

When we set $q = e^\eta$ and expand it around $\eta = 0$, we see that the Hamiltonian of the q-boson model reduces to the one of the free boson at the leading order of η . Therefore, q or η serve as a coupling constant. On the other hand, there exists a strong coupling limit $q \rightarrow 0$ and the q-boson model becomes the phase model. We will consider this model at the next section. There also exists a continuum limit because the q-boson model is the field theory on the lattice. In this limit, the q-boson model becomes the non-linear Schrödinger model.

2.1.2 The Algebraic Bethe Ansatz for the q-boson model

In this subsection, we apply the algebraic Bethe Ansatz to the q-boson model. If there exists a vacuum in a model which we would like to consider, we can apply the algebraic Bethe Ansatz to this model. Hence, we can apply the algebraic Bethe Ansatz to the q-boson model. Since a norm between eigenvectors of the Hamiltonian of the q-boson model becomes most important in the Gauge/Bethe correspondence, We will give a formula for the norm between the eigenstates of the Hamiltonian. See the Appendix B for the detailed derivation of this quantity.

We firstly define a L-matrix. The L-matrix of the q-boson model at the site n ($n = 1, \dots, L$) is defined by

$$\mathcal{L}_n(\mu) = \begin{pmatrix} 1 & \mu \beta_n^\dagger \\ \beta_n & \mu \end{pmatrix} \in \text{End}[\mathbb{C}^2(\mu)] \otimes \mathcal{H}_q \quad (2.8)$$

where $\mu \in \mathbb{C}$ is a spectral parameter and β_n and β_n^\dagger obey the q-boson algebra (2.6). Here, the L-matrix is a matrix in an auxiliary space \mathbb{C}^2 . This L-matrix satisfies the Yang-Baxter equation:

$$R(\mu, \nu)(\mathcal{L}(\mu) \otimes \mathcal{L}(\nu)) = (\mathcal{L}(\nu) \otimes \mathcal{L}(\mu))R(\mu, \nu), \quad (2.9)$$

with the R-matrix

$$R(\nu, \mu) = \begin{pmatrix} f(\nu, \mu) & 0 & 0 & 0 \\ 0 & g(\mu, \nu) & 1 & 0 \\ 0 & t & -g(\nu, \mu) & 0 \\ 0 & 0 & 0 & f(\nu, \mu) \end{pmatrix} \in \text{End}[\mathbb{C}^2(\mu) \otimes \mathbb{C}^2(\nu)] \quad (2.10)$$

where

$$f(\mu, \nu) = \frac{\mu t - \nu}{\mu - \nu}, \quad g(\mu, \nu) = \frac{(1-t)\nu}{\mu - \nu} \quad \text{and} \quad t = q^2. \quad (2.11)$$

The monodromy matrix is defined by

$$T(\mu) = \mathcal{L}_L(\mu) \mathcal{L}_{L-1}(\mu) \cdots \mathcal{L}_1(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}. \quad (2.12)$$

From the Yang-Baxter equation (2.9), the monodromy matrix satisfies a following relation:

$$R(\mu, \nu)(T(\mu) \otimes T(\nu)) = (T(\nu) \otimes T(\mu))R(\mu, \nu). \quad (2.13)$$

From this relation, we can derive 16 commutation relations for the monodromy matrix elements, $A(\mu), B(\mu), C(\mu), D(\mu)$:

$$A(\mu)A(\nu) = A(\nu)A(\mu) \quad (2.14)$$

$$B(\mu)B(\nu) = B(\nu)B(\mu) \quad (2.15)$$

$$C(\mu)C(\nu) = C(\nu)C(\mu) \quad (2.16)$$

$$D(\mu)D(\nu) = D(\nu)D(\mu) \quad (2.17)$$

$$(\mu - \nu)A(\mu)B(\nu) = (t\mu - \nu)B(\nu)A(\mu) + (1-t)\nu B(\mu)A(\nu) \quad (2.18)$$

$$t(\mu - \nu)A(\mu)C(\nu) = (\mu - t\nu)C(\nu)A(\mu) - (1-t)\mu C(\mu)A(\nu) \quad (2.19)$$

$$t(\mu - \nu)B(\mu)A(\nu) = (t\mu - \nu)A(\nu)B(\mu) + (1-t)\mu A(\mu)B(\nu) \quad (2.20)$$

$$t(\mu - \nu)B(\mu)D(\nu) = (\mu - t\nu)D(\nu)B(\mu) - (1-t)\mu D(\mu)B(\nu) \quad (2.21)$$

$$(\mu - \nu)C(\mu)A(\nu) = (\mu - t\nu)A(\nu)C(\mu) - (1-t)\nu A(\mu)C(\nu) \quad (2.22)$$

$$(\mu - \nu)C(\mu)D(\nu) = (t\mu - \nu)D(\nu)C(\mu) + (1-t)\nu D(\mu)C(\nu) \quad (2.23)$$

$$(\mu - \nu)D(\mu)B(\nu) = (\mu - t\nu)B(\nu)D(\mu) - (1-t)\nu B(\mu)D(\nu) \quad (2.24)$$

$$t(\mu - \nu)D(\mu)C(\nu) = (t\mu - \nu)C(\nu)D(\mu) + (1-t)\mu C(\mu)D(\nu) \quad (2.25)$$

$$C(\mu)B(\nu) - tB(\nu)C(\mu) = \frac{(1-t)\nu}{\mu - \nu} (D(\mu)A(\nu) - D(\nu)A(\mu)) \quad (2.26)$$

$$= \frac{(1-t)\nu}{\mu - \nu} (A(\nu)D(\mu) - A(\mu)D(\nu)) \quad (2.27)$$

$$[A(\mu), D(\nu)] = \frac{1-t}{\mu - \nu} (\nu B(\mu)C(\nu) - \mu B(\nu)C(\mu)) \quad (2.28)$$

$$= \frac{1-t}{t(\mu - \nu)} (\nu C(\nu)B(\mu) - \mu C(\mu)B(\nu)) \quad (2.29)$$

The algebra defined by this commutation relations is called as the Yang-Baxter algebra.

Taking trace with respect to the auxiliary space, the monodromy matrix becomes the transfer matrix

$$\tau(\mu) = \text{tr}T(\mu) = A(\mu) + D(\mu). \quad (2.30)$$

As we take a trace with respect to the auxiliary space for the both sides of (2.13), we can show that the transfer matrices at the different spectral parameters commute:

$$[\tau(\mu), \tau(\nu)] = 0. \quad (2.31)$$

When we expand the transfer matrix $\tau(\mu)$ as a power series, $\tau(\mu) = \sum_{a=0}^L H_a \mu^a$, the all operators $\{H_0, H_1, \dots, H_L\}$ commute. Therefore, the transfer matrix can be regarded as a generating function of the conserved charges. Noticing that H_0 and H_L are not independent conserved charges $H_0 = H_L = 1$, we see that the q-boson model possesses as many commuting conserved charges as a degree of freedom of the system. Hence the q-boson model is a quantum integrable system. In general, it becomes a sufficient condition for the quantum integrability that L and R-matrix which satisfy the Yang-Baxter equation exist. The Hamiltonian of the q-boson model (2.5) is expressed by the conserved charges as

$$H = -\frac{1}{2}(H_1 + H_{L-1}). \quad (2.32)$$

Next, we will construct the eigenvalues and the eigenvectors of the transfer matrix. First of all, we have to define the vacuum state. The vacuum state $|0\rangle$ and it's dual vacuum state $\langle 0|$ are defined by

$$C(\mu)|0\rangle = 0 \quad \text{and} \quad \langle 0|B(\mu) = 0, \quad (2.33)$$

because $C(\mu)$ is a creation operator and $B(\mu)$ is an annihilation operator. $a(\mu) = 1$ and $d(\mu) = \mu^L$ are the eigenvalues of operators $A(\mu)$ and $D(\mu)$ on the vacuum state $|0\rangle$, respectively:

$$A(\mu)|0\rangle = a(\mu)|0\rangle = |0\rangle, \quad D(\mu)|0\rangle = d(\mu)|0\rangle = \mu^L|0\rangle. \quad (2.34)$$

We consider a vector constructed by successive actions of operators $B(\lambda)$ on the vacuum state $|0\rangle$. Now, $\{\lambda_1, \dots, \lambda_M\}$ are generic complex numbers. Let us compute the action of the operators $A(\mu)$, $C(\mu)$ and $D(\mu)$ on the vector $\prod_{j=1}^M B(\lambda_j)|0\rangle$ by using the commutation relations (2.14)-(2.29). The final result is

$$\begin{aligned} A(\mu) \prod_{j=1}^M B(\lambda_j)|0\rangle &= a(\mu) \prod_{j=1}^M f(\mu, \lambda_j) \prod_{j=1}^M B(\lambda_j)|0\rangle \\ &+ \sum_{k=1}^M a(\lambda_k) g(\mu, \lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_k, \lambda_j) B(\mu) \prod_{\substack{j=1 \\ j \neq k}}^M B(\lambda_j)|0\rangle, \end{aligned} \quad (2.35)$$

$$\begin{aligned}
D(\mu) \prod_{j=1}^M B(\lambda_j)|0\rangle &= d(\mu) \prod_{j=1}^M f(\lambda_j, \mu) \prod_{j=1}^M B(\lambda_j)|0\rangle \\
&\quad - \sum_{k=1}^M d(\lambda_k) g(\mu, \lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_j, \lambda_k) B(\mu) \prod_{\substack{j=1 \\ j \neq k}}^M B(\lambda_j)|0\rangle, \tag{2.36}
\end{aligned}$$

$$\begin{aligned}
C(\mu) \prod_{j=1}^M B(\lambda_j)|0\rangle &= \sum_{k=1}^M \left\{ a(\lambda_k) d(\mu) g(\mu, \lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M \left(f(\lambda_j, \mu) f(\lambda_k, \lambda_j) \right) \right. \\
&\quad - a(\mu) d(\lambda_k) g(\mu, \lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M \left(f(\mu, \lambda_j) f(\lambda_j, \lambda_k) \right) \left. \right\} \prod_{\substack{j=1 \\ j \neq k}}^M B(\lambda_j)|0\rangle \\
&\quad - \sum_{\ell > k}^M \left\{ a(\lambda_k) d(\lambda_\ell) g(\mu, \lambda_\ell) g(\mu, \lambda_k) f(\lambda_k, \lambda_\ell) \prod_{\substack{j=1 \\ j \neq k, \ell}}^M \left(f(\lambda_j, \lambda_\ell) f(\lambda_k, \lambda_j) \right) \right. \\
&\quad \left. + a(\lambda_\ell) d(\lambda_k) g(\mu, \lambda_\ell) g(\mu, \lambda_k) f(\lambda_\ell, \lambda_k) \prod_{\substack{j=1 \\ j \neq k, \ell}}^M \left(f(\lambda_j, \lambda_k) f(\lambda_\ell, \lambda_j) \right) \right\} \\
&\quad \times B(\mu) \prod_{\substack{j=1 \\ j \neq k, \ell}}^M B(\lambda_j)|0\rangle. \tag{2.37}
\end{aligned}$$

In the same way, we consider a vector constructed by successive actions of operators $C(\lambda)$ on the dual vacuum state $\langle 0|$, $\langle 0| \prod_{j=1}^M C(\lambda_j)$ by using the commutation relations (2.14)-(2.29)¹. The action of the operators $A(\mu)$, $B(\mu)$ and $D(\mu)$ on the vector $\langle 0| \prod_{j=1}^M C(\lambda_j)$ is

$$\begin{aligned}
\langle 0| \prod_{j=1}^M C(\lambda_j) A(\mu) &= a(\mu) \prod_{j=1}^M f(\mu, \lambda_j) \langle 0| \prod_{j=1}^M C(\lambda_j) \\
&\quad - \sum_{k=1}^M a(\lambda_k) g(\lambda_k, \mu) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_k, \lambda_j) \cdot \langle 0| \prod_{\substack{j=1 \\ j \neq k}}^M C(\lambda_j) C(\mu), \tag{2.38}
\end{aligned}$$

$$\begin{aligned}
\langle 0| \prod_{j=1}^M C(\lambda_j) D(\mu) &= d(\mu) \prod_{j=1}^M f(\lambda_j, \mu) \langle 0| \prod_{j=1}^M C(\lambda_j) \\
&\quad + \sum_{k=1}^M d(\lambda_k) g(\lambda_k, \mu) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_j, \lambda_k) \langle 0| \prod_{\substack{j=1 \\ j \neq k}}^M C(\lambda_j) C(\mu), \tag{2.39}
\end{aligned}$$

¹Notice that the vector $\langle 0| \prod_{j=1}^M C(\lambda_j)$ is not the dual vector of $\prod_{j=1}^M B(\lambda_j)|0\rangle$ where $\{\lambda_1, \dots, \lambda_M\}$ are generic complex numbers. In other words, this means $(B(\lambda))^\dagger \neq C(\lambda)$.

$$\begin{aligned}
\langle 0 | \prod_{j=1}^M C(\lambda_j) B(\mu) &= \sum_{k=1}^M \left\{ -a(\lambda_k) d(\mu) g(\lambda_k, \mu) \prod_{\substack{j=1 \\ j \neq k}}^M \left(f(\lambda_j, \mu) f(\lambda_k, \lambda_j) \right) \right. \\
&\quad \left. + a(\mu) d(\lambda_k) g(\lambda_k, \mu) \prod_{\substack{j=1 \\ j \neq k}}^M \left(f(\mu, \lambda_j) f(\lambda_j, \lambda_k) \right) \right\} \langle 0 | \prod_{\substack{j=1 \\ j \neq k}}^M C(\lambda_j) \\
&\quad - \sum_{\ell > k}^M \left\{ a(\lambda_k) d(\lambda_\ell) g(\lambda_\ell, \mu) g(\lambda_k, \mu) f(\lambda_k, \lambda_\ell) \prod_{\substack{j=1 \\ j \neq k, \ell}}^M \left(f(\lambda_j, \lambda_\ell) f(\lambda_k, \lambda_j) \right) \right. \\
&\quad \left. + a(\lambda_\ell) d(\lambda_k) g(\lambda_\ell, \mu) g(\lambda_k, \mu) f(\lambda_\ell, \lambda_k) \prod_{\substack{j=1 \\ j \neq k, \ell}}^M \left(f(\lambda_j, \lambda_k) f(\lambda_\ell, \lambda_j) \right) \right\} \\
&\quad \times \langle 0 | \prod_{\substack{j=1 \\ j \neq k, \ell}}^M C(\lambda_j) C(\mu). \tag{2.40}
\end{aligned}$$

Let us construct eigenvalue and eigenstates of the transfer matrix. When the transfer matrix acts on the state $\prod_{j=1}^M B(\lambda_j)|0\rangle$, we obtain a following expression by using (2.35) and (2.36):

$$\begin{aligned}
\tau(\mu) \prod_{j=1}^M B(\lambda_j)|0\rangle &= \Lambda(\mu, \{\lambda\}) \prod_{j=1}^M B(\lambda_j)|0\rangle \\
&\quad + \sum_{k=1}^M \left\{ a(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_k, \lambda_j) - d(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^M f(\lambda_j, \lambda_k) \right\} \tag{2.41}
\end{aligned}$$

where

$$\Lambda(\mu, \{\lambda\}) = a(\mu) \prod_{j=1}^M f(\mu, \lambda_j) + d(\mu) \prod_{j=1}^M f(\lambda_j, \mu). \tag{2.42}$$

Suppose that the state $\prod_{j=1}^M B(\lambda_j)|0\rangle$ is eigenstates of the transfer matrix. Then, we find that the second term of (2.41) must vanish and the spectral parameters $\{\lambda_j\}_{j=1, \dots, M}$ must therefore satisfy equations

$$a(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^M f(\lambda_j, \lambda_k) = d(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^M f(\lambda_k, \lambda_j) \quad \text{for } j = 1, \dots, M. \tag{2.43}$$

This equations are referred to as Bethe Ansatz equations. Also, the spectral parameters which satisfy the Bethe Ansatz equations are called by Bethe roots. Thus, we can have

constructed the eigenvectors of the transfer matrix. At the moment, (2.42) becomes the eigenvalues of the transfer matrix. The Bethe Ansatz equations concretely are

$$\lambda_j^{-L} \prod_{\substack{k=1 \\ k \neq j}}^M \frac{\lambda_j t - \lambda_k}{\lambda_j - \lambda_k t} = 1 \quad \text{for } j = 1, \dots, M. \quad (2.44)$$

Notice that the Bethe roots assign a ground state or excited states in the q-boson model.

From now on, we consider properties of the Bethe Ansatz equations. For convenience, we change a parametrization of the Bethe roots as $\lambda_j = e^{2\pi i x_j}$ for $j = 1, \dots, M$ and of the coupling constant t as $t = e^{-2\pi\eta}$ where $\eta > 0$ because $0 \leq t < 1$. Then, the Bethe Ansatz equations become

$$e^{2\pi i L x_j} = \prod_{\substack{k=1 \\ k \neq j}}^M \frac{\sin[\pi(x_j - x_k + i\eta)]}{\sin[\pi(x_j - x_k - i\eta)]} \quad \text{for } j = 1, \dots, M. \quad (2.45)$$

From this equations, we can prove that the Bethe roots $\{x_1, \dots, x_M\}$ are real numbers by using a similar manner with the Bose gas model [27]. A logarithmic form of (2.45) is

$$2\pi i L x_j = 2\pi i I_j + \sum_{k=1}^M \log \frac{\sin[\pi(i\eta + (x_j - x_k))]}{\sin[\pi(i\eta - (x_j - x_k))]} \quad (2.46)$$

where I_j is (half-)integers when M is (even) odd. In a similar manner as the Bose gas model [27], we can also show the existence and uniqueness of the solutions of the Bethe Ansatz equations once we assign $\{I_1, \dots, I_M\}$.

We finally consider an inner product between the two vectors $\prod_{a=1}^M B(\mu_a)|0\rangle$ and $\langle 0|\prod_{a=1}^M C(\nu_a)$:

$$\langle 0|\prod_{a=1}^M C(\mu_a) \prod_{a=1}^M B(\nu_a)|0\rangle \quad (2.47)$$

where $\{\mu_1, \dots, \mu_M\}$ and $\{\nu_1, \dots, \nu_M\}$ are generic complex numbers. The inner product is rewritten as a determinant formula if either of $\{\mu_1, \dots, \mu_M\}$ or $\{\nu_1, \dots, \nu_M\}$ satisfy the Bethe Ansatz equations (2.44). This calculation essentially only use the commutation relations (2.14) - (2.29). See Appendix B for the detailed derivation.

We define a Bethe vector and the dual Bethe vector which is the eigenvector of the transfer matrix, as

$$|\psi(\{\lambda\}_M) = \prod_{a=1}^M B(\lambda_a)|0\rangle \quad \text{and} \quad \langle\psi(\{\lambda\}_M)| = \langle 0|\prod_{a=1}^M C(\lambda_a) \quad (2.48)$$

where $\{\lambda_1, \dots, \lambda_M\}$ satisfies the Bethe Ansatz equations (2.44)². When $\{\mu_1 \dots \mu_M\}$ are generic complex numbers, the inner product becomes (B.53):

$$\begin{aligned} \langle \psi(\{\lambda\}) | \prod_{a=1}^M B(\mu_a) | 0 \rangle &= \prod_{a=1}^M \left(\frac{\mu_a}{\lambda_a} \right) \cdot \langle 0 | \prod_{a=1}^M C(\mu_a) | \psi(\{\lambda\}) \rangle \\ &= \prod_{a=1}^M d(\lambda_a) \cdot \chi_M^{-1}(\{\mu\}, \{\lambda\}) \cdot \det_M \left(\frac{\partial}{\partial \lambda_j} \Lambda(\mu_k, \{\lambda\}) \right) \end{aligned} \quad (2.49)$$

where $\Lambda(\mu_k, \{\lambda\})$ is the eigenvalue of the transfer matrix (2.42) and $\chi_M(\{\mu\}, \{\lambda\})$ is the Cauchy determinant:

$$\chi_M(\{\mu\}, \{\lambda\}) = \frac{\prod_{a>b}^M (\lambda_a - \lambda_b)(\mu_b - \mu_a)}{\prod_{a,b=1}^M (\mu_a - \lambda_b)}. \quad (2.50)$$

Furthermore, when $\{\mu_1, \dots, \mu_M\}$ in (2.49) also satisfies the Bethe Ansatz equations (2.44), the scalar product becomes (2.70):

$$\begin{aligned} \langle \psi(\{\lambda\}_M) | \psi(\{\lambda\}_M) \rangle &= \langle 0 | \prod_{a=1}^M C(\lambda_a) \prod_{a=1}^M B(\lambda_a) | 0 \rangle \\ &= \frac{\prod_{a,b=1}^M (\lambda_a t - \lambda_b)}{\prod_{\substack{a,b=1 \\ a \neq b}}^M (\lambda_a - \lambda_b)} \cdot \det_M \Phi'_{j,k}(\{\lambda\}_M) \end{aligned} \quad (2.51)$$

where the Gaudin matrix $\Phi'_{j,k}(\{\lambda\}_M)$ is

$$\begin{aligned} \Phi'_{j,k}(\{\lambda\}_M) &= \frac{\partial}{\partial \lambda_k} \log \left\{ \lambda_j^{-L} \cdot \prod_{\substack{b=1 \\ b \neq j}}^M \frac{\lambda_j t - \lambda_b}{\lambda_j - \lambda_b t} \right\} \\ &= \delta_{j,k} \left\{ -\frac{L}{\lambda_j} + \sum_{b=1}^M \frac{(t^2 - 1)\lambda_b}{(\lambda_j t - \lambda_b)(\lambda_b t - \lambda_j)} \right\} - \frac{(t^2 - 1)\lambda_j}{(\lambda_j t - \lambda_k)(\lambda_k t - \lambda_j)}. \end{aligned} \quad (2.52)$$

From now on, we refer to this inner product as the Bethe norm throughout this thesis. This Bethe norm will become most important quantity when we study the correspondence between the q-boson model and a topological field theory.

We comment on the relations between the q-boson model and the infinite spin XXZ model. When the number of sites is even, the Bethe Ansatz equations (2.45) and the Bethe norm (2.51) coincide with the one of the infinite spin XXZ model under an appropriate rescaling of the parameters. See the algebraic Bethe Ansatz for the higher spin XXZ model, e.g. [30] and [31]. The agreement of the Bethe Ansatz equations in the q-boson

²Notice that the vector $\langle 0 | \prod_{j=1}^M C(\lambda_j)$ becomes the dual vector of $\prod_{j=1}^M B(\lambda_j) | 0 \rangle$ when $\{\lambda_1, \dots, \lambda_M\}$ satisfy the Bethe Ansatz equations, because $(B(\lambda))^{\dagger} = C(\lambda)$.

model and the infinite spin XXZ model is not accidental. This is because the q -oscillator representation is equivalent to the infinite limit of spin- s representation in the quantum group. In the case of $\mathfrak{su}_q(2)$, this fact is proved at [32].

Finally, we state applications for the integrable system. In the integrable system, one of the most challenging problem is a derivation of the correlation function for the integrable field theory or the integrable spin system at a continuum limit or a thermodynamic limit. In the XXZ model, the correlation function has calculated by taking advantage of the scalar product, the Bethe norm and the inverse scattering formula [33] at the thermodynamic limit, e.g. [34] and [35]. Therefore, one will be able to use the scalar product (2.49) and the Bethe norm (2.51) at the calculation of the correlation function in the q -boson model.

2.2 The phase model

In the previous section, we have defined the phase model by taking a strong coupling limit of $q \rightarrow 0$. Therefore, the phase model is a quantum integrable field theory on one-dimensional lattice and a strongly correlated boson system. In this section, we study the phase model. We also apply the algebraic Bethe Ansatz to this model as well as the q -boson model.

2.2.1 The phase model

In this subsection, we define the phase model. First of all, we define the phase algebra by taking the limit $q \rightarrow 0$ for the q -boson algebra. Therefore, the phase algebra Φ is an algebra such that operators $\{\hat{N}, \varphi, \varphi^\dagger\}$ obey

$$[\hat{N}, \varphi] = -\varphi, \quad [\hat{N}, \varphi^\dagger] = \varphi^\dagger, \quad \varphi\varphi^\dagger = 1. \quad (2.53)$$

The operators φ and φ^\dagger serve as an annihilation operator and a creation operator, respectively. Next, we define a Fock space \mathcal{F} for the phase algebra given by the equations (2.53) by acting the creation operator φ^\dagger on a vacuum $|0\rangle$ which is defined as $\varphi|0\rangle = 0$. The set $\{|m\rangle := (\varphi^\dagger)^m|0\rangle \mid m \in \mathbb{Z}_{\geq 0}\}$ forms the basis of the Fock space. The action of the operators $\hat{N}, \varphi^\dagger, \varphi$ on a state $|m\rangle$ also are

$$\hat{N}|m\rangle = m|m\rangle, \quad \varphi^\dagger|m\rangle = |m+1\rangle, \quad \varphi|m\rangle = |m-1\rangle. \quad (2.54)$$

The Hamiltonian of the phase model on the one-dimensional lattice with the total site number L is given by

$$H = -\frac{1}{2} \sum_{i=1}^L \left(\varphi_i \varphi_{i+1}^\dagger + \varphi_i^\dagger \varphi_{i+1} \right) \quad (2.55)$$

where we imposed a periodic boundary condition $L+1 \equiv 1$ and set the lattice spacing $\Delta = 1$. The operators $\{\varphi_i, \varphi_i^\dagger, \hat{N}_i\}_{i=1, \dots, L}$ obey the L -fold tensor product $\Phi^{\otimes L}$ of the phase

algebra (2.53). $\Phi^{\otimes L}$ is defined by

$$\begin{aligned} \varphi_i \varphi_j &= \varphi_j \varphi_i, & \varphi_i^\dagger \varphi_j^\dagger &= \varphi_j^\dagger \varphi_i^\dagger, & \hat{N}_i \hat{N}_j &= \hat{N}_j \hat{N}_i \\ \hat{N}_i \varphi_j - \varphi_j \hat{N}_i &= -\delta_{ij} \varphi_i, & \hat{N}_i \varphi_j^\dagger - \varphi_j^\dagger \hat{N}_i &= \delta_{ij} \varphi_i^\dagger \\ \varphi_i \varphi_i^\dagger &= 1, & \varphi_i \varphi_j^\dagger &= \varphi_j^\dagger \varphi_i & \text{if } i \neq j \\ \hat{N}_i (1 - \varphi_i^\dagger \varphi_i) &= 0 & &= (1 - \varphi_i^\dagger \varphi_i) \hat{N}_i. \end{aligned} \quad (2.56)$$

where the indices i, j label the sites of the lattice. Therefore, the Hamiltonian belongs to $\Phi^{\otimes L}$ and acts on the L -fold tensor products of the Fock space $\mathcal{F}^{\otimes L}$. The basis of $\mathcal{F}^{\otimes L}$ consists of $\{|m_1, \dots, m_L\rangle := |m_1\rangle \otimes \dots \otimes |m_L\rangle \mid m_i \in \mathbb{Z}_{\geq 0}\}$.

To better understand this model, let us change the operators obeying the phase algebra to the operators $\{a_i, a_i^\dagger, \hat{N}_i\}_{i=1, \dots, L}$ obeying the free boson algebra

$$[\hat{N}_i, a_j] = -\delta_{ij} a_j, \quad [\hat{N}_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad \hat{N}_i = a_i^\dagger a_i, \quad (2.57)$$

as follows

$$\varphi_i = \frac{1}{\sqrt{1 + \hat{N}_i}} a_i, \quad \varphi_i^\dagger = a_i^\dagger \frac{1}{\sqrt{1 + \hat{N}_i}} \quad (2.58)$$

where $(1 + \hat{N}_i)^{-1/2}$ is defined as formal power series. Substituting (2.58) into the Hamiltonian (2.55), we found that the Hamiltonian has infinite interaction terms in front of the hopping term. Therefore we found that the phase model is the strongly interacting system and the field theory with non-local interactions on the lattice as well as the q-boson model.

2.2.2 Algebraic Bethe Ansatz for the phase model

In this subsection, we apply the algebraic Bethe Ansatz to the phase model. As the all result is obtained by setting $q \rightarrow 0$ for the coupling constant of the q-boson model, we do not repeat the calculation in detail.

The L-matrix of the phase model at a site n ($n = 1, \dots, L$) is defined by

$$\mathcal{L}_n(\mu) = \begin{pmatrix} 1 & \mu \varphi_n^\dagger \\ \varphi_n & \mu \end{pmatrix} \in \text{End}[\mathbb{C}^2(\mu)] \otimes \Phi, \quad (2.59)$$

where $\mu \in \mathbb{C}$ is a spectral parameter. Here, the L-matrix is a matrix in an auxiliary space \mathbb{C}^2 . This L-matrix satisfies the Yang-Baxter equation

$$R(\mu, \nu)(\mathcal{L}(\mu) \otimes \mathcal{L}(\nu)) = (\mathcal{L}(\nu) \otimes \mathcal{L}(\mu))R(\mu, \nu), \quad (2.60)$$

with the R-matrix

$$R(\mu, \nu) = \begin{pmatrix} \frac{\mu}{\mu-\nu} & 0 & 0 & 0 \\ 0 & \frac{\nu}{\mu-\nu} & 1 & 0 \\ 0 & 0 & \frac{\mu}{\mu-\nu} & 0 \\ 0 & 0 & 0 & \frac{\mu}{\mu-\nu} \end{pmatrix} \in \text{End}[\mathbb{C}^2(\mu) \otimes \mathbb{C}^2(\nu)]. \quad (2.61)$$

The monodromy matrix is defined by

$$T(\mu) = \mathcal{L}_L(\mu)\mathcal{L}_{L-1}(\mu)\cdots\mathcal{L}_1(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}. \quad (2.62)$$

Taking trace with respect to the auxiliary space, the monodromy matrix becomes the transfer matrix

$$\tau(\mu) = \text{tr}T(\mu) = A(\mu) + D(\mu). \quad (2.63)$$

Eigenstates of the transfer matrix can be constructed by repeated actions of the operators $B(\lambda)$ on the vacuum state $|0\rangle$, that is to say, a state $\prod_{j=1}^M B(\lambda_j)|0\rangle$ is the eigenstate of the transfer matrix

$$\tau(\mu) \prod_{a=1}^M B(\lambda_a)|0\rangle = \Lambda(\mu, \{\lambda\}) \prod_{a=1}^M B(\lambda_a)|0\rangle \quad (2.64)$$

where

$$\Lambda(\mu, \{\lambda\}) = \prod_{a=1}^M \frac{\lambda_a}{\lambda_a - \mu} + \mu^L \prod_{a=1}^M \frac{\mu}{\mu - \lambda_a}, \quad (2.65)$$

if the spectrum parameters $\{\lambda_1, \dots, \lambda_M\}$ satisfy Bethe Ansatz equations

$$(-1)^{M-1} \cdot \lambda_a^{(M+L)} \cdot \prod_{b=1}^M \lambda_b^{-1} = 1. \quad (2.66)$$

When we set $\lambda_j = e^{2\pi i\phi_j}$, the logarithmic form of this equations becomes

$$(L + M)\phi_a - \sum_{a=1}^M \phi_a = I_a \quad (2.67)$$

where I_j is (half-)integers when M is (even) odd. We can easily solve this equation because this equations are linear algebraic equations unlike the q-boson. The solutions is given as

$$\phi_a = \frac{1}{M + L} \left(I_a + \frac{1}{L} \sum_{a=1}^M I_a \right). \quad (2.68)$$

The fact that the Bethe Ansatz equations can be exactly solved becomes important when we study the correspondence with the topological gauge theory.

Finally, we consider a Bethe norm

$$\langle 0 | \prod_{a=1}^M C(\lambda_a) \prod_{a=1}^M B(\lambda_a) | 0 \rangle \quad (2.69)$$

where $\{\lambda_1, \dots, \lambda_M\}$ satisfy the Bethe Ansatz equations. The Bethe norm in the phase model is also given by $q \rightarrow 0$ of (2.51) as

$$\langle \psi(\{\lambda\}_M) | \psi(\{\lambda\}_M) \rangle = \frac{\prod_{b=1}^M \lambda_b^{M-1}}{\prod_{\substack{a,b=1 \\ a \neq b}}^M (\lambda_a - \lambda_b)} L(L+M)^{M-1}. \quad (2.70)$$

This coincides with a result calculated by [18].

CHAPTER 3

G/G gauged Wess-Zumino-Witten model

In this chapter, we study the G/G gauged Wess-Zumino-Witten model on a genus- h Riemann surface Σ_h . We will establish the Gauge/Bethe correspondence for this model and the phase model introduced at Section 2.2. To see this, we apply the equivariant localization method to this model and calculate the partition function. Then, we will see that the localization configuration coincides with the Bethe Ansatz equations in the phase model and the partition function is related to the Bethe norm in the phase model. Finally, we consider relations with [20]. See [15] for the contents in this chapter.

In section 3.1, we define the G/G gauged WZW model on Σ_h . In section 3.2, we explain the equivariant localization method for this model with the gauge group $U(N)$. In section 3.3, we investigate relations between the $U(N)/U(N)$ gauged WZW model and the phase model. See Appendix A for the convention which we use in this chapter.

3.1 G/G gauged Wess-Zumino-Witten model

In this section, we introduce the G/G gauged WZW model on a genus- h Riemann surface Σ_h . Firstly, we consider the G WZW model which is a two dimensional conformal field theory. The action of the G WZW model is given by

$$kS_{\text{WZW}}(g) = -\frac{k}{4\pi} \int_{\Sigma_h} d^2z \text{Tr}(g^{-1} \partial_z g \cdot g^{-1} \partial_{\bar{z}} g) - ik\Gamma(g) \quad (3.1)$$

and the Wess-Zumino term $\Gamma(g)$ which is a topological term, is defined by

$$\Gamma(g) = \frac{1}{12\pi} \int_B d^3y \epsilon^{ijk} \text{Tr}(g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g). \quad (3.2)$$

Here, a field $g(z, \bar{z})$ is a map $g : \Sigma_h \rightarrow G$ from a genus- h Riemann surface to a compact Lie group G . B is a three dimensional manifold with the boundary $\partial B = \Sigma_h$.

One can construct the G/H gauged WZW model by gauging an anomaly free subgroup H of the global symmetry group G in the WZW model. The G/H gauged WZW model is regarded as an explicit Lagrangian realization of the coset construction in the CFT. When

especially $H = G$, the G/G gauged WZW model becomes a topological field theory [10], [11], [12]. Therefore, observables of this theory is topological invariants. For example, the partition function counts the dimensions of the space of conformal block, the number of conformal blocks in the G WZW model with level k on Σ_h . Also, a three point function of $g(z, \bar{z})$ gives fusion coefficients which is the number of the fusion with the WZW primary fields. Thus, one can consider that G/G gauged WZW model only has information of topological part in the WZW model with the global symmetry G . The partition function of the G/G gauged WZW model on Σ_h is defined by

$$Z_{\text{GWZW}}^G(\Sigma_h) = \int \mathcal{D}g \mathcal{D}A \mathcal{D}\lambda e^{-kS_{G/G}(g,A,\lambda)}. \quad (3.3)$$

where the action is defined as

$$\begin{aligned} S_{\text{GWZW}}(g, A, \lambda) &= S_{\text{WZW}}(g) - \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(\lambda_z \lambda_{\bar{z}}) \\ &\quad - \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(A_z \partial_{\bar{z}} g g^{-1} - A_{\bar{z}} g^{-1} \partial_z g - g^{-1} A_z g A_{\bar{z}} + A_z A_{\bar{z}}) \end{aligned} \quad (3.4)$$

where $S_{\text{WZW}}(g)$ is given by (3.1). Here, $A = A_z dz + A_{\bar{z}} d\bar{z}$ is a two-dimensional gauge field and $\lambda = \lambda_z dz + \lambda_{\bar{z}} d\bar{z}$ is a one-form adjoint fermion. We also denote the holomorphic part of A as $A^{(1,0)} = A_z dz$ and anti-holomorphic part as $A^{(0,1)} = A_{\bar{z}} d\bar{z}$ and so on. This model has the BRST symmetry which is generated by a scalar BRST charge Q defined by

$$QA = \lambda, \quad Q\lambda^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)}, \quad Q\lambda^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, \quad Qg = 0 \quad (3.5)$$

with $A^g = g^{-1}dg + gAg^{-1}$. The partition function (3.3) is invariant under the BRST transformation. The square of the BRST transformation generates gauge transformations

$$\begin{aligned} \mathcal{L}_g A^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)}, & \mathcal{L}_g A^{(0,1)} &= -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, \\ \mathcal{L}_g \lambda^{(1,0)} &= g^{-1} \lambda^{(1,0)} g - \lambda^{(1,0)}, & \mathcal{L}_g \lambda^{(0,1)} &= -g \lambda^{(0,1)} g^{-1} + \lambda^{(0,1)}, & \mathcal{L}_g g &= 0, \end{aligned} \quad (3.6)$$

where $Q^2 = \mathcal{L}_g$. Then, the partition function is of course invariant under this transformation.

Finally, we comment about relations between G/G gauged WZW model and other theories. As a first relation, we consider the BF theory. The partition function of the BF theory measures a volume of a moduli space of a flat connection. When we set $g = e^{2\pi i \phi/k}$ and expand them at the leading order of $1/k$, the G/G gauged WZW model on Σ_h reduces to the BF theory with the gauge group G on Σ_h [36]. Therefore, G/G gauged WZW model is considered as a non-linear deformation of the BF theory.

Further, there is a relation with the Chern-Simons (CS) theory. The Hilbert space of the CS theory with a gauge group G on $\mathbb{R} \times \Sigma_h$ is equivalent to the space of the conformal block for the G/G gauged WZW model on Σ_h [9]. Therefore, the partition function of both models coincides [13].

3.2 Localization

In this section, we calculate the partition function of the G/G gauged WZW model on the Riemann surface by using the equivariant localization method. This calculation is originally carried out by Blau and Thompson [13] in the case of a gauge group $SU(N)$. See also [37].

We consider the case of the gauge group $U(N)$ for simplicity and evaluate the partition function under this gauge group. First of all, we must gauge fix. We take a diagonal gauge on $t(z, \bar{z}) = g(z, \bar{z}) \in U(N)$ at (3.3), that is, t is an element of the maximal torus \mathbf{T} of the gauge group $U(N)$. Here, notice that we do not completely gauge fix yet and the abelian gauge symmetry remains as the residual gauge symmetry.

The functional of g that one obtains after having performed the path integral over the gauge fields is locally and pointwise conjugation invariant:

$$\mathcal{F}(g) = \int \mathcal{D}A \exp\{-kS_{GWZW}(g, A)\} = \mathcal{F}(h^{-1}gh). \quad (3.7)$$

Therefore, one can formally apply the infinite-dimensional Weyl integral formula to the partition function of the gauged WZW model:

$$\int Dg \mathcal{F}(g) = \frac{1}{|W|} \int Dt DADet_0(1 - \text{Ad}(t)) \exp(-kS_{GWZW}(t, A)). \quad (3.8)$$

where $|W| = N!$ is the order of Weyl group for the gauge group $U(N)$. Det_0 is the Faddeev-Popov determinant for the gauge fixing.

The action (3.4) under the diagonal gauge also becomes

$$\begin{aligned} S_{GWZW}(t, A) &= -\frac{1}{4\pi} \int_{\Sigma_h} d^2z \text{Tr}(t^{-1} \partial_z t \cdot t^{-1} \partial_{\bar{z}} t) - \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(A_z \partial_{\bar{z}} t t^{-1} - A_{\bar{z}} t^{-1} \partial_z t) \\ &\quad + \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(t^{-1} A_z t A_{\bar{z}} - A_z A_{\bar{z}}) + \Gamma(t). \end{aligned} \quad (3.9)$$

We set $t(z, \bar{z}) = \exp(2\pi i \sum_{j=1}^N \phi_j H^j)$ where H^1, \dots, H^N are the Cartan generators in the Lie algebra $\mathfrak{u}(N)$ and $0 \leq \phi_1, \dots, \phi_N < 1$. Then we expand the gauge field over the Cartan-Weyl basis $\{H^\alpha, E^\alpha, E^{-\alpha}\}$ where α is positive roots. Then, the first term of (3.9) becomes

$$-\frac{1}{4\pi} \int_{\Sigma_h} d^2z \text{Tr}(t^{-1} \partial_z t \cdot t^{-1} \partial_{\bar{z}} t) = 2\pi \sum_{\alpha=1}^N \int_{\Sigma_h} d^2z \partial_z \phi^\alpha \partial_{\bar{z}} \phi^\alpha \quad (3.10)$$

and the second and third terms of (3.9) become

$$\begin{aligned} -\frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(A_z \partial_{\bar{z}} t t^{-1} - A_{\bar{z}} t^{-1} \partial_z t) &= \sum_{\alpha=1}^N \int_{\Sigma_h} d^2z (A_z^\alpha \partial_{\bar{z}} \phi^\alpha - A_{\bar{z}}^\alpha \partial_z \phi^\alpha) \\ &= \sum_{j=1}^N \int_{\Sigma_h} d^2z \phi^j F_{z\bar{z}}^j \end{aligned} \quad (3.11)$$

where we have put the partial integral from the first line to the second line.

By making use of the Baker-Campbell-Hausdorff formula

$$\begin{aligned} \text{ad}A(B) &:= [A, B] \\ e^A B e^{-A} &= e^{\text{ad}A}(B) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots, \end{aligned} \quad (3.12)$$

the fourth term of (3.9) becomes

$$\frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(t^{-1} A_z t A_{\bar{z}}) = -\frac{1}{2\pi} \int_{\Sigma_h} d^2z \left(\sum_{a=1}^N A_z^a A_{\bar{z}}^a + \sum_{\alpha \in \Delta} e^{2\pi i \alpha(\phi)} A_z^{-\alpha} A_{\bar{z}}^{\alpha} \right) \quad (3.13)$$

where $\alpha(\phi) = \sum_{a=1}^N \alpha^a \phi^a$ and Δ is a set of the roots.

The fifth term of (3.9) becomes

$$-\frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr}(A_z A_{\bar{z}}) = \frac{1}{2\pi} \int_{\Sigma_h} d^2z \left(\sum_{a=1}^N A_z^a A_{\bar{z}}^a + \sum_{\alpha \in \Delta} A_z^{-\alpha} A_{\bar{z}}^{\alpha} \right). \quad (3.14)$$

Finally, we consider the WZ term. The WZ term $\Gamma(t)$ naively vanishes but as a topological term it only depends on the winding numbers of the field ϕ . The reason for the appearance of this contribution is, that maps from Σ to \mathbf{T} with non-trivial windings can not necessarily be extended to the interior N of Σ within \mathbf{T} , as some (half) of the non-contractable cycles of Σ become contractible in the handle-body N . The general form of this term is [38]

$$\Gamma(t) = \int_{\Sigma_h} d^2z \mu^{kl} \partial_z \phi_k \partial_{\bar{z}} \phi_l, \quad (3.15)$$

where μ^{kl} is some antisymmetric matrix. As we will show below that the non-trivial winding sectors do not contribute to the partition function, we will not have to be more precise about this term here.

When we put all the result (3.10), (3.11), (3.13) and (3.14) together, we obtain

$$\begin{aligned} S_{\text{GWZW}}(t, A) &= 2\pi \sum_{a=1}^N \int_{\Sigma_h} d^2z \partial_z \phi^a \partial_{\bar{z}} \phi^a + \sum_{a=1}^N \int_{\Sigma_h} d^2z \phi^a F_{z\bar{z}}^a \\ &\quad + \frac{1}{2\pi} \sum_{\alpha \in \Delta} \int_{\Sigma_h} d^2z A_z^{\alpha} (1 - e^{-2\pi i \alpha(\phi)}) A_{\bar{z}}^{-\alpha} + \Gamma(t). \end{aligned} \quad (3.16)$$

The first and last term vanish in (3.16) because zero modes of ϕ only contribute to the partition function, as we will show below. Thus, we see that the partition function of $U(N)/U(N)$ gauged WZW model reduces to the BF-type action.

Let us consider in (3.16):

$$\int \prod_{\alpha \in \Delta} \mathcal{D}A^{\alpha} e^{-\frac{k}{2\pi} \sum_{\alpha \in \Delta} \int_{\Sigma_h} d^2z A_z^{\alpha} M_{-\alpha} A_{\bar{z}}^{-\alpha}} \quad (3.17)$$

where $M_\alpha = 1 - e^{2\pi i\alpha(\phi)}$. We path integral with respect to the off-diagonal components of the gauge field $A_\alpha (\alpha \in \Delta)$. Noticing that $(A_z^\alpha)^* = A_{\bar{z}}^{-\alpha}$, we find that the path integration is factorized to a holomorphic $(1, 0)$ and an anti-holomorphic $(0, 1)$ part:

$$\begin{aligned} & \int \prod_{\alpha>0} \mathcal{D}A^\alpha \mathcal{D}A^{-\alpha} e^{-\frac{k}{2\pi} \int_{\Sigma_h} d^2z (A_z^\alpha M_{-\alpha} A_{\bar{z}}^{-\alpha} + A_{\bar{z}}^{-\alpha} M_\alpha A_z^\alpha)} \\ &= \prod_{\alpha>0} \int \mathcal{D}A_z^\alpha \mathcal{D}A_{\bar{z}}^{-\alpha} e^{-\frac{k}{2\pi} \int_{\Sigma_h} d^2z A_z^\alpha M_{-\alpha} A_{\bar{z}}^{-\alpha}} \times \int \mathcal{D}A_{\bar{z}}^\alpha \mathcal{D}A_z^{-\alpha} e^{-\frac{k}{2\pi} \int_{\Sigma_h} d^2z A_{\bar{z}}^\alpha M_\alpha A_z^{-\alpha}} \\ &= \prod_{\alpha>0} [\text{Det}_{(0,1)} M_{-\alpha}]^{-1} [\text{Det}_{(1,0)} M_\alpha]^{-1}. \end{aligned} \quad (3.18)$$

Similarly, the Faddeev-Popov determinant also is rewritten by

$$\text{Det}_0(1 - \text{Ad}(t)) = \prod_{\alpha>0} \text{Det}_0 M_\alpha \text{Det}_0 M_{-\alpha}. \quad (3.19)$$

As we put (3.18) and (3.19) together, we thus see that

$$\prod_{\alpha>0} \frac{\text{Det}_0 M_\alpha}{\text{Det}_{(1,0)} M_\alpha} \times \prod_{\alpha>0} \frac{\text{Det}_0 M_{-\alpha}}{\text{Det}_{(0,1)} M_{-\alpha}}. \quad (3.20)$$

This factorization property exhibits the chiral nature of the (gauged) WZW model.

From now on, let us evaluate the holomorphic part in (3.20):

$$\frac{\text{Det}_0 M_\alpha}{\text{Det}_{(1,0)} M_\alpha}. \quad (3.21)$$

We see that each functional determinants diverge when we recall that the diagonal gauge fixing remains partly. However, the ratio of them (3.21) becomes finite by some kind of supersymmetry between the ghost fields and gauge fields. To see this, we must consider the residual gauge invariance, the abelian gauge invariance and regularize it in a way such that we do not break the residual gauge invariance to evaluate (3.21). Here, we make use of a heat kernel (or ζ -function) regularization based on the \mathfrak{t} covariant Laplacian $\Delta_A = -(\partial_A^\dagger \partial_A + \partial_A \partial_A^\dagger)$ where A is the gauge field taking value at the Cartan subalgebra \mathfrak{t} . For an operator \mathcal{O} we set

$$\log \text{Det} \mathcal{O} = \text{Tr} (e^{-\epsilon \Delta_A} \log \mathcal{O}) \quad (3.22)$$

where Tr is a functional trace and the regularization parameter ϵ is a positive real number. Then, (3.21) becomes

$$\log \frac{\text{Det}_0 M_\alpha}{\text{Det}_{(1,0)} M_\alpha} = \text{Tr}_0 (e^{-\epsilon \Delta_A} \log M_\alpha) - \text{Tr}_{(1,0)} (e^{-\epsilon \Delta_A} \log M_\alpha). \quad (3.23)$$

We consider the case which M_α is constant for the convenience, although M_α actually is not. Then,

$$\log M_\alpha [\text{Tr}_0 e^{-\epsilon \Delta_A} - \text{Tr}_{(1,0)} e^{-\epsilon \Delta_A}] \quad (3.24)$$

where the Laplacian Δ_A acts to the right on one-forms taking values in $\mathfrak{g}_{(-\alpha)}$. Therefore,

$$\partial_A|_{(-\alpha)} = \partial^{(1,0)} - i \sum_{\ell=1}^N \alpha_\ell A_\ell^{(1,0)} \quad (3.25)$$

so that the ‘‘charge’’ is α_ℓ . Since the term in square bracket is just the index of the twisted Dolbeault complex, the Dolbeault complex coupled to a vector bundle $V_{-\alpha}$ with connection A , $0 \xrightarrow{\partial_A} \Omega^{(0,0)} \otimes V_{-\alpha} \xrightarrow{\partial_A} \Omega^{(1,0)} \otimes V_{-\alpha} \xrightarrow{\partial_A} 0$:

$$[\mathrm{Tr}_0 e^{-\epsilon \Delta_A} - \mathrm{Tr}_{(1,0)} e^{-\epsilon \Delta_A}] = \sum_{p=0}^1 (-1)^p b^{p,0} = \mathrm{Index} \partial_A. \quad (3.26)$$

We can calculate this index from the Hirzebruch-Riemann-Roch theorem

$$\mathrm{Index} \partial_A = \int_M \mathrm{Td}(T^{(1,0)}(M)) \mathrm{ch}(V_{-\alpha}) \quad (3.27)$$

where $\mathrm{Td}(T^{(1,0)}(M))$ is the Todd class of some manifold M and $\mathrm{ch}(V)$ is the Chern character of some vector space V . In two dimensions, this reduces to

$$\mathrm{Index} \bar{\partial}_A = \frac{1}{2} \chi(\Sigma_h) - c_1(V_{-\alpha}) \quad (3.28)$$

where $\chi(\Sigma_h)$ is the Euler number of the genus- h Riemann surface and $c_1(V)$ is the Chern class of V . Therefore, in the case at hand, one finds that (3.24) equals to

$$\mathrm{Index} \partial_A|_{(-\alpha)} \log M_\alpha = \left[\frac{1}{8\pi} \int_{\Sigma_h} R - \frac{1}{2\pi} \int_{\Sigma_h} \alpha_\ell F^\ell \right] \log M_\alpha \quad (3.29)$$

where R is the curvature two-form. In the case that M_α is constant, this term gives the Euler number of a genus- h Riemann surface, $2 - 2h$. Physically, The fact that the index theorem holds, owe a kind of supersymmetries between the ghost fields and the gauge fields and non-zero modes of both fields cancel out.

We similarly consider the anti-holomorphic part in (3.20):

$$\frac{\mathrm{Det}_0 M_{-\alpha}}{\mathrm{Det}_{(0,1)} M_{-\alpha}}. \quad (3.30)$$

In this case, we use the index of the twisted Dolbeault complex $0 \xrightarrow{\bar{\partial}_A} \Omega^{(0,0)} \otimes V_{-\alpha} \xrightarrow{\bar{\partial}_A} \Omega^{(0,1)} \otimes V_{-\alpha} \xrightarrow{\bar{\partial}_A} 0$. Here, we make use of a heat kernel regularization based on the \mathfrak{t} covariant Laplacian $\bar{\Delta}_A = -(\bar{\partial}_A^\dagger \bar{\partial}_A + \bar{\partial}_A \bar{\partial}_A^\dagger)$ unlike the holomorphic part. Also, $\bar{\partial}_A = \bar{\partial} - i\alpha_\ell A^\ell$. We use the Hirzebruch-Riemann-Roch theorem

$$[\mathrm{Tr}_0 e^{-\epsilon \bar{\Delta}_A} - \mathrm{Tr}_{(0,1)} e^{-\epsilon \bar{\Delta}_A}] = \mathrm{Index} \bar{\partial}_A. \quad (3.31)$$

Thus, (3.30) is

$$\begin{aligned} \text{Index } \bar{\partial}_A|_{(-\alpha)} \log M_{-\alpha} &= \left[\frac{1}{2} \chi(\Sigma_h) + c_1(V_{(-\alpha)}) \right] \log M_{-\alpha} \\ &= \left[\frac{1}{8\pi} \int_{\Sigma_h} R + \frac{i}{2\pi} \int_{\Sigma_h} \alpha_\ell F^\ell \right] \log M_{-\alpha} \end{aligned} \quad (3.32)$$

When M_α is not a constant, one simply has to move $\log M_\alpha$ into the integral, so that one obtains

$$\prod_{\alpha>0} \exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R \log M_\alpha M_{-\alpha} + \frac{1}{2\pi} \int_{\Sigma_h} \alpha_\ell F^\ell \log \frac{M_\alpha}{M_{-\alpha}} \right\} \quad (3.33)$$

We firstly consider the first term in (3.33). This term can be regarded as contributions to the partition function from the background gravity. When we define ‘‘dilaton’’ Φ as

$$\Phi = \sum_{\alpha>0} \log M_\alpha M_{-\alpha} = \log \text{Det}(\mathbf{1} - \text{Ad}(e^{2\pi i \phi})), \quad (3.34)$$

we recognize the first term of (3.33) as a dilaton like coupling to the metric:

$$\exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R \cdot \Phi \right\}. \quad (3.35)$$

If ϕ is constant, (3.35) becomes

$$\det(\mathbf{1} - \text{Ad}(e^{2\pi i \phi}))^{h-1} \quad (3.36)$$

where we have used a fact that the Euler character of the genus- h Riemann surface is $2 - 2h$.

Next, we consider the the second term in (3.33):

$$\prod_{\alpha>0} \exp \left\{ \sum_{\ell=1}^N \frac{1}{2\pi} \int_{\Sigma_h} \alpha_\ell F^\ell \log \frac{M_\alpha}{M_{-\alpha}} \right\}. \quad (3.37)$$

Since

$$\frac{M_\alpha}{M_{-\alpha}} = -e^{2\pi i \alpha(\phi)}, \quad (3.38)$$

(3.37) becomes

$$\exp \left\{ i \sum_{\alpha>0} \sum_{\ell=1}^N \int_{\Sigma_h} \alpha_\ell F^\ell (\alpha(\phi) + \log(-1)) \right\}. \quad (3.39)$$

Since the Killing-Cartan metric b of the Lie algebra g , restricted to the Cartan subalgebra \mathfrak{t} ,

$$b(X, Y) = -\text{tr}(\text{ad}(X)\text{ad}(Y)) \quad (3.40)$$

can be rewritten in terms of the positive roots as

$$b(X, Y) = 2 \sum_{\alpha > 0} \alpha(X)\alpha(Y), \quad (3.41)$$

the exponent of (3.39) becomes

$$\sum_{\alpha > 0} \alpha(F)\alpha(\phi) = \sum_{\ell=1}^N \frac{1}{2} b(\phi, \alpha_\ell) F^\ell = \sum_{\ell=1}^N \left(h\phi_\ell - \sum_{a=1}^N \phi_a \right) F^\ell \quad (3.42)$$

where $h = N$ is the dual Coxeter number of $\mathfrak{u}(N)$. Here, we used an explicit formula for the Killing-Cartan metric of $\mathfrak{u}(N)$ (A.8). Thus, we obtain

$$\exp \left\{ i \sum_{a=1}^N \int_{\Sigma_h} F_a \left((N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} \right) \right\}. \quad (3.43)$$

Thus, the level-shift $k \rightarrow k + N$ is produced by quantum effects.

Together with (3.35), (3.43) and the fermion bilinear term $\lambda \wedge \lambda$, the resulting expression of the partition function becomes

$$\begin{aligned} Z_{\text{GWZW}}(\Sigma_h) &= \int \prod_{a=1}^N \mathcal{D}\phi^a \prod_{a=1}^N \mathcal{D}A^a \exp \left\{ \frac{1}{8\pi} \int_{\Sigma_h} R\Phi + \frac{1}{4\pi} \int_{\Sigma_h} \lambda^a \wedge \lambda^a \right\} \\ &\quad \times \exp \left\{ i \sum_{a=1}^N \int_{\Sigma_h} F_a \left((N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} \right) \right\}. \end{aligned} \quad (3.44)$$

Here we note the fermion bilinear term $\lambda \wedge \lambda$. The effective action, the exponent of (3.44), is not invariant under the BRST transformation restricted to the abelian part. Therefore, we have to add appropriate counter terms to restore the BRST symmetry [1], [39] and [40]. However, such renormalization does not influence the later calculation at all because the fermion bilinear term enters in the effective action freely. The renormalization effect becomes crucial when we couple the theory to additional matters, as we will consider at Chapter 4.

By the Hodge decomposition theorem, F_b can be always decomposed to a harmonic part $F_b^{(0)}$ and an exterior derivative of a one-form da_b such that

$$F_b = F_b^{(0)} + da_b. \quad (3.45)$$

where n_b is an b -th diagonal $U(1)$ -charge of the background gauge field:

$$\frac{1}{2\pi} \int_{\Sigma_h} F_b^{(0)} = n_b \in \mathbb{Z}. \quad (3.46)$$

Integrating a_b by part puts delta functional constraints on $d\phi_a$, the fields $\phi_a(z, \bar{z})$ reduce to constant fields. Thus, we obtain

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \frac{1}{N!} \sum_{n_1, \dots, n_N = -\infty}^{\infty} \int \prod_{a=1}^N d\phi_a \prod_{\substack{a,b=1 \\ a \neq b}}^N (1 - e^{2\pi i(\phi_a - \phi_b)})^{1-h} \\ \times \exp \left\{ 2\pi i \sum_{a=1}^N n_a \left((N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} \right) \right\}. \quad (3.47)$$

We rewrite (3.47) by using the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n X} = \delta(X - m), \quad n \in \mathbb{Z} \quad (3.48)$$

as

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \frac{1}{N!} \sum_{m_1, \dots, m_N = -\infty}^{\infty} \int \prod_{a=1}^N d\phi_a \prod_{\substack{a,b=1 \\ a \neq b}}^N (1 - e^{2\pi i(\phi_a - \phi_b)})^{1-h} \\ \times \prod_{a=1}^N \delta \left((N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} - m_a \right). \quad (3.49)$$

The partition function (3.49) is invariant under the interchange $k \leftrightarrow N$ because the $U(N)/U(N)$ gauged WZW model on Σ_h has a property of the level-rank duality [41]. Therefore, we can rewrite (3.49) as

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \frac{1}{k!} \sum_{m_1, \dots, m_k = -\infty}^{\infty} \int \prod_{a=1}^k d\phi_a \prod_{\substack{a,b=1 \\ a \neq b}}^k (1 - e^{2\pi i(\phi_a - \phi_b)})^{1-h} \times \\ \times \prod_{a=1}^k \delta \left((N+k)\phi_a - \sum_{b=1}^k \phi_b + \frac{k-1}{2} - m_a \right). \quad (3.50)$$

Integrating (3.50) with respect to ϕ_a s, the partition function localizes to configurations which the constant fields ϕ_a satisfy constraints

$$(N+k)\phi_a - \sum_{b=1}^k \phi_b + \frac{k-1}{2} - m_a = 0. \quad (3.51)$$

Let us consider solutions of these equations. We immediately find that the solutions are

$$\phi_a = \frac{1}{k+N} \left(J_a + \frac{1}{N} \|J\| \right) \quad (3.52)$$

where $J_a = m_a - \frac{k-1}{2}$, $m_a \in \mathbb{Z}$ and $\|J\| = \sum_{a=1}^k J_a$ for $a = 1, \dots, k$. Note that range of the each field ϕ_a is $0 \leq \phi_a < 1$. The fields in this range only contribute to the partition function. Therefore, we count the number of solutions of these equations in the range $0 \leq \phi_a < 1$. We immediately notice that when $\phi_a = \phi_b$ for $a \neq b$ the configurations do not contribute to the partition function. All the ϕ_a are contained in the range, even if we interchange the all solutions ϕ_a . So we can set $\phi_1 < \phi_2 < \dots < \phi_k$ and a factor $k!$ in the partition function cancels out. The number of piecewise independent solutions of the equations in the range of $0 \leq \phi_a < 1$ is

$$\frac{(k + N - 1)!}{(N - 1)! k!}. \quad (3.53)$$

This number just coincides with the number of WZW primary fields of the $SU(N)_k$ WZW model. The each solution (3.52) is in one-to-one correspondence with the WZW primary fields of the $SU(N)_k$ WZW model or the highest weights of the integrable representation in the affine Lie algebra $\widehat{\mathfrak{su}}_k(N)$. When the set of this solutions is denoted by $\{\text{Sol}\}$, the partition function is

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \alpha \beta^{1-h} \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \left\{ \frac{\prod_{\substack{a,b=1 \\ a \neq b}}^k (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})}{\prod_{a=1}^k e^{2\pi i (k-1)\phi_a}} \right\}^{1-h} \quad (3.54)$$

where α and β are a genus independent and dependent constant, respectively.

Finally, we determine the normalization for the partition function of the G/G gauged WZW model on Σ_h which is compatible with the number of the conformal blocks in the G WZW model on Σ_h . The partition function of the G/G gauged WZW model on Σ_h also can be represented by the modular S-matrix for the character in the G WZW model as follows

$$Z_{\text{GWZW}}^G(\Sigma_h) = \sum_{\mathcal{R}} (\mathcal{S}_{0\mathcal{R}}^G)^{2-2h}, \quad (3.55)$$

where \mathcal{R} denotes an integrable highest weight representation in the affine Lie algebra $\widehat{\mathfrak{g}}$ corresponding to a WZW primary field in $U(N)$ WZW model and the summation runs through all the WZW primary fields in the G WZW model. Therefore, we determine the normalization such that the partition function (3.57) matches with (3.55) in $G = U(N)$. When $h = 1$, the partition function of the G/G gauged WZW model coincides with the number of the WZW primary fields in G WZW model. The genus independent normalization factor α is $(N + k)/N$ because $\{\text{Sol}\}$ only runs through the WZW primary fields in the $\widehat{\mathfrak{su}}(N)_k$ WZW model and the number of the WZW primary fields in the $\widehat{\mathfrak{u}}(N)_k$ WZW model is

$$\frac{(k + N)!}{N! k!}. \quad (3.56)$$

The resulting partition function of the $U(N)/U(N)$ GWZW model on Σ_h is

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \frac{N+k}{N} \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \left\{ \frac{1}{(k+N)^k} \frac{\prod_{\substack{a,b=1 \\ a \neq b}}^k (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})}{\prod_{a=1}^k e^{2\pi i (k-1)\phi_a}} \right\}^{1-h}. \quad (3.57)$$

Thus, we can have evaluated the partition function by the equivariant localization. In next section, we will show a relation between $U(N)/U(N)$ gauged WZW model and the phase model.

3.3 Gauge/Bethe Correspondence

In this section, we clarify connections between the $U(N)/U(N)$ gauged WZW model and the phase model. First of all we have to identify parameters of both theories. We identify the level k and the rank N of the gauge group $U(N)$ in the $U(N)/U(N)$ gauged WZW model with the total particle number M and the total site number L in the phase model, respectively. Under these parameter identifications, we can show that the constraints (3.51) coincide with the Bethe Ansatz equations (2.66). Taking the parameterization of the Bethe roots as $\lambda_a = e^{2\pi i \phi_a}$, the logarithm form of the Bethe Ansatz equations becomes

$$(N+k)\phi_a - \sum_{b=1}^k \phi_b + \frac{k-1}{2} = m_a \quad (3.58)$$

where $m_a \in \mathbb{Z}$ implies branches of the logarithm. Once we identify the constant field ϕ_a in the $U(N)/U(N)$ gauged WZW model with the Bethe roots ϕ_a in the phase model, we found that these equations coincide with the localization configurations (3.51) in the $U(N)/U(N)$ gauged WZW model.

Next, let us consider solutions of the Bethe Ansatz equations. The solutions are (3.52) because the Bethe Ansatz equations are equal to the localized configurations (3.51). Then we can show that piecewise independent solutions of the Bethe Ansatz equations coincide with the solutions to be included in the range of $0 \leq \phi_a < 1$ and to satisfy the condition $0 < \phi_1 < \phi_2 < \dots < \phi_k < 1$ in the $U(N)/U(N)$ gauged WZW model. Thus, we found that this solutions of the Bethe Ansatz equations coincide with $\{\text{Sol}\}$. The solutions (3.52) also imply the completeness of the state in the phase model because the number of the solutions is $(N+k-1)!/(N-1)!k!$.

Since the Bethe norm in the phase model (2.70) becomes

$$\langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle = \frac{\prod_{a=1}^k e^{2\pi i (k-1)\phi_a}}{\prod_{\substack{a,b=1 \\ a \neq b}}^k (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})} (k+N)^{k-1} N \quad (3.59)$$

under taking the parameterization of the Bethe roots as $\lambda_a = e^{2\pi i \phi_a}$, the partition function (3.57) of the $U(N)/U(N)$ gauged WZW model can be represented as

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \left(\frac{N+k}{N} \right)^h \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle^{h-1}. \quad (3.60)$$

Why the partition function of the $U(N)/U(N)$ gauged WZW model can be represented by the Bethe norm in the phase model? To understand this, we recall that the partition function is represented by using the modular S-matrix (3.55). Thus, we can expect that there is a relation between the modular S-matrix in $U(N)/U(N)$ gauged WZW model and the Bethe norm in the phase model. Actually, Korff and Stroppel constructed the Verlinde algebra in the $SU(N)$ WZW model on the sphere from a viewpoint of the phase model and showed that the modular S-matrix in $SU(N)$ WZW model coincides with the Bethe norm in the phase model [20]. So, let us derive the partition function of the $SU(N)/SU(N)$ gauged WZW model from the one of the $U(N)/U(N)$ case. There are two differences between these partition functions. Firstly, the modular S-matrices in each model are related to

$$\mathcal{S}_{0\mathcal{R}}^{\hat{u}(N)} = \sqrt{\frac{N}{N+k}} \mathcal{S}_{0R}^{\hat{su}(N)} \quad (3.61)$$

where \mathcal{R} and R denote the WZW primary field in the $U(N)$ and the $SU(N)$ WZW model, respectively [41]. Secondly, a range which the summation runs through is different because the number of the each WZW primary field is different. Taking account these two differences, we find that the partition function of the $SU(N)/SU(N)$ gauged WZW model is

$$Z_{\text{GWZW}}^{SU(N)}(\Sigma_h) = \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \left\{ \frac{1}{(k+N)^{k-1} N} \frac{\prod_{\substack{a,b=1 \\ a \neq b}}^k (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})}{\prod_{a=1}^k e^{2\pi i (k-1) \phi_a}} \right\}^{1-h} \quad (3.62)$$

and can be represented by the summation of the Bethe norm with respect to all the eigenstates of the transfer matrix in the phase model;

$$Z_{\text{GWZW}}^{SU(N)}(\Sigma_h) = \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle^{h-1}. \quad (3.63)$$

This shows that the modular S-matrix of the $SU(N)$ WZW model coincides with the Bethe norm;

$$\mathcal{S}_{0R}^{SU(N)} = \langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle. \quad (3.64)$$

This is considered as a reason why the partition function of the $U(N)/U(N)$ gauged WZW model can be represented by the Bethe norm in the phase model. Therefore, we found that the Gauge/Bethe correspondence between the G/G gauged WZW model and the phase model is also considered as the gauged WZW model realization of [20].

Finally, we comment relations between the CS theory and the phase model. The partition function of the G/G gauged WZW model coincides with the partition function of the CS theory with the gauge group G on $S^1 \times \Sigma_h$ [13]. We can apply equivariant localization methods to the CS theory in a similar way with G/G gauged WZW model. Thus in the CS theory with the gauge group $U(N)$ on $S^1 \times \Sigma_h$, the localization configurations

coincide with the Bethe Ansatz equations and the partition function is represented by the Bethe norm in the phase model:

$$Z_{\text{CS}}^{U(N)}(S^1 \times \Sigma_h) = \left(\frac{N+k}{N}\right)^h \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle^{h-1}. \quad (3.65)$$

Further, when the gauge group is $SU(N)$, the partition function of the CS theory is

$$Z_{\text{CS}}^{SU(N)}(S^1 \times \Sigma_h) = \sum_{\phi_1, \dots, \phi_k \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i \phi}\}_k) | \psi(\{e^{2\pi i \phi}\}_k) \rangle^{h-1}. \quad (3.66)$$

We have shown the Gauge/Bethe correspondence between CS theory on $S^1 \times \Sigma_h$ and the phase model. The equivariant localization for the CS theory on wider class manifolds (Seifert manifolds) is derived in [42], [43] and [44], see also [45] for generalization to the Chern-Simons-Matter theories. To describe the partition function of the CS theory on these manifolds, not only modular S-matrix but also modular T-matrix is needed.

Remark. Recall that we have interchanged the level k with the rank N in (3.50) because the partition function of the $U(N)/U(N)$ gauged WZW model has level-rank duality. In this circumstance, we have then investigated the relations between the $U(N)/U(N)$ or $SU(N)/SU(N)$ gauged WZW model and the phase model. This is because we can identify the WZW primary fields as the Bethe roots and the modular matrix in $SU(N)$ WZW model completely coincides the Bethe norm in the phase model in this circumstance.

However, this substitution is not indispensable when we consider the correspondence between the gauged WZW model and the phase model. To see this, let us return to (3.50). Integrating the delta function at (3.50) and setting a correct normalization, we obtain

$$Z_{\text{GWZW}}^{U(N)}(\Sigma_h) = \left(\frac{N+k}{k}\right)^h \sum_{\phi_1, \dots, \phi_N \in \{\widetilde{\text{Sol}}\}} \left\{ \frac{1}{(k+N)^N} \frac{\prod_{\substack{a,b=1 \\ a \neq b}}^N (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})}{\prod_{a=1}^N e^{2\pi i (N-1)\phi_a}} \right\}^{1-h}. \quad (3.67)$$

Here, $\{\widetilde{\text{Sol}}\}$ is defined by a set which ϕ_1, \dots, ϕ_N satisfy a constraint

$$(N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} = m_a \quad (3.68)$$

and the conditions $0 \leq \phi_1 < \phi_2 < \dots < \phi_N < 1$. The partition function of $U(N)/U(N)$ gauged WZW model can also reduce the one of the $SU(N)/SU(N)$ gauged WZW model as above:

$$Z_{\text{GWZW}}^{SU(N)}(\Sigma_h) = \left(\frac{N}{k}\right)^h \sum_{\phi_1, \dots, \phi_N \in \{\widetilde{\text{Sol}}\}} \left\{ \frac{1}{(k+N)^{N-1}k} \frac{\prod_{\substack{a,b=1 \\ a \neq b}}^N (e^{2\pi i \phi_a} - e^{2\pi i \phi_b})}{\prod_{a=1}^N e^{2\pi i (N-1)\phi_a}} \right\}^{1-h}. \quad (3.69)$$

From now on, we investigate relations between the $SU(N)/SU(N)$ gauged WZW model and the phase model by using this expression. Here, we identify the level k and the rank N with the total site number L and the total particle number M , respectively. Note that this identification is different from the above case. Under this identification, we thus see that the localization constraint (3.68) coincides the Bethe Ansatz equation in the phase model. The Bethe norm can further be expressed by

$$\langle \psi(\{e^{2\pi i\phi}\}_N) | \psi(\{e^{2\pi i\phi}\}_N) \rangle = \frac{\prod_{a=1}^N e^{2\pi i(N-1)\phi_a}}{\prod_{\substack{a,b=1 \\ a \neq b}}^N (e^{2\pi i\phi_a} - e^{2\pi i\phi_b})} (k + N)^{N-1} k. \quad (3.70)$$

Hence, we can express the partition function as a summation of the norm between the eigenstate in the phase model with respect to the all eigenstates up to a overall factor:

$$Z_{\text{GWZW}}^{SU(N)}(\Sigma_h) = \left(\frac{k}{N}\right)^h \sum_{\phi_1, \dots, \phi_N \in \{\widetilde{\text{Sol}}\}} \langle \psi(\{e^{2\pi i\phi}\}_N) | \psi(\{e^{2\pi i\phi}\}_N) \rangle^{1-h}. \quad (3.71)$$

Thus, we have established the Gauge/Bethe correspondence between the $SU(N)/SU(N)$ gauged WZW model and the phase model as the case of substituting the level with the rank. Notice that the number of the WZW primary fields in the $SU(N)$ WZW model does not coincide with the number of the elements in the set $\{\widetilde{\text{Sol}}\}$, $(N + k - 1)!/(k - 1)N!$. Therefore, we see that the WZW primary fields can not identify the elements $\{\phi_1, \dots, \phi_N\}$ in the set $\{\widetilde{\text{Sol}}\}$. Further, we find that the modular matrix in $SU(N)$ WZW model does not coincide with the Bethe norm in the phase model under the identification: $k \equiv L$ and $N \equiv M$.

Thus, we see that the identification $k \equiv M$ and $N \equiv L$ is more natural than the one $k \equiv L$ and $N \equiv M$. However, all models does not have the level-rank duality. In fact, such duality is unlikely to exist in the G/G gauged WZW model with additional matters, as see in next chapter. Therefore, this remark will become important for the Gauge/Bethe correspondence to work well.

G/G Gauged Wess-Zumino-Witten-Higgs model

In this chapter, we study a generalization of the Gauge/Bethe correspondence for the G/G gauged WZW model and the phase model in the previous chapter. In chapter 2, we have introduced the phase model as a $t = 0$ limit of the q -boson model. Since the Gauge/Bethe correspondence is a correspondence between some topological gauge theory and some integrable system, it is natural that a topological gauge theory which corresponds to the q -boson model exists. In this chapter, we will construct such a model. This model is the G/G gauged WZW model coupled to additional matters and is called by the G/G gauged WZW-Higgs model. In fact, we will show that this model corresponds to the q -boson model by utilizing the equivariant localization method as with Chapter 3. See [24].

In section 4.1, we firstly introduce the G/G gauged WZW-Higgs model. In section 4.2, we apply the localization to the $U(N)/U(N)$ gauged WZW-Higgs model and calculate the partition function. In section 4.3, we numerically give a value of the partition function. In section 4.4, we study the correspondence between the $U(N)/U(N)$ gauged WZW-Higgs model and the q -boson model. See Appendix A for the convention which we use in this chapter.

4.1 G/G gauged Wess-Zumino-Witten-Higgs model

In this section, we introduce the G/G gauged WZW-Higgs model on a genus h Riemann surface. It is a model of matters coupled to the G/G gauged WZW model on a genus h Riemann surface. The additional matters are an adjoint complex scalar boson Φ , an adjoint complex scalar fermion ψ , an adjoint 1-form auxiliary boson φ and an adjoint 1-form auxiliary fermion χ ¹.

From now on, let us construct the action of the G/G gauged WZW-Higgs model on a genus h Riemann surface. Since this model is a topological field theory, the matters should enter in the action as a BRST-exact term. Therefore, we firstly define the BRST

¹Notice that this matter contents is different from a matter contents of [5]. In [5], the additional matters are an adjoint 1-form boson etc.

transformation generated by a BRST charge $Q_{(g,t)}$:

$$\begin{aligned} Q_{(g,t)}A &= \lambda, & Q_{(g,t)}\lambda^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)}, & Q_{(g,t)}\lambda^{(0,1)} &= -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}, \\ Q_{(g,t)}g &= 0, & Q_{(g,t)}\Phi &= \psi, & Q_{(g,t)}\Phi^\dagger &= \psi^\dagger, & Q_{(g,t)}\psi &= tg^{-1}\Phi g - \Phi, \\ Q_{(g,t)}\psi^\dagger &= -tg\Phi^\dagger g^{-1} + \Phi^\dagger, & Q_{(g,t)}\chi^{(1,0)} &= \varphi^{(1,0)}, & Q_{(g,t)}\chi^{(0,1)} &= \varphi^{(0,1)}, \\ Q_{(g,t)}\varphi^{(1,0)} &= tg^{-1}\chi^{(1,0)}g - \chi^{(1,0)}, & Q_{(g,t)}\varphi^{(0,1)} &= -tg\chi^{(0,1)}g^{-1} + \chi^{(0,1)} \end{aligned} \quad (4.1)$$

where $0 \leq t < 1$. This is a natural generalization of the BRST transformation for the G/G gauged WZW model.

Next, we define the partition function:

$$Z_{\text{GWZWH}}(\Sigma_h, t) = \int DAD\lambda D\Phi Dg D\psi D\varphi D\chi e^{-kS_{\text{GWZWH}}(\Sigma_h, t)} \quad (4.2)$$

where the action is defined as

$$S_{\text{GWZWH}}(\Sigma_h, t) = S_{\text{GWZW}}(\Sigma_h) + \frac{i}{4\pi} \int_{\Sigma_h} \text{Tr}(\lambda \wedge \lambda) + S_{\text{matter}}(\Sigma_h, t). \quad (4.3)$$

where $S_{\text{GWZW}}(\Sigma_h)$ is the action of the G/G gauged WZW model (3.4). Here, the matter part of (4.3) is represented as BRST-exact form:

$$S_{\text{matter}}(\Sigma_h, t) = Q_{(g,t)} \cdot \mathcal{R} \quad (4.4)$$

where \mathcal{R} is defined as

$$\mathcal{R} = \frac{1}{4\pi} \int_{\Sigma_h} \{d\mu \text{Tr}(\Phi\psi^\dagger - \Phi^\dagger\psi) + \mathcal{R}_1 + \mathcal{R}_2\}. \quad (4.5)$$

Here, \mathcal{R}_1 and \mathcal{R}_2 are defined as

$$\mathcal{R}_1 := \text{Tr} \{ \chi^{(0,1)} \wedge (\nabla^{(1,0)}\Phi - \Phi X + X\Phi) \}, \quad (4.6)$$

$$\mathcal{R}_2 := \text{Tr} \{ \chi^{(1,0)} \wedge (\nabla^{(0,1)}\Phi^\dagger - Y\Phi^\dagger + \Phi^\dagger Y) \} \quad (4.7)$$

where X and Y are defined by

$$X := \sum_{n=0}^{\infty} X_n := \sum_{n=0}^{\infty} g^{-n} (g^{-1} \nabla^{(1,0)} g) g^n, \quad (4.8)$$

$$Y := \sum_{n=0}^{\infty} Y_n := \sum_{n=0}^{\infty} g^n (\nabla^{(0,1)} g \cdot g^{-1}) g^{-n}. \quad (4.9)$$

Here, we define a covariant derivative as $\nabla^{(1,0)}X = \partial^{(1,0)}X + [A^{(1,0)}, X]$ and so on. The square of the BRST transformation $Q_{(g,t)}$ generates the bosonic transformation $\mathcal{L}_{(g,t)}$:

$$\begin{aligned} \mathcal{L}_{(g,t)}A^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)}, & \mathcal{L}_{(g,t)}A^{(0,1)} &= -(A^{g^{-1}})^{(0,1)} + A^{(0,1)} \\ \mathcal{L}_{(g,t)}\lambda^{(1,0)} &= g^{-1}\lambda^{(1,0)}g - \lambda^{(1,0)}, & \mathcal{L}_{(g,t)}\lambda^{(0,1)} &= -g\lambda^{(0,1)}g^{-1} + \lambda^{(0,1)}, & \mathcal{L}_{(g,t)}g &= 0 \\ \mathcal{L}_{(g,t)}\Phi &= tg^{-1}\Phi g - \Phi, & \mathcal{L}_{(g,t)}\Phi^\dagger &= -tg\Phi^\dagger g^{-1} + \Phi^\dagger, \\ \mathcal{L}_{(g,t)}\psi &= tg^{-1}\psi g - \psi, & \mathcal{L}_{(g,t)}\psi^\dagger &= -tg\psi^\dagger g^{-1} + \psi^\dagger, \\ \mathcal{L}_{(g,t)}\chi^{(1,0)} &= tg^{-1}\chi^{(1,0)}g - \chi^{(1,0)}, & \mathcal{L}_{(g,t)}\chi^{(0,1)} &= -tg\chi^{(0,1)}g^{-1} + \chi^{(0,1)} \\ \mathcal{L}_{(g,t)}\varphi^{(1,0)} &= tg^{-1}\varphi^{(1,0)}g - \varphi^{(1,0)}, & \mathcal{L}_{(g,t)}\varphi^{(0,1)} &= -tg\varphi^{(0,1)}g^{-1} + \varphi^{(0,1)} \end{aligned} \quad (4.10)$$

where $Q_{(g,t)}^2 = \mathcal{L}_{(g,t)}$. The matter part of the action (4.4) is invariant under the transformation $\mathcal{L}_{(g,t)}$. The bosonic transformation $\mathcal{L}_{(g,1)}$ generates the finite gauge transformation at $t = 1$:

$$\begin{aligned}
\mathcal{L}_{(g,1)}A^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)}, & \mathcal{L}_{(g,1)}A^{(0,1)} &= -(A^{g^{-1}})^{(0,1)} + A^{(0,1)} \\
\mathcal{L}_{(g,1)}\lambda^{(1,0)} &= g^{-1}\lambda^{(1,0)}g - \lambda^{(1,0)}, & \mathcal{L}_{(g,1)}\lambda^{(0,1)} &= -g\lambda^{(0,1)}g^{-1} + \lambda^{(0,1)}, \\
\mathcal{L}_{(g,1)}g &= 0 \\
\mathcal{L}_{(g,1)}\Phi &= g^{-1}\Phi g - \Phi, & \mathcal{L}_{(g,1)}\Phi^\dagger &= -g\Phi^\dagger g^{-1} + \Phi^\dagger, \\
\mathcal{L}_{(g,1)}\psi &= g^{-1}\psi g - \psi, & \mathcal{L}_{(g,1)}\psi^\dagger &= -g\psi^\dagger g^{-1} + \psi^\dagger, \\
\mathcal{L}_{(g,1)}\chi^{(1,0)} &= g^{-1}\chi^{(1,0)}g - \chi^{(1,0)}, & \mathcal{L}_{(g,1)}\chi^{(0,1)} &= -g\chi^{(0,1)}g^{-1} + \chi^{(0,1)} \\
\mathcal{L}_{(g,1)}\varphi^{(1,0)} &= g^{-1}\varphi^{(1,0)}g - \varphi^{(1,0)}, & \mathcal{L}_{(g,1)}\varphi^{(0,1)} &= -g\varphi^{(0,1)}g^{-1} + \varphi^{(0,1)}
\end{aligned} \tag{4.11}$$

where $Q_{(g,1)}^2 = \mathcal{L}_{(g,1)}$. Similarly, the bosonic transformation $\mathcal{L}_{(1,t)}$ at $g \in G = 1$ generates the finite $U(1)$ transformation:

$$\begin{aligned}
\mathcal{L}_{(1,t)}A &= 0, & \mathcal{L}_{(1,t)}\lambda &= 0, & \mathcal{L}_{(1,t)}g &= 0, \\
\mathcal{L}_{(1,t)}\Phi &= t\Phi - \Phi, & \mathcal{L}_{(1,t)}\Phi^\dagger &= -t\Phi^\dagger + \Phi^\dagger, \\
\mathcal{L}_{(1,t)}\psi &= t\psi - \psi, & \mathcal{L}_{(1,t)}\psi^\dagger &= -t\psi^\dagger + \psi^\dagger, \\
\mathcal{L}_{(1,t)}\chi^{(1,0)} &= t\chi^{(1,0)} - \chi^{(1,0)}, & \mathcal{L}_{(1,t)}\chi^{(0,1)} &= -t\chi^{(0,1)} + \chi^{(0,1)}, \\
\mathcal{L}_{(1,t)}\varphi^{(1,0)} &= t\varphi^{(1,0)} - \varphi^{(1,0)}, & \mathcal{L}_{(1,t)}\varphi^{(0,1)} &= -t\varphi^{(0,1)} + \varphi^{(0,1)}.
\end{aligned} \tag{4.12}$$

where $Q_{(1,t)}^2 = \mathcal{L}_{(1,t)}$. Thus we see that $\mathcal{L}_{(g,t)}$ generates the gauge and $U(1)$ transformation.

For convenience, we explicitly rewrite the action (4.4) by carrying out the BRST transformation as follows.

$$\begin{aligned}
S_{\text{matter}}(\Sigma_h, t) &= \frac{1}{2\pi} \int_{\Sigma_h} d\mu \text{Tr} (\Phi\Phi^\dagger + \psi\psi^\dagger - t\Phi^\dagger g^{-1}\Phi g) \\
&+ \frac{1}{4\pi} \int_{\Sigma_h} \text{Tr} \{ \varphi^{(0,1)} \wedge (\nabla^{(1,0)}\Phi + [X, \Phi]) - \chi^{(0,1)} \wedge (\nabla^{(1,0)}\psi + [X, \psi]) \\
&+ \varphi^{(1,0)} \wedge (\nabla^{(0,1)} - [Y, \Phi^\dagger]) - \chi^{(1,0)} \wedge (\nabla^{(0,1)} - [Y, \psi^\dagger]) \}.
\end{aligned} \tag{4.13}$$

We see from this that a interaction term between the fields of the G/G gauged WZW model and the additional matters disappears when we set $t = 0$. Hence, the G/G gauged WZW-Higgs model becomes the G/G gauged WZW model at $t = 0$. We can regard this model as some kind of a one-parameter deformation of the G/G gauged WZW model.

Since the action is written as the action of the G/G gauged WZW model plus the BRST-exact term, the G/G gauged WZW-Higgs model will become the topological field theory. Thus, the partition function will be a topological invariant. In chapter 3, we have seen that the partition function of the G/G gauged WZW model counts the number of the conformal block of the G WZW model. Therefore, we can expect that the partition function of the G/G gauged WZW-Higgs model counts the number of the building block of some underlying field theory. However, we do not know what field theory is.

In next section, we will calculate the partition function of the $U(N)/U(N)$ gauged WZW-Higgs model by using the equivariant localization method like the case of the $U(N)/U(N)$ gauged WZW model.

4.2 Localization

From now on, we set the gauge group G as $U(N)$ for simplicity. We evaluate the partition function of the $U(N)/U(N)$ gauged WZW-Higgs model by using the equivariant localization method. However, we can not directly evaluate the partition function with the action (4.3). To simplify the calculations it is useful to consider the more general action given by

$$\begin{aligned} S_{\text{matter}}(\Sigma_h, t) &= Q_{(g,t)} \cdot \left[\frac{1}{4\pi} \int_{\Sigma_h} \left\{ d\mu \text{Tr} (\Phi\psi^\dagger - \Phi^\dagger\psi) + \tau_1 (\mathcal{R}_1 + \mathcal{R}_2) - \tau_2 \text{Tr}(\chi \wedge *\varphi) \right\} \right] \end{aligned} \quad (4.14)$$

where $d\mu = -d^2z$ is a volume form. For $\tau_1 = 1, \tau_2 = 0$, (4.14) matches (4.4). From a viewpoint of cohomological localization for the path integral, one can expect that the partition function for the $\tau_1 = 1, \tau_2 = 0$ coincides with one for the $\tau_1 = 0, \tau_2 = 1$. Therefore, we consider the case of $\tau_1 = 0, \tau_2 = 1$ from here. In this case, the action (4.14) becomes

$$\begin{aligned} S_{\text{matter}}(\Sigma_h, t) &= Q_{(g,t)} \cdot \left[\frac{1}{4\pi} \int_{\Sigma_h} \left\{ d\mu \text{Tr}(\Phi\psi^\dagger - \Phi^\dagger\psi) - \text{Tr}(\chi \wedge *\varphi) \right\} \right] \\ &= \frac{1}{2\pi} \int_{\Sigma_h} d\mu \text{Tr} \left\{ \Phi\Phi^\dagger - t\Phi g\Phi^\dagger g^{-1} + \psi\psi^\dagger \right\} \\ &\quad - \frac{1}{2\pi} \int_{\Sigma_h} d^2z \text{Tr} (\varphi_z \varphi_{\bar{z}} - \chi_z \chi_{\bar{z}} + t\chi_z g\chi_{\bar{z}} g^{-1}). \end{aligned} \quad (4.15)$$

The action will become quadratic in terms of Φ , φ , ψ and χ after we take a diagonal gauge. Thus, we can evaluate the partition function in a similar manner with Chapter 3.

Let us take a diagonal gauge $g(z, \bar{z}) \equiv \exp \left\{ 2\pi i \sum_{a=1}^N \phi_a(z, \bar{z}) H^a \right\}$. Then, the partition function under the diagonal gauge becomes

$$\begin{aligned} Z_{\text{GWZWH}}(\Sigma_h, t) &= \frac{1}{|W|} \int DAD\lambda D\phi DM \text{Det}(1 - \text{Ad}(e^{2\pi i\phi})) \\ &\quad \times \exp \left\{ -kS_{\text{GWZW}}(\phi, A) - \frac{ik}{4\pi} \int_{\Sigma_h} \text{Tr}(\lambda \wedge \lambda) - kS_{\text{matter}}(\phi, \Phi, \psi, \chi, \varphi) \right\} \end{aligned} \quad (4.16)$$

where the measure $DM = D\Phi D\Phi^\dagger D\psi D\psi^\dagger D\varphi D\chi$. Also, $\text{Det}(1 - \text{Ad}(e^{2\pi i\phi}))$ is the Faddeev-Popov determinant for the diagonal gauge fixing as with Chapter 3. Here, the

action (4.15) becomes

$$S_{\text{GWZWH}}(\Sigma_h, t) = S_{\text{GWZW}}(\Sigma_h) + \frac{i}{4\pi} \int_{\Sigma_h} \text{Tr}(\lambda \wedge \lambda) + S_{\text{matter}}(\Sigma_h, t) \quad (4.17)$$

where the gauged WZW part is given by (3.16) and the matter part is

$$\begin{aligned} S_{\text{matter}}(\Sigma_h, t) &= \frac{1}{2\pi} \int_{\Sigma_h} d\mu \text{Tr}(\psi \psi^\dagger) - \frac{1}{2\pi} \int_{\Sigma_h} d^2 z \text{Tr}(\varphi_z \varphi_{\bar{z}}) + \\ &+ \frac{1}{2\pi} \int_{\Sigma_h} d^2 z \left\{ (1-t) \sum_{a=1}^N \Phi_a \Phi_a^\dagger + \sum_{\alpha \in \Delta} (1 - te^{2\pi i \alpha(\phi)}) \Phi_{-\alpha} \Phi_\alpha^\dagger \right. \\ &\left. + (1-t) \sum_{a=1}^N \chi_{\bar{z}}^a \chi_z^a + \sum_{\alpha \in \Delta} (1 - te^{2\pi i \alpha(\phi)}) \chi_{\bar{z}}^\alpha \chi_z^{-\alpha} \right\}. \end{aligned} \quad (4.18)$$

Firstly, we can evaluate the path integral with respect to $\chi_{z,\alpha}$ and $\chi_{\bar{z},\alpha}$ as

$$\begin{aligned} &\int \mathcal{D}\chi^\alpha \mathcal{D}\chi^{-\alpha} \prod_{\alpha \in \Delta} \exp \left\{ -\frac{k}{2\pi} \int d^2 z \chi_{\bar{z}}^\alpha (1 - te^{2\pi i \alpha(\phi)}) \chi_z^{-\alpha} \right\} \\ &= \int \mathcal{D}\chi^\alpha \mathcal{D}\chi^{-\alpha} \prod_{\alpha > 0} \exp \left\{ -\frac{k}{2\pi} \int d^2 z (\chi_{\bar{z}}^\alpha M_\alpha(t) \chi_z^{-\alpha} + \chi_{\bar{z}}^{-\alpha} M_{-\alpha}(t) \chi_z^\alpha) \right\} \\ &= \prod_{\alpha > 0} \text{Det}_{(1,0)} M_\alpha(t) \cdot \text{Det}_{(1,0)} M_{-\alpha}(t). \end{aligned} \quad (4.19)$$

where $M_\alpha(t) = 1 - te^{2\pi i \alpha(\phi)}$. Similarly, we can evaluate the path integral with respect to Φ_α and Φ_α^\dagger as

$$\begin{aligned} &\int \mathcal{D}\Phi_\alpha \mathcal{D}\Phi_{-\alpha} \mathcal{D}\Phi_\alpha^\dagger \mathcal{D}\Phi_{-\alpha}^\dagger \prod_{\alpha \in \Delta} \exp \left\{ -\frac{k}{2\pi} \int d^2 z \Phi_\alpha^\dagger (1 - te^{2\pi i \alpha(\phi)}) \Phi_{-\alpha} \right\} \\ &\int \mathcal{D}\Phi_\alpha \mathcal{D}\Phi_{-\alpha} \mathcal{D}\Phi_\alpha^\dagger \mathcal{D}\Phi_{-\alpha}^\dagger \prod_{\alpha > 0} \exp \left\{ -\frac{k}{2\pi} \int d^2 z (\Phi_\alpha^\dagger M_\alpha(t) \Phi_{-\alpha} + \Phi_{-\alpha}^\dagger M_{-\alpha}(t) \Phi_\alpha) \right\} \\ &= \prod_{\alpha > 0} [\text{Det}_0 M_\alpha(t)]^{-1} \cdot [\text{Det}_0 M_{-\alpha}(t)]^{-1}. \end{aligned} \quad (4.20)$$

Putting together with (4.19) and (4.20), the contributions to the partition function from Φ and χ become

$$\prod_{\alpha > 0} \frac{\text{Det}_{(1,0)} M_\alpha(t)}{\text{Det}_0 M_\alpha(t)} \times \prod_{\alpha > 0} \frac{\text{Det}_{(1,0)} M_{-\alpha}(t)}{\text{Det}_0 M_{-\alpha}(t)}. \quad (4.21)$$

We can evaluate this ratio of the functional determinant by using the Hirzebruch-Riemann-Roch theorem for the twisted Dolbeault complex as well as the case of the gauged WZW

model in Chapter 3. Here, we utilize the complex $0 \xrightarrow{\partial_A} \Omega^{(0,0)} \otimes V_{-\alpha} \xrightarrow{\partial_A} \Omega^{(1,0)} \otimes V_{-\alpha} \xrightarrow{\partial_A} 0$ at the first part and the complex $0 \xrightarrow{\partial_A} \Omega^{(0,0)} \otimes V_{\alpha} \xrightarrow{\partial_A} \Omega^{(1,0)} \otimes V_{\alpha} \xrightarrow{\partial_A} 0$ at the second part.

Thus, we obtain

$$\prod_{\alpha>0} \exp \left\{ \frac{1}{8\pi} R \log M_{\alpha}(t) M_{-\alpha}(t) + \frac{1}{2\pi} \int_{\Sigma_h} \alpha_{\ell} F^{\ell} \log \frac{M_{\alpha}(t)}{M_{-\alpha}(t)} \right\}. \quad (4.22)$$

Evaluating the path integration in terms of Φ^a , Φ_a^{\dagger} , χ_z^a and $\chi_{\bar{z}}^a$, we obtain

$$\prod_{a=1}^N (1-t)^{-h+1}. \quad (4.23)$$

The contribution to the partition function from φ and ψ also cancel out. Since the action of the gauged WZW part is equal to (3.16), we obtain

$$\begin{aligned} & \int \prod_{a=1}^N \mathcal{D}\phi^a \prod_{a=1}^N \mathcal{D}A^a \left(\frac{\prod_{\substack{a,b=1 \\ a \neq b}}^N (1 - e^{2\pi i(\phi_a - \phi_b)})}{\prod_{a,b=1}^N (1 - te^{2\pi i(\phi_a - \phi_b)})} \right)^{1-h} \times \exp \left\{ + \frac{1}{4\pi} \int_{\Sigma_h} \lambda^a \wedge \lambda^a \right\} \\ & \times \exp \left\{ i \sum_{a=1}^N \int_{\Sigma_h} F_a \left((N+k)\phi_a - \sum_{b=1}^N \phi_b + \frac{N-1}{2} \right) \right\}. \end{aligned} \quad (4.24)$$

Here, we have used the fact that the constant modes of $\{\phi_1, \dots, \phi_N\}$ only contribute to the partition function as we will show below.

Thus, the partition function of the $U(N)/U(N)$ gauged WZW-Higgs model on a genus- h Riemann surface becomes

$$\begin{aligned} Z_{GWZWH}(\Sigma_h, t) &= \int \prod_{a=1}^N D\phi_a D\lambda_a D A_a \left(\frac{\prod_{\substack{a,b=1 \\ a \neq b}}^N (1 - e^{2\pi i(\phi_a - \phi_b)})}{\prod_{a,b=1}^N (1 - te^{2\pi i(\phi_a - \phi_b)})} \right)^{1-h} \\ & \times \exp \left\{ i \sum_{a=1}^N \int_{\Sigma_h} \left(\beta_a(\phi) F_a + \frac{k}{4\pi} \lambda_a \wedge \lambda_a \right) \right\} \end{aligned} \quad (4.25)$$

where $\beta_a(\phi)$ is defined by

$$\beta_a(\phi) = k\phi_a - \frac{i}{2\pi} \sum_{\substack{b=1 \\ b \neq a}}^N \log \left(\frac{e^{2\pi i\phi_a} - te^{2\pi i\phi_b}}{te^{2\pi i\phi_a} - e^{2\pi i\phi_b}} \right). \quad (4.26)$$

When we define an abelianized effective action by

$$S_{\text{eff}}(\phi, A, \lambda) = -i \sum_{a=1}^N \int_{\Sigma_h} \left(\beta_a(\phi) F_a + \frac{k}{4\pi} \lambda_a \wedge \lambda_a \right), \quad (4.27)$$

we see that this is not invariant under a following abelianized BRST transformation:

$$QA_a = \lambda_a, \quad Q\lambda_a = 2\pi d\phi_a, \quad Q\phi_a = 0 \quad (4.28)$$

where Q is an abelianized BRST charge. Although the effective action (4.27) should be invariant under the abelianized BRST transformation, it is not. Therefore, we have to add appropriate counter terms to restore the BRST symmetry by requiring the effective action such that it satisfies descent equations.

Now, we explain the descent equations and how to restore the BRST invariance of the action. Firstly, we define a local operator $\mathcal{O}^{(0)}$ as

$$\mathcal{O}^{(0)} = W(\phi) \quad (4.29)$$

where $W(\phi)$ is an arbitrary function of ϕ_1, \dots, ϕ_N on the Riemann surface. Let $\mathcal{O}^{(n)}$, ($n = 0, 1, 2$) be n -form valued local operators which satisfy a following relation

$$d\mathcal{O}^{(n-1)} = Q\mathcal{O}^{(n)}. \quad (4.30)$$

(4.30) is called by the descent equations. Note that the 3-form local operator $\mathcal{O}^{(3)}$ does not exist because we consider the Riemann surface as the base manifold. Then, we see that the integration of $\mathcal{O}^{(n)}$ over a n -cycle γ_n , namely $\int_{\gamma_n} \mathcal{O}^{(n)}$, becomes the BRST-closed operator under the abelianized BRST transformation (4.28):

$$Q \cdot \int_{\gamma_n} \mathcal{O}^{(n)} = 0. \quad (4.31)$$

In fact, we can construct the BRST-closed operators as follow:

$$\mathcal{O}^{(0)} = W(\phi) \quad (4.32)$$

$$\mathcal{O}^{(1)} = \frac{1}{2\pi} \sum_{a=1}^N \frac{\partial W(\phi)}{\partial \phi_a} \lambda_a \quad (4.33)$$

$$\mathcal{O}^{(2)} = \frac{1}{8\pi^2} \sum_{a,b=1}^N \frac{\partial^2 W(\phi)}{\partial \phi_a \partial \phi_b} \lambda_a \wedge \lambda_b + \frac{1}{2\pi} \sum_{a=1}^N \frac{\partial W(\phi)}{\partial \phi_a} F_a. \quad (4.34)$$

In our case, by defining the function $W(\phi)$ as

$$\frac{1}{2\pi} \frac{\partial W(\phi)}{\partial \phi_a} = \beta_a(\phi). \quad (4.35)$$

the operator $\mathcal{O}^{(2)}$ becomes

$$\mathcal{O}^{(2)} = \sum_{a=1}^N \left(\beta_a(\phi) F_a + \frac{1}{4\pi} \sum_{b=1}^N \frac{\partial \beta_b(\phi)}{\partial \phi_a} \lambda_a \wedge \lambda_b \right). \quad (4.36)$$

To restore the BRST invariance in the effective action (4.27), we must replace (4.27) with (4.36):

$$S_{\text{eff}}(\phi, A, \lambda) = - \sum_{a=1}^N \int_{\Sigma_h} \left(\beta_a(\phi) F_a + \frac{1}{4\pi} \sum_{b=1}^N \frac{\partial \beta_b(\phi)}{\partial \phi_a} \lambda_a \wedge \lambda_b \right). \quad (4.37)$$

As a result, we have restored the BRST symmetry in the effective theory. Hence, the partition function becomes

$$\begin{aligned} Z_{GWZWH}(\Sigma_h, t) &= \frac{1}{|W|} \int \prod_{a=1}^N D\phi_a D\lambda_a DA_a \left(\frac{\prod_{\substack{a,b=1 \\ a \neq b}}^N (1 - e^{2\pi i(\phi_a - \phi_b)})}{\prod_{a,b=1}^N (1 - te^{2\pi i(\phi_a - \phi_b)})} \right)^{1-h} \\ &\times \exp \left\{ i \sum_{a=1}^N \int_{\Sigma_h} \left(\beta_a(\phi) F_a + \frac{1}{4\pi} \sum_{b=1}^N \frac{\partial \beta_b(\phi)}{\partial \phi_a} \lambda_a \wedge \lambda_b \right) \right\}. \end{aligned} \quad (4.38)$$

By using the Hodge decomposition theorem, the two-form F_b can be decomposed to a harmonic part $F_b^{(0)}$ and an exterior derivative of a one-form da_b such that

$$F_b = F_b^{(0)} + da_b \quad (4.39)$$

where k_b is an b -th diagonal $U(1)$ -charges of the background gauge fields:

$$\frac{1}{2\pi} \int_{\Sigma_h} F_b^{(0)} = k_b. \quad (4.40)$$

Integrating a_b by part puts delta functional constraints on $d\phi_a$, the fields $\phi_a(z, \bar{z})$ reduce to constant fields.

We also decompose λ into $\lambda_a = \lambda_a^{(0)} + \delta\lambda_a$ where $\lambda_a^{(0)}$ is a harmonic 1-form and $\delta\lambda$ is fluctuation orthogonal to $\lambda_a^{(0)}$ by using the Hodge decomposition theorem. Determinants from integral of $\delta\lambda_b$ are completely canceled with Jacobians induced from the integral of a_b . Since the number of fermionic zero-mode of each $\lambda_a^{(0)}$ equal to the number of the harmonic form $2h$ on the genus- h Riemann surface, the path integration with respect to $\lambda_1^{(0)}, \dots, \lambda_N^{(0)}$ gives an additional factor

$$\mu_q(\phi)^h = \left| \det \left(\frac{\partial \beta_b(\phi)}{\partial \phi_a} \right) \right|^h. \quad (4.41)$$

Thus, the resulting expression of the partition function becomes

$$\begin{aligned} Z_{GWZWH}(\Sigma_h, t) &= \frac{1}{|W|} \sum_{k_1, \dots, k_N = -\infty}^{\infty} \int \prod_{a=1}^N d\phi_a \mu_q(\phi)^h e^{i \sum_{a=1}^N k_a \beta_a(\phi)} \\ &\times \left(\frac{1}{(1-t)^N} \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - te^{2\pi i \phi_b}} \right)^{1-h}. \end{aligned} \quad (4.42)$$

Since we ignore the overall factor for the partition function in our calculation, we can replace the functional determinant in $\mu_q(\phi)$ with the determinant:

$$\mu_q(\phi) = \left| \det \left(\frac{\partial \beta_b(\phi)}{\partial \phi_a} \right) \right|. \quad (4.43)$$

By using the Poisson resummation formula, we rewrite (4.42) as

$$\begin{aligned} Z_{GWZWH}(\Sigma_h, t) &= \frac{1}{|W|} \sum_{\ell_1, \dots, \ell_N = -\infty}^{\infty} \int \prod_{a=1}^N d\phi_a \prod_{a=1}^N \delta(\beta_a(\phi) - \ell_a) \mu_q(\phi)^h \\ &\quad \times \left(\frac{1}{(1-t)^N} \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}} \right)^{1-h}. \end{aligned} \quad (4.44)$$

Here, let us utilize a property about the delta function

$$\delta(f(x)) = \sum_a \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad (4.45)$$

where x_i is solutions of $f(x) = 0$. Then the delta function in the partition function becomes

$$\prod_{a=1}^N \delta(\beta_a(\phi) - \ell_a) = \mu_q(x)^{-1} \prod_{a=1}^N \delta(\phi_a - x_a) \quad (4.46)$$

where x_1, \dots, x_N satisfy

$$k\phi_a - \frac{i}{2\pi} \sum_{\substack{b=1 \\ b \neq a}}^N \log \left(\frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{t e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right) = \ell_a. \quad (4.47)$$

If a number of solutions exists in (4.47), one must sum up all solutions in a region $0 \leq \phi_1, \dots, \phi_N < 1$. In our case, we can show that the solution is unique up to permutations. The partition function is invariant under the permutation and the contribution to the partition function from the permutation therefore cancel out the order of the Weyl group $|W|$.

By integrating with respect to ϕ_1, \dots, ϕ_N , we obtain a final result for the partition function:

$$Z_{GWZWH}(\Sigma_h, t) = \sum_{\{\phi_1, \dots, \phi_N\} \in \{\text{Sol}\}} \left\{ (1-t)^N \mu_q(\phi) \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right\}^{h-1}. \quad (4.48)$$

Here, $\{\text{Sol}\}$ is a set of the solution which satisfies $0 \leq \phi_1 < \dots < \phi_N < 1$ and the constraint

$$2\pi i k \phi_a + \sum_{\substack{b=1 \\ b \neq a}}^N \log \left(\frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{t e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right) = 2\pi i \ell_a. \quad (4.49)$$

Also, we explicitly can express $\mu_q(\phi)$ as

$$\begin{aligned} \mu_q(\phi) &= \det_N \frac{\partial \beta_b(\phi)}{\partial \phi_a} \\ &= \det_N \left[\left\{ k - \sum_{c=1}^N \frac{(t^2 - 1) e^{2\pi i (\phi_b + \phi_c)}}{(t e^{2\pi i \phi_b} - e^{2\pi i \phi_c})(t e^{2\pi i \phi_c} - e^{2\pi i \phi_b})} \right\} \delta_{a,b} \right. \\ &\quad \left. + \frac{(t^2 - 1) e^{2\pi i (\phi_a + \phi_b)}}{(t e^{2\pi i \phi_a} - e^{2\pi i \phi_b})(t e^{2\pi i \phi_b} - e^{2\pi i \phi_a})} \right]. \end{aligned} \quad (4.50)$$

Thus, we see that the path integral of $U(N)/U(N)$ gauged WZW-Higgs model reduces to the finite sum of the solutions which satisfies the localization configuration.

Finally, we comment about a normalization of the partition function. The partition function with a general normalization becomes

$$Z_{GWZWH}(\Sigma_h, t) = \alpha(t) \beta(t)^{1-h} \sum_{\{\phi_1, \dots, \phi_N\} \in \{\text{Sol}\}} \left\{ (1-t)^N \mu_q(x) \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right\}^{h-1} \quad (4.51)$$

where $\alpha(t)$ and $\beta(t)$ are a genus independent and dependent function of t , respectively. Note that this partition function should coincide with (3.57) at a limit $t \rightarrow 0$ at least. However, we can not completely determine the normalization of the partition function of $U(N)/U(N)$ gauged WZW-Higgs model unlike the gauged WZW model.

4.3 Numerical Simulation

In this section, we numerically investigate the partition function of the $SU(N)/SU(N)$ gauged WZW-Higgs model at level k . We have not determined the normalization of the partition function as we have discussed it at previous section. Therefore, we assume that the normalization of the partition function of the gauged WZW-Higgs model coincides with the one of the gauged WZW model. In other words, we assume that the partition function of the $U(N)/U(N)$ gauged WZW-Higgs model becomes

$$\begin{aligned} &Z_{GWZWH}^{U(N)}(\Sigma_h, t) \\ &= \left(\frac{k+N}{k} \right)^h \sum_{\{\phi_1, \dots, \phi_N\} \in \{\text{Sol}\}} \left\{ (1-t)^N \mu_q(x) \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right\}^{h-1}. \end{aligned} \quad (4.52)$$

In the same way, we assume that the partition function of the $SU(N)/SU(N)$ gauged WZW-Higgs model becomes

$$Z_{GWZWH}^{SU(N)}(\Sigma_h, t) = \left(\frac{N}{k}\right)^h \sum_{\{\phi_1, \dots, \phi_N\} \in \{\text{Sol}\}} \left\{ (1-t)^N \mu_q(x) \prod_{\substack{a,b=1 \\ a \neq b}}^N \frac{e^{2\pi i \phi_a} - t e^{2\pi i \phi_b}}{e^{2\pi i \phi_a} - e^{2\pi i \phi_b}} \right\}^{h-1}. \quad (4.53)$$

From now on, we calculate a value of the partition function of the $SU(N)/SU(N)$ gauged WZW-Higgs model under this normalization by utilizing the Mathematica ².

Firstly, let us consider the case of genus-1, a torus. In the gauged WZW model, the partition function counts the number of the WZW primary fields and is $(N+k-1)!/(N-1)!k!$. In the gauged WZW-Higgs model, we will expect that the partition function counts the number of some fields in an underlying theory and becomes integer value. In fact, we found that the partition function is not modified from the gauged WZW model by the numerical simulation:

$$Z_{GWZWH}^{SU(N)}(T^2, t) = \frac{(N+k-1)!}{(N-1)!k!}. \quad (4.54)$$

Next, we investigate the partition function on a sphere, a genus-0. By the numerical simulation, we conjecture that the partition function behaves as

$$Z_{GWZWH}^{SU(N)}(S^2, t) = \frac{1}{\prod_{a=1}^N (1-t^a)}. \quad (4.55)$$

Notice that this does not depend on the level k and coincides with the partition function of the gauged WZW model in a limit $t \rightarrow 0$.

In the case of genus-2 and above, we can not conjecture how the partition function behaves in arbitrary k and N . Therefore, we consider special cases where $N = 2$, $k =$ arbitrary, $h = 2$ and $N = k = 2$, $h =$ arbitrary. We list the result in the former and later case at Table 4.1 and Table 4.2, respectively.

In the former case, we conjecture that from Table 4.1 the partition function becomes

$$Z_{GWZWH}^{SU(2)}(\Sigma_2, k, t) = (1-t)^2 \left(\frac{(k+3)(k+2)(k+1)}{6} - \frac{(k-7)k(k+1)}{3}t + \frac{(k-3)(k-2)(k-1)}{6}t^2 \right). \quad (4.56)$$

²We must solve the localization constraints (4.49) to numerically find the partition function. Note that we change the localization constraint (4.49) as

$$2\pi k x_j = 2\pi I_j - \sum_{k=1}^N \left(2 \tan^{-1} \left\{ \frac{\tan(\pi(x_j - x_k))}{\tan(\pi\eta)} \right\} + 2\pi \left[x_j - x_k + \frac{1}{2} \right] \right)$$

where $[\dots]$ is the Gauss' symbol because it is necessary to choice correct branches of the logarithm. See [46]

In the later case, we also conjecture that from Table 4.2 the partition function becomes

$$Z_{\text{GWZWH}}^{SU(2)}(\Sigma_h, k = 2, t) = 2^{h-1}(2^h + 1)(1 - t)^{2h-2}(1 + t)^{h-1}. \quad (4.57)$$

We can not conjecture a general form in other case but list the result of the other case at Table 4.3. As see Table 4.1, Table 4.2 and Table 4.3, we see that all expansion coefficients in terms of t of the partition function are integer. The partition function itself change but this nature does not change, even if one changes the normalization such that the partition function of the gauged WZW-Higgs model becomes one of the gauged WZW model at the limit $t \rightarrow 0$. Therefore, this implies that the partition function is a topological invariant.

Genus	k	N	Partition Function
2	2	2	$(1 - t)^2(10 + 10t)$
	3		$(1 - t)^2(20 + 16t)$
	4		$(1 - t)^2(35 + 20t + t^2)$
	5		$(1 - t)^2(56 + 20t + 4t^2)$
	6		$(1 - t)^2(84 + 14t + 10t^2)$
	7		$(1 - t)^2(120 + 20t^2)$
	8		$(1 - t)^2(165 - 24t + 35t^2)$
	9		$(1 - t)^2(220 - 60t + 56t^2)$
	10		$(1 - t)^2(286 - 110t + 84t^2)$
	50		$(1 - t)^2(23426 - 36550t + 18424t^2)$

Table 4.1: The partition function of the $SU(2)/SU(2)$ gauged WZW-Higgs model with the level k on the genus-2 Riemann surface

Genus	k	N	Partition Function
2	2	2	$10(1 - t)^2(1 + t)$
3			$36(1 - t)^4(1 + t)^2$
4			$136(1 - t)^6(1 + t)^3$
5			$528(1 - t)^8(1 + t)^4$
10			$524800(1 - t)^{18}(1 + t)^9$

Table 4.2: The partition function of the $SU(2)/SU(2)$ gauged WZW-Higgs model with the level $k = 2$ on the genus- h Riemann surface

Genus	$L = k$	$M = N$	Partition Function
2	2	3	$(1 - t)^3(45 + 99t + 99t^2 + 45t^3)$
	3	3	$(1 - t)^3(166 + 332t + 252t^2 + 86t^3 + t^4)$
	4	3	$(1 - t)^3(504 + 810t + 396t^2 + 126t^3 + 36t^4)$
	5	3	$(1 - t)^3(1332 + 1512t + 369t^2 + 243t^3 + 144t^4)$
	6	3	$(1 - t)^3(3168 + 2046t + 112t^2 + 593t^3 + 339t^4 + 5t^5 + t^6)$
	7	3	$(1 - t)^3(6930 + 1188t + 162t^2 + 1188t^3 + 648t^4 + 18t^5 + 9t^6)$
	2	4	$4(1 - t)^4(1 + t)^2(35 + 50t + 86t^2 + 50t^3 + 35t^4)$
	3	4	$16(1 - t)^4(1 + t)(56 + 134t + 177t^2 + 128t^3 + 54t^4 + 17t^5 + t^6)$
	4	4	$2(1 - t)^4(2340 + 7020t + 8761t^2 + 5628t^3 + 2167t^4 + 1076t^5 + 615t^6 + 164t^7 + 5t^8)$
	5	4	$16(1 - t)^4(1314 + 3114t + 2381t^2 + 605t^3 + 359t^4 + 526t^5 + 249t^6 + 67t^7 + 10t^8)$
	2	5	$25(1 - t)^5(1 + t)^2(1 + t + t^2)(14 + 23t + 43t^2 + 48t^3 + 43t^4 + 23t^5 + 14t^6)$
3	3	2	$8(1 - t)^4(3 + 2t)(5 + 4t)$
	4	2	$(1 - t)^4(329 + 280t + 86t^2 + 8t^3 + t^4)$
	2	3	$27(1 - t)^6(1 + t)^2(3 + 4t + 3t^2)(5 + 6t + 5t^2)$
	3	3	$(1 - t)^6(4390 + 17560t + 29296t^2 + 26428t^3 + 14020t^4 + 4480t^5 + 772t^6 + 10t^7 + t^8)$
4	3	2	$16(1 - t)^6(2 + t)(5 + 4t)^2$
5	3	2	$32(1 - t)^8(5 + 4t)^2(7 + 8t + 2t^2)$

Table 4.3: The partition function of the $SU(N)/SU(N)$ gauged WZW-Higgs model with the level k on the genus- h Riemann surface

4.4 Gauge/Bethe Correspondence

In this section, we consider the Gauge/Bethe correspondence which is a correspondence between the $U(N)/U(N)$ gauged WZW-Higgs model and the q-boson model.

Firstly, let us see that the localization configuration in $U(N)/U(N)$ gauged WZW-Higgs model agrees with the Bethe Ansatz equation in the q-boson model. We change a parametrization of a coupling constant t as $t = e^{-2\pi\eta}$ at the localization configuration (4.49) to manifest the Bethe Ansatz equations in the q-boson model. Then, we obtain a following expression for the localization configuration:

$$2\pi i k x_j = 2\pi i I_j + \sum_{k=1}^N \log \frac{\sin[\pi(i\eta + (x_j - x_k))]}{\sin[\pi(i\eta - (x_j - x_k))]} \quad (4.58)$$

where I_j is (half-)integers when N is (even) odd. We identify the level k , the rank N of the gauge group $U(N)$ and the coupling constant η in the $U(N)/U(N)$ gauged WZW-Higgs model with the total particle number L , the total site number M and the coupling constant η in the q-boson model, respectively. Further, we identify the Cartan part ϕ_1, \dots, ϕ_N of a field g in the gauged WZW-Higgs model as the Bethe roots x_1, \dots, x_N in the q-boson model. Under these parameter identifications, we see that the constraints (4.58) coincide with the Bethe Ansatz equations (2.46) in the q-boson model.

Next, let us investigate relations between a set of piecewise independent solutions of the Bethe Ansatz equations for the q-boson model and a set $\{\text{Sol}\}$ of x_1, \dots, x_N which contributes to the partition function of the gauged WZW-Higgs model. It is necessary for the Bethe states to form a complete system that the number of the piecewise independent solutions of the Bethe Ansatz equations for the q-boson model is $(N+k-1)/(N-1)k!$. Although it is nontrivial which this number coincides with the number of elements of the set $\{\text{Sol}\}$, we can numerically confirm that the number of the elements of the set $\{\text{Sol}\}$ is $(N+k-1)/(N-1)k!$ and coincides with the number of the piecewise independent solutions of the Bethe Ansatz equations for the q-boson model. This circumstances is equal to the one of the relation between the $U(N)/U(N)$ gauged WZW model and the phase model. Thus, we can have established an identification with $\{\text{Sol}\}$ and the independent solutions of the Bethe Ansatz equation for the q-boson model.

Finally, we consider the partition function for the $U(N)/U(N)$ gauged WZW-Higgs model. Under above identification, the Bethe norm in the q-boson model (2.51) becomes

$$\langle \psi(\{e^{2\pi i x}\}_N) | \psi(\{e^{2\pi i x}\}_N) \rangle = \frac{\prod_{a,b=1}^N (e^{2\pi i x_a} t - e^{2\pi i x_b})}{\prod_{\substack{a,b=1 \\ a \neq b}}^N (e^{2\pi i x_a} - e^{2\pi i x_b})} \cdot \det_N \Phi'_{j,k}(\{x\}_N) \quad (4.59)$$

where the Gaudin matrix is

$$\Phi'_{a,b}(\{e^{2\pi i x}\}_N) = \delta_{a,b} \left\{ -k e^{-2\pi i x_b} + \sum_{c=1}^N \frac{(t^2 - 1) e^{2\pi i x_c}}{(t e^{2\pi i x_a} - e^{2\pi i x_c})(t e^{2\pi i x_c} - e^{2\pi i x_a})} \right\} \\ - \frac{(t^2 - 1) e^{2\pi i x_a}}{(t e^{2\pi i x_a} - e^{2\pi i x_b})(t e^{2\pi i x_b} - e^{2\pi i x_a})}. \quad (4.60)$$

Thus, the partition function of the $U(N)/U(N)$ gauged WZW-Higgs model on a genus- h Riemann surface is expressed by a summation of the norm between the eigenstates of the Hamiltonian in the q-boson model in terms of the all eigenstates:

$$Z_{\text{GWZWH}}^{U(N)}(\Sigma_h, t) = \left(\frac{N+k}{k}\right)^h \sum_{x_1, \dots, x_N \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i x}\}_N) | \psi(\{e^{2\pi i x}\}_N) \rangle^{h-1}. \quad (4.61)$$

As a result, we found that the $U(N)/U(N)$ gauged WZW-Higgs model corresponds to the q-boson model.

Further, we can reduce the partition function (4.61) to the $SU(N)/SU(N)$ gauged WZW-Higgs model:

$$Z_{\text{GWZWH}}^{SU(N)}(\Sigma_h, t) = \left(\frac{N}{k}\right)^h \sum_{x_1, \dots, x_N \in \{\text{Sol}\}} \langle \psi(\{e^{2\pi i x}\}_N) | \psi(\{e^{2\pi i x}\}_N) \rangle^{h-1}. \quad (4.62)$$

This circumstances is also equal to the one of the relation between the gauged WZW model and the phase model. We see that the $SU(N)/SU(N)$ gauged WZW-Higgs model also corresponds to the q-boson model. This correspondence is just one parameter deformation of a correspondence between the $SU(N)/SU(N)$ or $U(N)/U(N)$ gauged WZW model and the phase model. We find that ‘‘Gauge/Bethe correspondence’’ also work well in this situation.

Finally, we consider why does ‘‘Gauge/Bethe correspondence’’ for $SU(N)/SU(N)$ gauged WZW-Higgs model and the q-boson model work well. We consider this through a perspective of the axiom of the topological field theory. It is well known that the topological field theory has the axiomatic formulation given by Atiyah [47] and Segal [48]. See [49] and [50] for reviews. Especially, it is well known that the 2-dimensional topological field theory is equivalent to the commutative Frobenius algebra. Recently, C.Korff constructed a new commutative Frobenius algebra from the q-boson model [23]. Thus, we expect that there is a relation between $SU(N)/SU(N)$ gauged WZW-Higgs model and topological field theory equivalent to this commutative Frobenius algebra. We can actually derive the formula (4.54) and (4.55) given at previous section by using appropriate cutting/gluing relations, in other words the commutative Frobenius algebra. Therefore, the $SU(N)/SU(N)$ gauged WZW-Higgs model can regard as a Lagrangian realization of the commutative Frobenius algebra constructed by C.Korff.

CHAPTER 5

Conclusion

In this thesis, we have studied the relation between the 2-dimensional topological gauge theory and the integrable system. We especially have studied the relation between the $U(N)/U(N)$ or $SU(N)/SU(N)$ gauged WZW model and the phase model and between the $U(N)/U(N)$ or $SU(N)/SU(N)$ gauged WZW-Higgs model and the q-boson model.

In the former case, we found that the localization configurations (3.51) coincide with the Bethe Ansatz equations (3.58), once the diagonal group elements, the level and the rank of the gauge group $U(N)$ in the $U(N)/U(N)$ gauged WZW model are identified with the Bethe roots, the total site number and the total particle number in the phase model, respectively. We also showed that the partition function of the $U(N)/U(N)$ and the $SU(N)/SU(N)$ gauged WZW model is represented as the summation of the Bethe norm with respect to the all eigenstates of the transfer matrix in the phase model. This is because the modular S-matrix in the $SU(N)$ WZW model coincides with the Bethe norm. This is also considered as the gauged WZW model realization involving a generalization to a higher genus case of [20]. We further found that the partition function of the CS theory on $S^1 \times \Sigma_h$ is also related to norms of Hamiltonian eigenstates for the phase model. These relations are summarized in the table 5.1.

Phase model	$U(N)/U(N)$ GWZW model/ the $U(N)$ CS theory
Bethe root	Diagonal group element/ Holonomy along S^1 direction
Bethe Ansatz equation	Configuration of (3.51)
Total site number	Rank of the gauge group $U(N)$
Total particle number	Level
Bethe norm	Modular S-matrix $\mathcal{S}_{0\hat{\rho}}$
Partition function	Summation of Bethe norm with respect to the all eigenstates of the Hamiltonian

Table 5.1: Dictionary in the Gauge/Bethe correspondence between $U(N)/U(N)$ gauged WZW model and the phase model

Note that this correspondence similarly works well for the case of an interchange between the level and the rank. However, the Bethe norm no longer correspond to the modular S-matrix.

In the later case, we found that the localization configurations (4.58) coincide with the Bethe Ansatz equations (2.46), once the diagonal group elements, the level, the rank of the gauge group $U(N)$ and the coupling constant in the $U(N)/U(N)$ gauged WZW-Higgs model are identified with the Bethe roots, the total particle number, the total site number and the coupling constant in the q-boson model, respectively. We also showed that the partition function of the $U(N)/U(N)$ and the $SU(N)/SU(N)$ gauged WZW-Higgs model is represented as the summation of the Bethe norm with respect to the all eigenstates of the transfer matrix in the q-boson model. These relations are summarized in the table 5.2.

q-boson model	$U(N)/U(N)$ GWZW-Higgs model
Bethe root	Diagonal group element
Bethe Ansatz equation	Localization Configuration (4.58)
Total site number	Lank
Total particle number	Rank of the gauge group $U(N)$
Partition function	Summation of Bethe norm with respect to the all eigenstates of the Hamiltonian

Table 5.2: Dictionary in the Gauge/Bethe correspondence between $U(N)/U(N)$ gauged WZW-Higgs model and the q-boson model

Further, we numerically calculated the value of the partition function. Since the G/G gauged WZW-Higgs model is a topological field theory, we have checked that the expansion coefficients of the partition function in terms of the coupling constant became integers as expected. This quantity may be a new topological invariant.

Finally, let us consider the results of this thesis from a more general perspective. Any two-dimensional topological field theory is equivalent to a commutative Frobenius algebra. In the special case, the commutative Frobenius algebra is constructed from the some integrable system. In fact, the commutative Frobenius algebra is constructed from the phase model and the q-boson model in [20] and [23]. We showed that the commutative Frobenius algebra constructed from the phase model and from the q-boson model correspond to the $SU(N)/SU(N)$ gauged WZW model and $SU(N)/SU(N)$ gauged WZW-Higgs model, respectively. Thus, we can think that this is a mathematical reason how the Gauge/Bethe correspondence works well.

APPENDIX A

Convention

In this Appendix, we summarize the convention about the differential form and the Lie algebra which we use in Chapter 3 and 4.

Differential form We firstly summarize the convention about the differential form. The convention which we use is as follows:

Euclid signature:	$(+, +)$
n form field f :	$f = \frac{1}{n!} f_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$
Coordinate:	$z = \frac{1}{\sqrt{2}}(x + iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x - iy)$ $\Leftrightarrow x = \frac{1}{\sqrt{2}}(z + \bar{z}), \quad y = \frac{1}{\sqrt{2}i}(z - \bar{z})$
Partial derivative:	$\partial_z = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y)$
Integral:	$dzd\bar{z} = dx dy$
Metric:	$g_{\mu\nu} = \delta_{\mu\nu}$ for $\mu, \nu = x, y$ $g_{z\bar{z}} = g_{\bar{z}z} = 1, g_{zz} = g_{\bar{z}\bar{z}} = 0$
Complete anti-symmetric tensor:	$\epsilon_{xy} = -\epsilon_{yx} = \epsilon^{xy} = -\epsilon^{yx} = -1$ $\epsilon_{\bar{z}z} = -\epsilon_{z\bar{z}} = \epsilon^{\bar{z}z} = -\epsilon^{z\bar{z}} = i$
Hodge operator:	$*dx^\mu = \epsilon_{\mu\nu} dx^\nu$ for $\mu, \nu = x, y$ $*dz = idz, \quad *d\bar{z} = -id\bar{z}$ $*(dx^\mu \wedge dx^\nu) = \epsilon^{\mu\nu}$ $*1 = \frac{1}{2} \epsilon_{\mu\nu} dx^\mu \wedge dx^\nu = d^2x$ $(*)^2 = (-1)^{p(2-p)}$, when $*$ act on the p -form
Co-derivative operator:	$d^\dagger = - * d *$

(A.1)

Lie algebra Let us summarize the convention for a Lie algebra \mathfrak{g} , especially $\mathfrak{u}(N)$. We take generators T_a ($a = 1, \dots, \dim \mathfrak{g}$) in the orthogonal basis of the Lie algebra as an anti-Hermitic. Therefore, these generators satisfy

$$[T_a, T_b] = f_{ab}^c T_c \quad (\text{A.2})$$

where f_{ab}^c is structure constants.

In the Cartan-Weyl basis, we denote Cartan generators and ladder operators as $H^i, i = 1, \dots, r$ where r is the rank of the Lie algebra and E^α where $\alpha = (\alpha^1, \dots, \alpha^r)$ is a root, respectively. Here, we take the Cartan generators as Hermitic. Under the Hermitic conjugation, the ladder operator also becomes

$$E^{-\alpha} = (E^\alpha)^\dagger. \quad (\text{A.3})$$

These operators satisfy following commutation relations

$$[H^a, H^b] = 0, \quad [H^a, E^\alpha] = \alpha^a E^\alpha \quad (\text{A.4})$$

and

$$\begin{aligned} [E^\alpha, E^\beta] &= N_{\alpha,\beta} E^{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \Delta \\ &= \frac{2}{|\alpha|^2} \alpha \cdot H, \quad \text{if } \alpha = -\beta \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (\text{A.5})$$

where $N_{\alpha,\beta}$ is a constant and Δ is a set of the roots.

We regard X as a generic operator taking value in the Lie algebra \mathfrak{g} . Then, X can be expanded by the Cartan-Weyl basis as

$$X = \sum_{a=1}^N X_a (iH^a) + \sum_{\alpha \in \Delta} X_\alpha (iE^\alpha). \quad (\text{A.6})$$

Finally, we define the Killing-Cartan form as

$$b(X, Y) = -\text{Tr}(\text{ad}(X)\text{ad}(Y)). \quad (\text{A.7})$$

When Lie algebra is $\mathfrak{u}(N)$, the Killing-Cartan form can be written as

$$b(X, Y) = 2(h\text{Tr}(XY) - \text{Tr}X \cdot \text{Tr}Y) \quad (\text{A.8})$$

where h is the dual Coxeter number and N in the case of $\mathfrak{u}(N)$. Also, the trace Tr is defined as

$$\begin{aligned} \text{Tr}(H^a H^b) &= \delta^{ab} \\ \text{Tr}(E^\alpha E^\beta) &= \frac{2}{|\alpha|^2} \delta_{\alpha+\beta, 0} \end{aligned} \quad (\text{A.9})$$

where $|\alpha|^2 = 2$ in the case of $\mathfrak{u}(N)$.

APPENDIX B

Inner product in the q-boson model

In this appendix, we show (2.49) and (2.51), an inner product between the eigenstates of the transfer matrix in the q-boson model

$$S_M(\{\mu\}|\{\lambda\}) = \langle 0 | \prod_{j=1}^M C(\mu_j) \prod_{j=1}^M B(\lambda_j) | 0 \rangle \quad (\text{B.1})$$

where the parameters $\{\mu_1, \dots, \mu_M\}$ and $\{\lambda_1, \dots, \lambda_M\}$ are arbitrary complex numbers which do not satisfy the Bethe Ansatz equations. One can calculate the inner product by using various methods. In [51], [52] and [53], they firstly has calculated this inner product in the XXZ model or the 6-vertex model. In this appendix, we follow Slavnov's derivation [29] of the inner product based on the commutation relations of the Yang-Baxter algebra, (2.14) - (2.29). This method has the advantage of being able to apply a wide class of models. Therefore, we apply this method to the q-boson model and calculate the inner product (B.1). See also [27].

B.1 Inner product between general states

From now on, we consider the inner product between general states, that is, the case which the parameters $\{\lambda\}$ and $\{\mu\}$ in (B.1) are generic complex parameters. This inner product formally is calculated by using the commutation relations (2.14) - (2.29) and (2.33) and (2.34). We see that after use of the commutation relations (2.26) the parameters $\{\mu\}$ and $\{\lambda\}$ first become arguments of the vacuum eigenvalues a and d . Therefore, the most general form of the final result is

$$S_M(\{\mu\}|\{\lambda\}) = \sum_{\substack{\alpha \cup \bar{\alpha} \\ \gamma \cup \bar{\gamma}}} \prod_{j \in \gamma} a(\mu_j) \prod_{k \in \bar{\gamma}} d(\mu_k) \prod_{k \in \bar{\alpha}} a(\lambda_k) \prod_{j \in \alpha} d(\lambda_j) \\ \times K_M(\{\mu\}_\gamma, \{\mu\}_{\bar{\gamma}} | \{\lambda\}_\alpha \{\lambda\}_{\bar{\alpha}}). \quad (\text{B.2})$$

Here, we explain the notation used in this formula. The family $\{\lambda\}$ of parameters is partitioned into two disjoint subsets $\{\lambda\} = \{\lambda\}_\alpha \cup \{\lambda\}_{\bar{\alpha}}$. Similarly, $\{\mu\} = \{\mu\}_\gamma \cup \{\mu\}_{\bar{\gamma}}$. These partitions are independent, except for the condition $\{\lambda\}_\alpha = \{\mu\}_\gamma = n$, where

$n = 0, 1, \dots, M$. The partitions of the parameters $\{\lambda\}$ and $\{\mu\}$ automatically induce two partitions of the indices $1, \dots, M$, into $\{\lambda\}_\alpha \cup \{\lambda\}_{\bar{\alpha}}$ and into $\{\mu\} = \{\mu\}_\gamma \cup \{\mu\}_{\bar{\gamma}}$. In each of the subsets the parameters are ordered in a natural way, for example, $\{\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_n}\}$ if $\alpha_1 < \alpha_2, \dots < \alpha_n$, and so on. The sum in the formula (B.2) is taken over all partitions of the indicated form. Similar notation is used below throughout this appendix. Also, $K_M(\{\mu\}_\gamma, \{\mu\}_{\bar{\gamma}} | \{\lambda\}_\alpha \{\lambda\}_{\bar{\alpha}})$ denotes the coefficient appearing when the operators are permuted. Therefore, it depends on the R-matrix but not on the vacuum eigenvalues of the operators A and D . Our purpose will be to find an explicit form for this coefficient below.

We show that an arbitrary coefficient $K_M(\{\mu\}_\gamma, \{\mu\}_{\bar{\gamma}} | \{\lambda\}_\alpha \{\lambda\}_{\bar{\alpha}})$ in the formula (B.2) can be expressed in terms of the leading coefficient $K_M(\{\mu\} | \{\lambda\})$ and the conjugate leading coefficient $\bar{K}_M(\{\mu\} | \{\lambda\})$ defined by

$$K_M(\{\mu\} | \{\lambda\}) := K_M(\{\mu\}_\gamma, \emptyset | \{\lambda\}_\alpha, \emptyset) \quad (\text{B.3})$$

$$\bar{K}_M(\{\mu\} | \{\lambda\}) := K_M(\emptyset, \{\mu\}_{\bar{\gamma}} | \emptyset, \{\lambda\}_{\bar{\alpha}}). \quad (\text{B.4})$$

Here, the leading coefficient means the coefficient in (B.2) corresponding to the partition $\{\mu\}_{\bar{\gamma}} = \{\lambda\}_{\bar{\alpha}} = \emptyset$. Similarly, the conjugate leading coefficient means the coefficient in (B.2) corresponding to the empty partition $\{\mu\}_\gamma = \{\lambda\}_\alpha = \emptyset$.

To this end, we fix some partitions $\{\mu\} = \{\mu\}_\gamma \cup \{\mu\}_{\bar{\gamma}}$ and $\{\lambda\} = \{\lambda\}_\alpha \cup \{\lambda\}_{\bar{\alpha}}$ and find the coefficient corresponding to the given specific partitions. Using (2.15) and (2.16), we can reorder the operators B and C as follows:

$$\langle 0 | \prod_{j=1}^M C(\mu_j) \prod_{j=1}^M B(\lambda_j) | 0 \rangle = \langle 0 | \prod_{j \in \gamma} C(\mu_j) \prod_{k \in \bar{\gamma}} C(\mu_k) \cdot \prod_{k \in \alpha} B(\lambda_k) \prod_{j \in \bar{\alpha}} B(\lambda_j) | 0 \rangle. \quad (\text{B.5})$$

For the convenience, we rewrite the commutation relation (2.27) as the form

$$C(\mu)B(\lambda) = tB(\lambda)C(\mu) + g(\mu, \lambda)(A(\lambda)D(\mu) - A(\mu)D(\lambda)) \quad (\text{B.6})$$

where

$$g(\mu, \lambda) = \frac{(1-t)\lambda}{\mu - \lambda}. \quad (\text{B.7})$$

Here, we call the first and second terms on the right-hand side of (B.6) the first and second commutation schemes, respectively.

Let us consider an arbitrary operator $C(\mu_s)$ with a argument $\mu_s \in \{\mu\}_\gamma$ and begin moving it to the right using the relation (B.6). Suppose that during the commutation with the product $\prod_{j \in \alpha} B(\lambda_j)$ we always use the first scheme. Then, we obtain a state

$$\prod_{j \in \alpha} B(\lambda_j) \cdot C(\mu_s) \cdot \prod_{k \in \bar{\alpha}} B(\lambda_k) | 0 \rangle. \quad (\text{B.8})$$

In general, it is clear that the action of the operator $C(\mu_s)$ on the vector $\prod_{k \in \bar{\alpha}} B(\lambda_k) | 0 \rangle$ gives terms proportional to $a(\mu_s)d(\lambda_\ell)$, to $a(\mu_\ell)d(\mu_s)$ or to $a(\lambda_\ell)d(\lambda_{\ell'})$, where $\lambda_\ell, \lambda_{\ell'} \in$

$\{\lambda\}_{\bar{\alpha}}$. However, the coefficient with the partition which we have fixed contains the functions $d(\mu)$ and $d(\lambda)$ only for $\mu \in \{\mu\}_{\bar{\gamma}}$ and $\lambda \in \{\lambda\}_{\alpha}$. This is because the resulting partition is $\{\mu\} = \{\mu\}_{\gamma} \cup \{\mu\}_{\bar{\gamma}}$ and $\{\lambda\} = \{\lambda\}_{\alpha} \cup \{\lambda\}_{\bar{\alpha}}$, and the resulting coefficient must be proportional to

$$\prod_{j \in \gamma} a(\mu_j) \prod_{k \in \bar{\gamma}} d(\mu_k) \prod_{k \in \bar{\alpha}} a(\lambda_k) \prod_{j \in \alpha} d(\lambda_j). \quad (\text{B.9})$$

Hence, the state (B.8) does not contribute to the coefficient with the partition which we have fixed. As a result, we see that in the course of commutation of each of the operators $C(\mu_s)$, $\mu_s \in \{\mu\}_{\gamma}$, with the product $\prod_{j \in \alpha} B(\lambda_j)$, we must use the second scheme at least once:

$$\begin{aligned} \prod_{j \in \gamma} C(\mu_j) \cdot \prod_{j \in \alpha} B(\lambda_j) &= \prod_{\substack{j \in \gamma \\ j \neq s}} C(\mu_j) \sum_{\ell=1}^n t^{\ell-1} g(\mu_s, \lambda_{\ell}) B(\lambda_{\alpha_1}) \cdots B(\lambda_{\alpha_{\ell-1}}) \\ &\quad \times [A(\mu_s)D(\lambda_{\alpha_{\ell}}) - A(\lambda_{\alpha_{\ell}})D(\mu_s)] B(\lambda_{\alpha_{\ell+1}}) \cdots B(\lambda_{\alpha_n}) + \mathcal{L} \end{aligned} \quad (\text{B.10})$$

where we have denoted by \mathcal{L} all the terms that do not contribute to the desired coefficient. Using the relations (2.20), (2.22) and (2.24), we now move the operators A to the leftmost position and the operators D to the rightmost position. Repeating this procedure for all the operators $C(\mu)$ with $\{\mu\}_{\gamma}$, we finally obtain a formula analogous to (B.2) with the single difference that instead of the functions a and d we get the operators A and D :

$$\begin{aligned} \prod_{j \in \gamma} C(\mu_j) \cdot \prod_{j \in \alpha} B(\lambda_j) &= \sum_{\substack{\alpha_+ \cup \alpha_- \\ \gamma_+ \cup \gamma_-}} \prod_{j \in \gamma_+} A(\mu_j) \prod_{k \in \alpha_-} A(\lambda_k) \prod_{j \in \gamma_+} D(\lambda_j) \prod_{k \in \gamma_-} D(\mu_k) \\ &\quad \times K_n(\{\mu\}_{\gamma_+}, \{\mu\}_{\gamma_-} | \{\lambda\}_{\alpha_+}, \{\lambda\}_{\alpha_-}) + \mathcal{L} \end{aligned} \quad (\text{B.11})$$

where the summation is carried out here over all partitions of $\{\lambda\}_{\alpha}$ into two subsets $\{\lambda\}_{\alpha} = \{\lambda\}_{\alpha_+} \cup \{\lambda\}_{\alpha_-}$ and of $\{\mu\}_{\gamma}$ into two subsets $\{\mu\}_{\gamma} = \{\mu\}_{\gamma_+} \cup \{\mu\}_{\gamma_-}$.

Suppose that $\{\mu\}_{\gamma_-} \neq \emptyset$. Then, when an operator $D(\mu_s)$ with $\mu_s \in \{\mu\}_{\gamma_-}$ is commuted with the product $\prod_{j \in \bar{\alpha}} B(\lambda_j)$, we obtain terms proportional to either $d(\mu_s)$ or $d(\mu_{\ell})$ with $\lambda_{\ell} \in \{\lambda\}_{\bar{\lambda}}$. Since neither of these functions can occur in the final answer, we conclude that $\{\mu\}_{\gamma_-} = \emptyset$, and therefore also $\{\lambda\}_{\alpha_-} = \emptyset$. Consequently

$$\prod_{j \in \gamma} C(\mu_j) \cdot \prod_{j \in \alpha} B(\lambda_j) = \prod_{j \in \gamma} A(\mu_j) \cdot \prod_{j \in \alpha} D(\lambda_j) \cdot K_n(\{\mu\}_{\gamma} | \{\lambda\}_{\alpha}) + \mathcal{L} \quad (\text{B.12})$$

where $K_n(\{\mu\}_{\gamma} | \{\lambda\}_{\alpha})$ is the leading coefficient depending on the families $\{\mu\}_{\gamma}$ and $\{\lambda\}_{\alpha}$. As a result, we obtain

$$\begin{aligned} &\langle 0 | \prod_{j=1}^M C(\mu_j) \prod_{j=1}^M B(\lambda_j) | 0 \rangle \\ &= K_n(\{\mu\}_{\gamma} | \{\lambda\}_{\alpha}) \cdot \langle 0 | \prod_{k \in \bar{\gamma}} C(\mu_k) \prod_{j \in \gamma} A(\mu_j) \cdot \prod_{j \in \alpha} D(\lambda_j) \prod_{k \in \bar{\alpha}} B(\lambda_k) | 0 \rangle + \mathcal{L} \end{aligned} \quad (\text{B.13})$$

We move all the operators D to the rightmost position and the operators A to the leftmost position. Here we can only the first commutation scheme such that the operators A and D must preserve their arguments to obtain the term proportional to (B.9). Thus, we obtain

$$\begin{aligned} \langle 0 | \prod_{j=1}^M C(\mu_j) \prod_{j=1}^M B(\lambda_j) | 0 \rangle &= K_n(\{\mu\}_\gamma | \{\lambda\}_\alpha) \cdot \prod_{j \in \gamma} a(\mu_j) \prod_{j \in \alpha} d(\lambda_j) \\ &\prod_{a \in \alpha} \prod_{b \in \bar{\alpha}} f(\lambda_b, \lambda_a) \prod_{a \in \gamma} \prod_{b \in \bar{\gamma}} f(\mu_a, \mu_b) \langle 0 | \prod_{k \in \bar{\gamma}} C(\mu_k) \cdot \prod_{k \in \bar{\alpha}} B(\lambda_k) | 0 \rangle. \end{aligned} \quad (\text{B.14})$$

The contribution in the remaining inner product must be given by the term proportional to the conjugate leading coefficient. We finally obtain

$$\begin{aligned} K_M(\{\mu\}_\gamma, \{\mu\}_{\bar{\gamma}} | \{\lambda\}_\alpha, \{\lambda\}_{\bar{\alpha}}) &= \prod_{a \in \alpha} \prod_{b \in \bar{\alpha}} f(\lambda_b, \lambda_a) \prod_{a \in \gamma} \prod_{b \in \bar{\gamma}} f(\mu_a, \mu_b) \\ &\times K_n(\{\mu\}_\gamma | \{\lambda\}_\alpha) \bar{K}_{M-n}(\{\mu\}_{\bar{\gamma}} | \{\lambda\}_{\bar{\alpha}}). \end{aligned} \quad (\text{B.15})$$

Thus we can have proved that an arbitrary coefficient can be expressed in terms of the leading and conjugate leading coefficient.

B.1.1 The leading coefficient

We derive a recurrence relation for the leading coefficient and find an explicit formula for the leading coefficient K_M by solving it. To this end, we must single out the unique term in (B.2) corresponding to the partition $\{\mu\}_\gamma = \{\mu\}$, $\{\lambda\}_\alpha = \{\lambda\}$. Let us consider the action of the operator $C(\mu_M)$ on the vector $\prod_{j=1}^M B(\lambda_j) | 0 \rangle$. Using the formula (2.37), we obtain

$$\begin{aligned} C(\mu_M) \prod_{j=1}^M B(\lambda_j) | 0 \rangle &= - \sum_{\ell=1}^M a(\mu_M) d(\lambda_\ell) g(\mu_M, \lambda_\ell) \prod_{\substack{j=1 \\ j \neq \ell}}^M (f(\mu_M, \lambda_j) f(\lambda_j, \lambda_\ell)) \cdot \prod_{\substack{j=1 \\ j \neq \ell}}^M B(\lambda_j) | 0 \rangle + \mathcal{L}. \end{aligned} \quad (\text{B.16})$$

Multiplying the equality (B.16) by the dual vector $\langle 0 | \prod_{j=1}^{M-1} C(\mu_j)$, we immediately obtain a recurrence relation for the leading coefficient:

$$\begin{aligned} K_M(\{\mu\} | \{\lambda\}) &= - \sum_{\ell=1}^M g(\mu_M, \lambda_\ell) \prod_{\substack{a=1 \\ a \neq \ell}}^M (f(\mu_M, \lambda_a) f(\lambda_a, \lambda_\ell)) \cdot K_{M-1}(\{\mu \neq \mu_M\} | \{\lambda \neq \lambda_\ell\}). \end{aligned} \quad (\text{B.17})$$

This relation together with the initial condition

$$K_1(\mu_1 | \lambda_1) = -g(\mu_1, \lambda_1) \quad (\text{B.18})$$

uniquely fixes the leading coefficient and enables one to compute it recursively. However, we can find an explicit formula for the leading coefficient for any M .

Proposition B.1.1. *The leading coefficient $K_M(\{\mu\}|\{\lambda\})$ is given explicitly by the formula*

$$K_M(\{\mu\}|\{\lambda\}) = (-1)^M \prod_{a=1}^M \left\{ (1-t)\lambda_a \right\} \cdot \frac{\prod_{a,b=1}^M (t\mu_a - \lambda_b)}{\prod_{a>b}^M (\mu_a - \mu_b)(\lambda_b - \lambda_a)} \cdot \det_M \left(t(\mu_a, \lambda_b) \right) \quad (\text{B.19})$$

where

$$t(\mu, \lambda) = \frac{1}{(\mu - \lambda)(\mu t - \lambda)}. \quad (\text{B.20})$$

To prove Proposition B.1.1, we need the following lemma.

Lemma B.1.1. *Let*

$$u_k = \frac{\prod_{a=1}^M (\mu_k - \lambda_a)}{\prod_{\substack{a=1 \\ a \neq k}}^M (\mu_k - \mu_a)}. \quad (\text{B.21})$$

Then,

$$\sum_{k=1}^M u_k \cdot t(\mu_k, \lambda_j) = \frac{\prod_{\substack{a=1 \\ a \neq k}}^M (t\lambda_a - \lambda_j)}{\prod_{a=1}^M (t\mu_a - \lambda_j)}. \quad (\text{B.22})$$

The proof of Lemma B.1.1

Let us define G_j for $j = 1, \dots, M$ as

$$G_j = \sum_{k=1}^M u_k t(\mu_k, \lambda_j). \quad (\text{B.23})$$

We consider the auxiliary integral

$$I = \int_C \frac{dz}{2\pi i} \frac{1}{(z - \lambda_j)(tz - \lambda_j)} \prod_{a=1}^M \frac{z - \lambda_a}{z - \mu_a} \quad (\text{B.24})$$

where the contour of integral C is a circle with a radius $|z| = \infty$. Then, we conclude that $I = 0$. On the other hand, poles of the integrand are $z = \lambda_j/t$ and $z = \mu_1, \dots, \mu_M$ and the integral (B.24) is equal to the sum of the residues inside the contour. The sum of residues at the points $z = \mu_k$ is equal to G_j . Also, the residue at the point $z = \lambda_j/t$ is equal to

$$-\frac{\prod_{\substack{a=1 \\ a \neq k}}^M (t\lambda_a - \lambda_j)}{\prod_{a=1}^M (t\mu_a - \lambda_j)}. \quad (\text{B.25})$$

Equating the total sum of the residues to zero, we arrive at the equality

$$G_j = \frac{\prod_{\substack{a=1 \\ a \neq k}}^M (t\lambda_a - \lambda_j)}{\prod_{a=1}^M (t\mu_a - \lambda_j)}, \quad (\text{B.26})$$

as was to be proved. \square

Proof of Proposition B.1.1

Let us prove this proposition by using the induction. When $M = 1$, (B.19) coincides the initial condition (B.18). When $M = m - 1$, we assume that K_{m-1} satisfies (B.19). Then, $K_m(\{\mu\}|\{\lambda\})$ becomes

$$\begin{aligned} K_m(\{\mu\}|\{\lambda\}) &= - \sum_{\ell=1}^m g(\mu_m, \lambda_\ell) \prod_{\substack{a=1 \\ a \neq \ell}}^m \left(f(\mu_m, \lambda_a) f(\lambda_a, \lambda_\ell) \right) \cdot K_{m-1}(\{\mu \neq \mu_m\}|\{\lambda \neq \lambda_\ell\}) \\ &= (-1)^m \prod_{a=1}^m \{(1-t)\lambda_a\} \cdot \frac{\prod_{a,b=1}^m (t\mu_a - \lambda_b)}{\prod_{a>b}^m (\mu_a - \mu_b)(\lambda_b - \lambda_a)} \\ &\quad \times \frac{1}{u_m} \cdot \sum_{\ell=1}^m (-1)^{m+\ell} G_\ell \cdot \det_{m-1} t(\mu_i, \lambda_j) \Big|_{\{\mu \neq \mu_m\}, \{\lambda \neq \lambda_\ell\}}. \end{aligned} \quad (\text{B.27})$$

On the other hand, we consider the matrix $t(\mu_j, \lambda_k)$. To the last row of the matrix, let us add all the other rows, multiplied by the coefficients u_k/u_m :

$$\det_m t(\mu_j, \lambda_k) = \prod_{j=1}^m \frac{u_m}{u_j} \cdot \det_m \begin{pmatrix} \frac{u_1}{u_m} t(\mu_1, \lambda_1) & \dots & \frac{u_1}{u_m} t(\mu_1, \lambda_M) \\ \vdots & \dots & \vdots \\ \frac{u_{m-1}}{u_m} t(\mu_{m-1}, \lambda_1) & \dots & \frac{u_{m-1}}{u_m} t(\mu_{m-1}, \lambda_m) \\ \sum_{j=1}^m \frac{u_j}{u_m} t(\mu_j, \lambda_1) & \dots & \sum_{j=1}^m \frac{u_j}{u_m} t(\mu_j, \lambda_m) \end{pmatrix}. \quad (\text{B.28})$$

Then, by Lemma B.1.1, the last row turns out to be equal to G_j/u_m . Expanding this determinant by the last row, we obtain

$$\det_m t(\mu_j, \lambda_k) = \frac{1}{u_m} \sum_{\ell=1}^m (-1)^{m+\ell} \cdot G_\ell \cdot \det_{m-1} t(\mu_j, \lambda_k) \Big|_{\{\mu \neq \mu_m\}, \{\lambda \neq \lambda_\ell\}}. \quad (\text{B.29})$$

Substituting this relation to (B.27), we see that K_m satisfies (B.19). Thus we have proved the Proposition B.1.1. \square

B.1.2 The conjugate leading coefficient

Next, we consider the conjugate leading coefficient $\bar{K}_M(\{\mu\}|\{\lambda\})$. The recurrence relation for the conjugate leading coefficient is

$$\begin{aligned} \bar{K}_M(\{\mu\}|\{\lambda\}) &= \sum_{\ell=1}^M g(\mu_M, \lambda_\ell) \prod_{\substack{a=1 \\ a \neq \ell}}^M \left(f(\lambda_a, \mu_M) f(\lambda_\ell, \lambda_a) \right) \cdot \bar{K}_{M-1}(\{\mu \neq \mu_M\}|\{\lambda \neq \lambda_\ell\}). \end{aligned} \quad (\text{B.30})$$

The initial condition is

$$\bar{K}_1(\mu|\lambda) = g(\mu, \lambda). \quad (\text{B.31})$$

Then, from the recurrence relation, we arrive at a following proposition.

Proposition B.1.2. *The conjugate leading coefficient $\bar{K}_M(\{\mu\}|\{\lambda\})$ is given explicitly by the formula*

$$\bar{K}_M(\{\mu\}|\{\lambda\}) = (-1)^M \prod_{a=1}^M \left\{ (1-t)\lambda_a \right\} \cdot \frac{\prod_{a,b=1}^M (t\lambda_a - \mu_b)}{\prod_{a>b}^M (\mu_a - \mu_b)(\lambda_b - \lambda_a)} \cdot \det_M(t(\lambda_a, \mu_b)) \quad (\text{B.32})$$

We can prove this proposition by means of the induction and a following Lemma as well as the case of the leading coefficient:

Lemma B.1.2.

$$\sum_{k=1}^M v_k t(\lambda_\ell, \mu_k) = J_\ell \quad (\text{B.33})$$

where

$$v_k = \frac{\prod_{a=1}^M (\lambda_a - \mu_k)}{\prod_{a \neq k}^M (\mu_a - \mu_k)} \quad \text{and} \quad J_\ell = \frac{\prod_{a \neq k}^M (t\lambda_\ell - \mu_a)}{\prod_{a=1}^M (t\lambda_\ell - \mu_a)}. \quad (\text{B.34})$$

As a result, we obtain the final answer for the inner product:

$$\begin{aligned} S_M(\{\mu\}|\{\lambda\}) &= \prod_{a=1}^M \{(t-1)\lambda_a\} \cdot \prod_{a>b}^M (c(\lambda_a, \lambda_b)c(\mu_b, \mu_a)) \\ &\times \sum_{\substack{\alpha \cup \bar{\alpha} \\ \gamma \cup \bar{\gamma}}} (-1)^{P_\alpha + P_\gamma} \prod_{j \in \gamma} a(\mu_j) \prod_{j \in \bar{\gamma}} d(\mu_j) \prod_{j \in \alpha} d(\lambda_j) \prod_{j \in \bar{\alpha}} a(\lambda_j) \cdot \det_{\substack{j \in \alpha \\ k \in \gamma}} t(\mu_k, \lambda_j) \cdot \det_{\substack{j \in \bar{\alpha} \\ k \in \bar{\gamma}}} t(\lambda_j, \mu_k) \\ &\times \prod_{a \in \alpha} \prod_{b \in \gamma} h(\mu_b, \lambda_a) \cdot \prod_{a \in \bar{\alpha}} \prod_{b \in \bar{\gamma}} h(\lambda_a, \mu_b) \cdot \prod_{a \in \alpha} \prod_{b \in \bar{\alpha}} h(\lambda_b, \lambda_a) \cdot \prod_{a \in \gamma} \prod_{b \in \bar{\gamma}} h(\mu_a, \mu_b) \end{aligned} \quad (\text{B.35})$$

where

$$c(\lambda, \mu) = \frac{1}{\lambda - \mu}, \quad h(\lambda, \mu) = t\lambda - \mu. \quad (\text{B.36})$$

Here, P_α and P_γ are the parties of the permutations

$$\begin{aligned} P(\alpha_1, \dots, \alpha_n, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-n}) &= 1, \dots, M \\ P(\gamma_1, \dots, \gamma_n, \bar{\gamma}_1, \dots, \bar{\gamma}_{M-n}) &= 1, \dots, M. \end{aligned} \quad (\text{B.37})$$

B.2 Inner product of an eigenstate with an arbitrary state

Let us now consider the inner product for the case when one of the states is an eigenvector of the transfer matrix. Hence, we suppose that the parameters $\{\lambda\}$ of the state $\prod_{j=1}^M B(\lambda_j)|0\rangle$ satisfy the Bethe Ansatz equations of the q-boson model (2.44). Therefore we can express the function $a(\lambda_j)$ in terms of $d(\lambda_j)$ as

$$a(\lambda_j) = d(\lambda_j)(-1)^{M-1} \prod_{k=1}^M \frac{h(\lambda_k, \lambda_j)}{h(\lambda_j, \lambda_k)} \quad (\text{B.38})$$

Substituting this into (B.35), we obtain

$$\begin{aligned} & \langle 0 | \prod_{j=1}^M C(\mu_j) | \psi(\{\lambda\}) \rangle \\ &= \prod_{a=1}^M \{(t-1)\lambda_a\} \cdot \prod_{a>b}^M (c(\lambda_a, \lambda_b)c(\mu_b, \mu_a)) \cdot \prod_{j=1}^M d(\lambda_j) \cdot \\ & \times \sum_{\substack{\alpha \cup \bar{\alpha} \\ \gamma \cup \bar{\gamma}}} (-1)^{P_\alpha + P_\gamma} (-1)^{nM-n} \prod_{j \in \gamma} a(\mu_j) \prod_{j \in \bar{\gamma}} d(\mu_j) \cdot \det_{\substack{j \in \alpha \\ k \in \gamma}} t(\mu_k, \lambda_j) \cdot \det_{\substack{j \in \bar{\alpha} \\ k \in \bar{\gamma}}} t(\lambda_j, \mu_k) \\ & \times \prod_{a \in \alpha} \prod_{b \in \gamma} h(\mu_b, \lambda_a) \cdot \prod_{a \in \bar{\alpha}} \prod_{b \in \bar{\gamma}} h(\lambda_a, \mu_b) \cdot \prod_{a \in \alpha} \prod_{b \in \bar{\alpha}} h(\lambda_a, \lambda_b) \cdot \prod_{a \in \gamma} \prod_{b \in \bar{\gamma}} h(\mu_a, \mu_b) \end{aligned} \quad (\text{B.39})$$

where $\#\{\alpha\}$.

Here, we introduce an auxiliary function $G_M^{(n)}$ depending for fixed n and M ($0 \leq n \leq M$) on three families $\{\xi_1, \dots, \xi_n\}$, $\{\nu_1, \dots, \nu_{M-n}\}$ and $\{\lambda_1, \dots, \lambda_M\}$ of complex variables for the convenience:

$$\begin{aligned} & G_M^{(n)}(\{\xi\}, \{\nu\}, \{\lambda\}) \\ &= \sum_{\substack{\alpha \cup \bar{\alpha} \\ \#\{\alpha\}=n}} (-1)^{P_\alpha} \det_{\substack{j \in \alpha \\ k=1, \dots, n}} t(\xi_k, \lambda_j) \cdot \det_{\substack{j \in \bar{\alpha} \\ k=1, \dots, M-n}} t(\lambda_j, \nu_k) \\ & \times \left\{ \prod_{a=1}^n \prod_{b=1}^{M-n} h(\xi_a, \nu_b) \cdot \prod_{a \in \alpha} \prod_{b \in \bar{\alpha}} h(\lambda_a, \lambda_b) - \prod_{a=1}^n \prod_{b \in \bar{\alpha}} h(\xi_a, \lambda_b) \prod_{a=1}^{M-n} \prod_{b \in \alpha} h(\lambda_b, \nu_a) \right\} \\ & \times \prod_{a=1}^n \prod_{b \in \alpha} h(\xi_a, \lambda_b) \prod_{a=1}^{M-n} \prod_{b \in \bar{\alpha}} h(\lambda_b, \nu_a). \end{aligned} \quad (\text{B.40})$$

Then, we can show that for the arbitrary families $\{\xi\}$, $\{\nu\}$ and $\{\lambda\}$ of complex numbers this function is

$$G_M^{(n)}(\{\xi\}, \{\nu\}, \{\lambda\}) = 0. \quad (\text{B.41})$$

Let us set the partition $\{\mu\}_\gamma = \{\xi_1, \dots, \xi_n\}$ and $\{\mu\}_{\bar{\gamma}} = \{\nu_1, \dots, \nu_{M-n}\}$ at (B.40) and make use of (B.41). Then, we obtain

$$\begin{aligned}
 & \langle 0 | \prod_{j=1}^M C(\mu_j) | \psi(\{\lambda\}) \rangle \\
 &= \prod_{a=1}^M \{(t-1)\lambda_a\} \cdot \prod_{a>b}^M (c(\lambda_a, \lambda_b)c(\mu_b, \mu_a)) \cdot \prod_{j=1}^M d(\lambda_j) \\
 & \times \sum_{\substack{\alpha \cup \bar{\alpha} \\ \gamma \cup \bar{\gamma}}} (-1)^{P_\alpha + P_\gamma} (-1)^{nM-n} \prod_{j \in \gamma} a(\mu_j) \prod_{j \in \bar{\gamma}} d(\mu_j) \cdot \det_{\substack{j \in \alpha \\ k \in \gamma}} t(\mu_k, \lambda_j) \cdot \det_{\substack{j \in \bar{\alpha} \\ k \in \bar{\gamma}}} t(\lambda_j, \mu_k) \\
 & \times \prod_{a=1}^M \prod_{b \in \gamma} h(\mu_b, \lambda_a) \cdot \prod_{a=1}^M \prod_{b \in \bar{\gamma}} h(\lambda_a, \mu_b). \tag{B.42}
 \end{aligned}$$

Further, we use the Laplace formula for the determinant of a sum of two matrices $U(\mu_j, \lambda_k)$ and $V(\mu_j, \lambda_k)$ at (B.42)

$$\det_M (U(\mu_k, \lambda_j) V(\mu_k, \lambda_j)) = \sum_{\substack{\alpha \cup \bar{\alpha} \\ \gamma \cup \bar{\gamma}}} (-1)^{P_\alpha + P_\gamma} \cdot \det_{\substack{k \in \alpha \\ j \in \gamma}} U(\mu_k, \lambda_j) \cdot \det_{\substack{k \in \bar{\alpha} \\ j \in \bar{\gamma}}} V(\mu_k, \lambda_j) \tag{B.43}$$

where

$$U(\mu_k, \lambda_j) = (-1)^{M-1} (t-1) \lambda_j \cdot a(\mu_k) \cdot t(\mu_k, \lambda_j) \cdot \prod_{a=1}^M h(\mu_k, \lambda_a), \tag{B.44}$$

$$V(\mu_k, \lambda_j) = (t-1) \cdot \lambda_j \cdot d(\mu_k) \cdot t(\lambda_j, \mu_k) \cdot \prod_{a=1}^M h(\lambda_a, \mu_k). \tag{B.45}$$

Thus, we arrive at the following assertion.

Proposition B.2.1. *Suppose that the parameter family $\{\lambda\}$ satisfies the system of the Bethe Ansatz equation (2.44) and let the parameters $\{\mu\}$ be arbitrary complex numbers. Then,*

$$\begin{aligned}
 & \langle 0 | \prod_{j=1}^M C(\mu_j) | \psi(\{\lambda\}) \rangle \\
 &= \prod_{a=1}^M d(\lambda_a) \cdot \prod_{a>b}^M \{c(\lambda_a, \lambda_b)c(\mu_b, \mu_a)\} \cdot \prod_{a=1}^M \{(t-1)\lambda_a\} \cdot \det_M H(\lambda_j, \mu_k) \tag{B.46}
 \end{aligned}$$

where

$$H(\lambda_j, \mu_k) = \frac{1}{\mu_k - \lambda_j} \left\{ a(\mu_k) \prod_{\substack{a=1 \\ a \neq j}}^M h(\mu_k, \lambda_a) - (-1)^{M-1} d(\mu_k) \prod_{\substack{a=1 \\ a \neq j}}^M h(\lambda_a, \mu_k) \right\}. \tag{B.47}$$

The matrix $H(\lambda_j, \mu_k)$ turns out to be closely related to the eigenvalues of the transfer matrix (2.42) as

$$H(\lambda_j, \mu_k) = \frac{1}{(t-1)\mu_k} \cdot \prod_{a=1}^M \frac{1}{c(\mu_k, \lambda_a)} \cdot \frac{\partial \Lambda(\mu_k, \{\lambda\})}{\partial \lambda_j}. \quad (\text{B.48})$$

Thus, the formula (B.46) can be rewritten in the form

$$\begin{aligned} & \langle 0 | \prod_{j=1}^M C(\mu_j) | \psi(\{\lambda\}) \rangle \\ &= (-1)^M \prod_{a=1}^M d(\lambda_a) \cdot \prod_{a=1}^M \frac{\lambda_a}{\mu_a} \cdot \mathcal{X}_M^{-1}(\{\mu\}, \{\lambda\}) \cdot \det_M \left(\frac{\partial}{\partial \lambda_j} \Lambda(\mu_k, \{\lambda\}) \right) \end{aligned} \quad (\text{B.49})$$

where \mathcal{X}_M is the Cauchy determinant:

$$\mathcal{X}_M^{-1}(\{\mu\}, \{\lambda\}) = \det_M \left(\frac{1}{\mu - \lambda} \right) = \frac{\prod_{a>b}^M (\lambda_a - \lambda_b)(\mu_b - \mu_a)}{\prod_{a,b=1}^M (\mu_a - \lambda_b)}. \quad (\text{B.50})$$

One can treat similarly the case in which a dual vector is an eigenstate of the transfer matrix. As a result, we can show

$$\begin{aligned} & \langle \psi(\{\lambda\}) | \prod_{j=1}^M B(\mu_j) | 0 \rangle \\ &= (-1)^M \prod_{a=1}^M d(\lambda_a) \cdot \mathcal{X}_M^{-1}(\{\mu\}, \{\lambda\}) \cdot \det_M \left(\frac{\partial}{\partial \lambda_j} \Lambda(\mu_k, \{\lambda\}) \right). \end{aligned} \quad (\text{B.51})$$

Thus, we find that the relation between (B.49) and (B.51) is

$$\langle \psi | \prod_{j=1}^M B(\mu_j) | \psi(\{\lambda\}) | 0 \rangle = \prod_{a=1}^M \frac{\mu_a}{\lambda_a} \cdot \langle 0 | \prod_{j=1}^M C(\mu_j) | \psi(\{\lambda\}) \rangle. \quad (\text{B.52})$$

Unlike the case of the XXZ Heisenberg model, (B.51) completely does not coincide with (B.49). This is because we have carried out the calculation by using the anti-symmetric R-matrix (2.10).

Finally, we derive a formula for the squared norm of an eigenstate of the transfer matrix. We set $\{\mu\} = \{\lambda\}$ for the scalar product (B.49) or (B.51). Noticing that H becomes $0/0$ in this limit, we arrive at a following proposition:

Proposition B.2.2. *Suppose that the parameter family $\{\lambda\}$ satisfies the system of the Bethe Ansatz equation (2.44). Then,*

$$\begin{aligned} \langle \psi(\{\lambda\}_M) | \psi(\{\lambda\}_M) \rangle &= \langle 0 | \prod_{a=1}^M C(\lambda_a) \prod_{a=1}^M B(\lambda_a) | 0 \rangle \\ &= \frac{\prod_{a,b=1}^M (\lambda_a t - \lambda_b)}{\prod_{\substack{a,b=1 \\ a \neq b}}^M (\lambda_a - \lambda_b)} \cdot \det_M \Phi'_{j,k}(\{\lambda\}_M) \end{aligned} \quad (\text{B.53})$$

where the Gaudin matrix $\Phi'_{j,k}(\{\lambda\}_M)$ is

$$\begin{aligned}\Phi'_{j,k}(\{\lambda\}_M) &= \frac{\partial}{\partial \lambda_k} \log \left\{ \frac{a(\lambda_j)}{d(\lambda_j)} \cdot \prod_{\substack{b=1 \\ b \neq j}}^M \frac{f(\lambda_j, \lambda_b)}{f(\lambda_b, \lambda_j)} \right\} \\ &= \delta_{j,k} \left\{ -\frac{L}{\lambda_j} + \sum_{b=1}^M \frac{(t^2 - 1)\lambda_b}{(\lambda_j t - \lambda_b)(\lambda_b t - \lambda_j)} \right\} - \frac{(t^2 - 1)\lambda_j}{(\lambda_j t - \lambda_k)(\lambda_k t - \lambda_j)}. \quad (\text{B.54})\end{aligned}$$

As a result, we can have showed the expression (2.49) and (2.51) for an inner product between the eigenstates of the transfer matrix in the q-boson model.

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