Math. Z. 248, 773–794 (2004) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-004-0680-x

Motivic cohomology over Dedekind rings

Thomas Geisser[*]

Department of Mathematics, KAP108, University of Southern California, 3620 Vermont Av., Los Angeles, CA 90089, USA

Received: 20 February 2003; in final form: 16 November 2003 /
Published online: 10 August 2004 –

Abstract. We study properties of Bloch's higher Chow groups on smooth varieties over Dedekind rings. We prove the vanishing of $\mathcal{H}^i(\mathbb{Z}(n))$ for $i > n$, and the existence of a Gersten resolution for $\mathcal{H}^i(\mathbb{Z}/p^r(n))$, if the residue characteristic is p. We also show that the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture $\mathbb{Z}(n)_{\text{Zar}} \cong \tau_{\leq n+1} R\epsilon_* \mathbb{Z}(n)_{\text{ét}}$, an identification $\mathbb{Z}/m(n)_{\text{ét}} \cong \mu_m^{\otimes n}$, for m invertible, and a Gersten resolution with (arbitrary) finite coefficients. Over a complete discrete valuation ring of mixed characteristic $(0, p)$, we construct a map from motivic cohomology to syntomic cohomology, which is a quasi-isomorphism provided the Bloch-Kato conjecture holds.

1. Introduction

Let X be an (essentially) smooth scheme over a base B, and define motivic cohomology $H^*(X, \mathbb{Z}(n))$ and étale motivic cohomology $H^*_{\text{ét}}(X, \mathbb{Z}(n))$ as the hypercohomology of Bloch's cycle complex $\mathbb{Z}(n)$ for the Zariski and étale topology, respectively. If B is the spectrum of a field, then these groups agree with the cohomology of Bloch's cycle complex on the one hand [1], and with motivic cohomology groups of Voevodsky on the other hand [30]. In case B is a discrete valuation ring, Levine [16, 17] has established several properties, most notably the localization property, which assures that cohomology and hypercohomology of $\mathbb{Z}(n)$ agree. The purpose of this article is to establish additional sheaf theoretic properties of $\mathbb{Z}(n)$ if the base is a Dedekind ring. Fix an essentially smooth scheme X over the spectrum B of a Dedekind ring. We prove the following conditional Gersten resolution:

[*] Supported in part by JSPS, NSF Grant. No. 0070850, and the Alfred P. Sloan Foundation

Theorem 1.1. *Assume that for every discrete valuation ring V, essentially of finite type over B, with quotient field K, the map $H^s(V,\mathbb{Z}(n)) \to H^s(K,\mathbb{Z}(n))$ is injective. Then there is an exact sequence of Zariski sheaves on X,*

$$0 \to \mathcal{H}^s(\mathbb{Z}(n)_{Zar}) \to \bigoplus_{x\in X^{(0)}} (i_x)_* H^s(k(x),\mathbb{Z}(n)) \to \bigoplus_{x\in X^{(1)}} (i_x)_* H^{s-1}(k(x), \mathbb{Z}(n-1)) \to \cdots.$$

The analog statement holds with arbitrary coefficients. The proof is based on the modification of Quillen's argument [24] by Gillet-Levine [11]. As corollaries of the proof, we obtain that the complex $\mathbb{Z}(n)$ is acyclic in degrees above n, and a Gersten resolution for $\mathcal{H}^s(\mathbb{Z}/p^r(n))$ if B has residue characteristic p.

The Bloch-Kato conjecture [3] states that for any field F, and m relatively prime to the characteristic of F, the symbol map from Milnor K-theory to Galois cohomology,

$$K_n^M(F)/m \to H^n_{\text{ét}}(F,\mu_m^{\otimes n})$$

is surjective. The conjecture is known for m a power of 2 by Voevodsky [31], and in general a proof has been announced in [32]. Let $\epsilon : X_{\text{ét}} \to X_{\text{Zar}}$ be the change of site map.

Theorem 1.2. *The Bloch-Kato conjecture implies the following:*

1. *(Purity) If $i : Y \to X$ is the inclusion of one of the closed fibers, then the canonical map*
$$\mathbb{Z}(n-1)_{ét}[-2] \to \tau_{\leq n+1} Ri^!\mathbb{Z}(n)_{ét} \tag{1}$$
is a quasi-isomorphism. If $n \neq 0$, then truncation is unnecessary after inverting the residue characteristic.
2. *(Beilinson-Lichtenbaum) The canonical map induces a quasi-isomorphism*
$$\mathbb{Z}(n)_{Zar} \xrightarrow{\sim} \tau_{\leq n+1} R\epsilon_*\mathbb{Z}(n)_{ét}.$$
3. *(Rigidity) Let R be a henselian local ring of a smooth scheme over B, k the residue field of R, and $m \in \mathbb{N}$ be invertible in k. Then the canonical map is an isomorphism*
$$H^i(R,\mathbb{Z}/m(n)) \xrightarrow{\sim} H^i(k,\mathbb{Z}/m(n)).$$
4. *(Étale sheaf) If $m \in \mathbb{N}$ is invertible in B, then there is a quasi-isomorphism of complexes of étale sheaves*
$$\mathbb{Z}/m(n)_{ét} \simeq \mu_m^{\otimes n}[0].$$
5. *(Gersten resolution) For any m, there is an exact sequence:*
$$0 \to \mathcal{H}^s(\mathbb{Z}/m(n)_{Zar}) \to \bigoplus_{x\in X^{(0)}} (i_x)_* H^s(k(x),\mathbb{Z}/m(n)) \to \bigoplus_{x\in X^{(1)}} (i_x)_* H^{s-1}(k(x),\mathbb{Z}/m(n-1)) \to \cdots.$$

Note that the combination of (2), (4) and (5) gives a Gersten resolution for the sheaf $R^s\epsilon_*\mu_m^{\otimes n}$ for $s \le n$, and m invertible on B.

If X is a smooth scheme over a discrete valuation ring Λ of characteristic $(0, p)$, with generic fiber $j : u \to X$ and closed fiber $i : y \to x$ then (1) implies the existence of a distinguished triangle

$$i^*\mathbb{Z}/p^r(n)_{\text{ét}} \to \tau_{\le n} i^* Rj_*\mu_{p^r}^{\otimes n} \to \nu_r^{n-1}[-n], \tag{2}$$

where ν_r^{n-1} is the logarithmic de Rham-Witt sheaf of Bloch-Illusie-Milne. If Λ is complete and its residue field has a finite p-base, then by results of Kato and Kurihara [15], this property is shared by the syntomic complex $S_r(n)$ of Fontaine-Messing [6], whenever the latter is defined. In general, we define $S_r(n)$ as the cone of the map $\kappa : \tau_{\le n} i^* Rj_*\mu_{p^r}^{\otimes n} \to \nu_r^{n-1}[-n]$ defined by Bloch-Kato [3]. The following theorem shows that motivic cohomology can be thought of as a generalization of syntomic cohomology, see Milne [19, Remark 2.7] and Schneider [27].

Theorem 1.3. *There is a unique map $i^*\mathbb{Z}/p^r(n)_{ét} \to S_r(n)$ in the derived category of étale sheaves on Y which is compatible with the maps of both complexes to $\tau_{\le n} i^* Rj_*\mu_{p^r}^{\otimes n}$. The map is a quasi-isomorphism provided that the Bloch-Kato conjecture with mod p^r-coefficients holds.*

This can be used to better understand the relationship between various proofs of the crystalline conjecture, and to construct a cycle map to syntomic cohomology.

Notation. Throughout the paper, B is the spectrum of a Dedekind ring, and X an equidimensional scheme, essentially smooth over B. By $X^{(i)}$ we denote the set of points of codimension i of X. For a closed point $b \in B^{(1)}$ we denote by $i_b : X_b \to X$ the closed embedding of the fiber of X over b, and by $\tilde{X}_b$ the fiber over the localization of B at b.

For an integral scheme V over B we define the dimension of V to be the Krull dimension of V if V is contained in one of the closed fibers, and to be the Krull dimension of the generic fiber plus one in case V is flat over B, see the discussion in [25, §4.2].

Acknowledgements. This paper relies heavily on the work of M. Levine and discussions with him. The paper benefited from discussions with W. Niziol and K. Sato, and comments of the referee. Finally, I would like to thank T. Saito and the Tokyo University Department of Mathematics for their hospitality.

2. Some homological algebra

Let $X_{\text{ét}}$, X_{Nis} and X_{Zar} be the category of étale schemes over X equipped with the étale, Nisnevich and Zariski topology, respectively. In case the site has infinite cohomological dimension (which may happen for the étale topology), we calculate derived functors of unbounded complexes using the method of Spaltenstein [28]. Let

$$X_{\text{ét}} \xrightarrow{\epsilon} X_{\text{Zar}}, \qquad X_{\text{ét}} \xrightarrow{\alpha} X_{\text{Nis}} \xrightarrow{\beta} X_{\text{Zar}}$$

be the canonical maps of sites. By adjointness, the sheafification functors ϵ^*, α^*, β^* are exact and preserve colimits, and the forgetful functors ϵ_*, α_*, β_* are left exact, preserve injectives and limits.

Let $i : Y \to X$ be a closed subscheme with open complement $j : U \to X$. For any of the three topologies above, there are mutually adjoint functors $i^* \vdash i_* \vdash i^!$ and $j_! \vdash j^* \vdash j_*$. In particular, $i^!$, j_* are left exact, i_*, j^* are exact, and in our situation i^* and $j_!$ are exact. Consequently, i_*, $i^!$, j^*, j_* preserve limits and injectives, whereas i^*, i_*, $j_!$, j^* preserve colimits. The remaining functors are covered by the following:

Lemma 2.1. *The functors $\epsilon_*, \alpha_*, \beta_*, j_*, i^!$, and the derived functors $R\epsilon_*$, $R\alpha_*$, $R\beta_*$, Rj_*, $Ri^!$ commute with pseudo-filtered colimits of sheaves.*

Proof. Since the sites we consider are noetherian, the presheaf colimit of a pseudo-filtered system of sheaves is a sheaf [20, III 3.6(b)], and the result for the functors ϵ_*, α_*, β_*, and j_* follows. The derived functors can be calculated with flabby sheaves [20, Cor. 1.14], and a pseudo-filtered colimit of injective sheaves is flabby [20, III 3.6(c)], hence the result for the derived functors.

The result for $i^! = \ker i^* \to i^* j_* j^*$ follows from the left exactness of pseudo-filtered colimits and the commutation of i^*, j_*, j^* with pseudo-filtered colimits. For an injective sheaf I, and V étale over X, the inclusion $V \times_X U \to V$ of Zariski open sets induces a surjection $I(V) \to I(V \times_X U)$. This property is preserved by colimits of sheaves, hence a pseudo-filtered colimit of injective sheaves is acyclic for $i^!$, and we can calculate $Ri^!$ with it. □

Proposition 2.2. *a) The following maps are isomorphisms of derived functors*

$$\begin{array}{ll} \epsilon^* j^* \xrightarrow{\sim} j^* \epsilon^* & R\epsilon_* Rj_* \xrightarrow{\sim} Rj_* R\epsilon_* \\ \epsilon^* i^* \xrightarrow{\sim} i^* \epsilon^* & R\epsilon_* i_* \xrightarrow{\sim} i_* R\epsilon_* \\ \epsilon^* i_* \xrightarrow{\sim} i_* \epsilon^* & R\epsilon_* Ri^! \xrightarrow{\sim} Ri^! R\epsilon_* \\ \epsilon^* j_! \xrightarrow{\sim} j_! \epsilon^* & R\epsilon_* j^* \xrightarrow{\sim} j^* R\epsilon_*. \end{array}$$

The analog statements for α and β hold.

b) If the étale cohomological dimension of each residue field of Y is finite, then

$$i^* R\alpha_* \xrightarrow{\sim} R\alpha_* i^*.$$

Proof. a) The transformations on the left exist by the universal property of sheafification, and the transformation on the right are their adjoints. Thus it suffices to verify one isomorphism for each pair. Since the functors on the right hand side preserve injectives, it suffices to verify the isomorphisms of functors before deriving. The statements $\epsilon^* j^* \cong j^* \epsilon^*$ and $\epsilon_* i_* \cong i_* \epsilon_*$ are obvious. The other two statements follow because

$$(\epsilon^* i_* \mathcal{F})_{\bar{x}} \cong (i_* \epsilon^* \mathcal{F})_{\bar{x}} \cong \begin{cases} \mathcal{F}_{\bar{x}} & x \in Y \\ 0 & x \notin Y, \end{cases}$$

and similarly

$$(\epsilon^* j_! \mathcal{F})_{\bar{x}} \cong (j_! \epsilon^* \mathcal{F})_{\bar{x}} \cong \begin{cases} \mathcal{F}_{\bar{x}} & x \in U \\ 0 & x \notin U. \end{cases}$$

b) Let $(R, \mathfrak{m})$ be a henselian local ring. The map $\iota : R \to R/\mathfrak{m}$ induces an equivalence between the categories of finite étale R-algebras and the category of finite étale $R/\mathfrak{m}$-algebras [20, I.4.4]. Since any étale cover of R has a refinement by a finite étale cover [20, I.4.2 (c)], [20, III.3.3] implies that for every sheaf $\mathcal{F}$, the restriction map $H^s_{\text{ét}}(R, \mathcal{F}) \xrightarrow{\sim} H^s_{\text{ét}}(R/\mathfrak{m}, \iota^*\mathcal{F})$ is an isomorphism. The same applies to R replaced by R/I for $I \subseteq \mathfrak{m}$ an ideal of R, so we get $H^s_{\text{ét}}(R, \mathcal{F}) \cong H^s_{\text{ét}}(R/I, i^*\mathcal{F})$. By the usual spectral sequence argument, this extends to complexes of sheaves if the étale cohomological dimension of Spec $R/\mathfrak{m}$ is finite.

Now let $y \in Y \subseteq X$, $\mathcal{F}^\cdot$ be a complex of étale sheaves on X, $R = \mathcal{O}^h_{X,y}$ be the henselian local ring of y in X and $R/I = \mathcal{O}^h_{Y,y}$ be the henselian local ring of y in Y. Then

$$\begin{aligned}(i^* R^s \alpha_* \mathcal{F}^\cdot)_y = H^s_{\text{Nis}}(R/I, i^* R\alpha_*\mathcal{F}^\cdot) \xleftarrow{\sim} H^s_{\text{Nis}}(R, R\alpha_*\mathcal{F}^\cdot) \cong H^s_{\text{ét}}(R, \mathcal{F}^\cdot)\\ \xrightarrow{\sim} H^s_{\text{ét}}(R/I, i^*\mathcal{F}^\cdot) \cong H^s_{\text{Nis}}(R/I, R\alpha_* i^*\mathcal{F}^\cdot) = (R^s\alpha_* i^*\mathcal{F}^\cdot)_y.\end{aligned}$$

□

Remark. The other functors do not commute in general. For example, if R is a local ring with closed subscheme $i : \mathrm{Spec} R/I \to \mathrm{Spec} R$, and if R^h_I is the henselization of R along I, then

$$\begin{aligned}H^s_{\text{Zar}}(R/I, i^* R\epsilon_*\mathcal{F}) &\cong H^s_{\text{Zar}}(R, R\epsilon_*\mathcal{F}) \cong H^s_{\text{ét}}(R, \mathcal{F})\\ H^s_{\text{Zar}}(R/I, R\epsilon_* i^*\mathcal{F}) &\cong H^s_{\text{ét}}(R/I, i^*\mathcal{F}) = H^s_{\text{ét}}(R^h_I, \mathcal{F}).\end{aligned}$$

The failure of the natural transformations $\epsilon^* Rj_* \to Rj_*\epsilon^*$ and $\epsilon^* Ri^! \to Ri^!\epsilon^*$ to be quasi-isomorphisms are closely related to the Beilinson-Lichtenbaum conjecture.

Lemma 2.3. *If $Z = Z_1 \coprod Z_2$ is a disjoint union of closed sets, then*

$$i_{Z_1*} Ri^!_{Z_1} \oplus i_{Z_2*} Ri^!_{Z_2} \xrightarrow{\sim} i_{Z*} Ri^!_Z$$

Proof. Let U_1 and U_2 be the complements of Z_1 and Z_2, respectively. Then for every sheaf $\mathcal{F}$ on $X_{\text{ét}}$ and every V étale over X, there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{F}(V) & \to & \mathcal{F}(V) \oplus \mathcal{F}(V) & \to & \mathcal{F}(V) \to 0 \\ & \| & & \downarrow & & \downarrow \\ 0 \to & \mathcal{F}(V) & \to & \mathcal{F}(U_1 \times_X V) \oplus \mathcal{F}(U_2 \times_X V) & \to & \mathcal{F}(U_1 \times_X U_2 \times_X V) \end{array}$$

By the snake lemma, the kernel of the two right vertical maps agree, and we get

$$i_{Z_1*} i^!_{Z_1} \oplus i_{Z_2*} i^!_{Z_2} \xrightarrow{\sim} i_{Z*} i^!_Z.$$

Since the functors involved preserve injectives, the lemma follows. □

Lemma 2.4. *Let X be essentially of finite type over B, $j : X_K \to X$ be the inclusion of the generic fiber, and let $C^\cdot$ be a complex of sheaves on X. Then there is an distinguished triangle*

$$\cdots \to \bigoplus_{b\in B^{(1)}} i_{b*}Ri_b^! C^\cdot \to C^\cdot \to Rj_*j^*C^\cdot \to \cdots. \tag{3}$$

In particular, there is a long exact sequence

$$\cdots \to \bigoplus_{b\in B^{(1)}} H^i_{X_b}(X, C^\cdot) \to H^i(X, C^\cdot) \to H^i(X_K, C^\cdot) \to \cdots.$$

Proof. Let $j_U : X_U \to X$ be the embedding of the fiber of an open subset over U of B. By the previous lemma, the decomposition triangle gives a distinguished triangle

$$\cdots \to \bigoplus_{b\notin U} i_{b*}Ri_b^! C^\cdot \to C^\cdot \to Rj_{U*}j_U^*C^\cdot \to \cdots.$$

Now we take the colimit over decreasing open sets U. Since our sites are noetherian, the presheaf direct limit of a pseudo-filtered direct system of sheaves is a sheaf [20, III 3.6(b)]. So for every sheaf $\mathcal{F}$ on X and every étale V over X,

$$j_*j^*\mathcal{F}(V) \xrightarrow{\sim} \operatorname*{colim}_U(j_{U*}j_U^*\mathcal{F}(V)) \xrightarrow{\sim} (\operatorname*{colim}_U j_{U*}j_U^*\mathcal{F})(V).$$

Exactness of the colimit implies that

$$\operatorname*{colim}_U Rj_{U*}j_U^*C^\cdot \cong R(\operatorname*{colim}_U j_{U*}j_U^*)C^\cdot \cong Rj_*j^*C^\cdot.$$

The exact sequence of cohomology groups follows, because on a noetherian site cohomology commutes with filtered colimits of sheaves [20, III 3.6(d)]. □

3. Motivic Cohomology

We define motivic cohomology as the hyper-cohomology of Bloch's cycle complex. Let $D_i = \mathbb{Z}[t_0, \ldots, t_i]/(\sum_i t_i - 1)$, and $\Delta_i = \operatorname{Spec} D_i$ be the algebraic i-simplex. For an equidimensional scheme X, let $z^n(X, i)$ be the free abelian group on closed integral subschemes of codimension n of $X \times \Delta^i$, which intersect all faces properly. Intersecting with faces defines the structure of a simplicial abelian group, and hence gives a (homological) complex $z^n(X, *)$. The complex $z^n(X, *)$ is covariant for proper morphisms (with the appropriate shift in weight and degree) and contravariant for flat morphisms [1, 10].

Lemma 3.1. *The presheaves $z^n(-, i)$ are sheaves for the étale topology on X.*

Proof. The presheaf $z^n(-, i)$ is clearly a sheaf for the Zariski topology, hence by [20, II.1.5] we only need to treat the case of a covering of the form $V = \operatorname{Spec} B \to U = \operatorname{Spec} A$. We have to show that the sequence

$$0 \to z^n(\operatorname{Spec} A, i) \to z^n(\operatorname{Spec} B, i) \to z^n(\operatorname{Spec} B \otimes_A B, i)$$

is exact. Since $V \to U$ is a covering, exactness on the left is obvious, and we only have to show exactness in the middle.

If $A \to B$ is flat or unramified, then so is $A \otimes D_i \to B \otimes D_i$, hence the pull-back preserves the property of intersecting faces properly, and we can ignore this condition. By flatness, the minimal primes $\mathcal{P}$ of $B \otimes D_i$ above a prime $\mathfrak{p}$ of $A \otimes D_i$ determine $\mathfrak{p}$, and we can localize $A \otimes D_i$ and $B \otimes D_i$ at a prime $\mathfrak{p} \subseteq A \otimes D_i$ of height n. We can also divide by $\mathfrak{p}$, and show exactness of $z^n(\mathrm{Spec}\, k) \xrightarrow{s} z^n(\mathrm{Spec} B) \xrightarrow{f,g} z^n(\mathrm{Spec} B \otimes_k B)$ for k a field and B a semi-local, Artinian étale k-algebra.

Since B is unramified over k, it is a product $\prod_i k_i$ of fields. Similarly, $k_i \otimes_k k_j$ is the product $\prod_l k_{ij}^l$ of fields. Let p, p_i and p_{ij}^l be the unique prime ideal of k, k_i and k_{ij}^l, respectively. Then $s(p) = \sum_i p_i$, the map $f : B \otimes_k k \xrightarrow{\mathrm{id} \otimes s} B \otimes_k B$ sends p_i to $\sum_{j,l} p_{ij}^l$ and the map $g : k \otimes_k B \xrightarrow{s \otimes \mathrm{id}} B \otimes_k B$ sends p_i to $\sum_{j,l} p_{ji}^l$. If $f(\sum_i c_i p_i) = g(\sum_i c_i p_i)$, then looking at the coefficient of p_{ij}^l, we get $c_i = c_j$, hence $\sum_i c_i p_i = c_1 \sum_i p_i = c_1 s(p)$. □

Remark. One can show with a more careful analysis that $z^n(-, i)$ is a sheaf for any Grothendieck topology such that all coverings are flat and have reduced fibers. On the other hand, if $s : V \to U$ is a covering such that the pull-back $s^*[u]$ of a closed irreducible subscheme u is of the form $d \cdot \sum_i [v_i]$ for $d > 1$, then $\sum_i [v_i]$ is in the equalizer of the two maps induced from the projections $V \times_U V \to V$, but not in the image of s^*.

The complex of sheaves $\mathbb{Z}(n)_t$ on the site X_t, where $t \in \{\mathrm{et}, \mathrm{Nis}, \mathrm{Zar}\}$, is defined as the cohomological complex with $z^n(-, 2n - i)$ in degree i. For an abelian group A we define $A(n)$ to be $\mathbb{Z}(n) \otimes A$. Since $\mathbb{Z}(n)$ is a complex of sheaves for the étale topology,

$$\epsilon^* \mathbb{Z}(n)_{\mathrm{Zar}} = \mathbb{Z}(n)_{\text{ét}}, \quad \epsilon_* \mathbb{Z}(n)_{\text{ét}} = \mathbb{Z}(n)_{\mathrm{Zar}}, \tag{4}$$

and similarly for $\alpha^* \vdash \alpha_*$ and $\beta^* \vdash \beta_*$. Consequently,

$$\epsilon^* i^! \mathbb{Z}(n)_{\mathrm{Zar}} \cong i^! \epsilon^* \mathbb{Z}(n)_{\mathrm{Zar}} \qquad \epsilon^* j_* \mathbb{Z}(n)_{\mathrm{Zar}} \cong j_* \epsilon^* \mathbb{Z}(n)_{\mathrm{Zar}}.$$

We define $H_t^i(X, \mathbb{Z}(n))$ as the hyper-cohomology of $\mathbb{Z}(n)_t$ and omit t for the Zariski topology. A flat map $f : X \to Y$ induces via the pull back of cycles a map of sheaves $f^* \mathbb{Z}(n)^Y \to \mathbb{Z}(n)^X$, hence by adjointness

$$H_t^i(Y, \mathbb{Z}(n)) \to H_t^i(Y, Rf_* \mathbb{Z}(n)) \cong H_t^i(X, \mathbb{Z}(n)).$$

Theorem 3.2. *(Levine) Let X be essentially of finite type over the spectrum of a discrete valuation ring.*

a) If $i : Z \to X$ is a closed subscheme of codimension c with open complement $j : U \to X$, then the exact sequence

$$0 \to z^{q-c}(Z, *) \xrightarrow{i_*} z^q(X, *) \xrightarrow{j^*} z^q(U, *) \tag{5}$$

forms a distinguished triangle in the derived category of abelian groups.

b) Motivic cohomology $H^s(X, \mathbb{Z}(n))$ agrees with the cohomology of the complex of abelian groups $H_{2n-s}(z^n(X, *))$.

Proof. The first statement is [16, Theorem 1.7]. For (b), we employ the criterion of Brown-Gersten [4] [5, Theorem 7.5.1]. Let $\mathbb{Z}(n) \xrightarrow{\eta} I^*$ be an injective resolution (for the Zariski topology). Then η is an isomorphism on stalks. On the other hand, localization implies that the cohomology of $\mathbb{Z}(n)$ satisfies the Mayer-Vietoris property, whereas hypercohomology, i.e. the cohomology of I^*, automatically has this property. Hence η induces a quasi-isomorphism on global sections. □

We now return to our assumption that B is the spectrum of a Dedekind ring.

Corollary 3.3. *Let $X \xrightarrow{p} B$ be essentially of finite type.*

a) With the notation of the theorem, there is a distinguished triangle in the derived category of Zariski sheaves on X,

$$0 \to i_*\mathbb{Z}(n-c)^Z[-2c] \to \mathbb{Z}(n)^X \to j_*\mathbb{Z}(n)^U.$$

b) There is an isomorphism of hyper-cohomology groups

$$H^s_{Zar}(B, p_*\mathbb{Z}(n)) \cong H^s(X, \mathbb{Z}(n)) = H^s_{Zar}(B, Rp_*\mathbb{Z}(n)).$$

Proof. a) We verify the exactness stalkwise. Since the cycle complex commutes with direct limits of rings, it suffices to verify the statement for the local ring at a point $x \in X$. If x is in the generic fiber, then $\mathcal{O}_{X,x}$ is essentially of finite type over a field, the situation considered by Bloch [1]. If x is in a closed fiber, then $\mathcal{O}_{X,x}$ satisfies the hypothesis of the Theorem 3.2.

b) To see that the natural map $p_*\mathbb{Z}(n) \to Rp_*\mathbb{Z}(n)$ is a quasi-isomorphism, consider the stalk at $b \in B$. Then by Theorem 3.2,

$$\mathcal{H}^s(p_*\mathbb{Z}(n))_b = H_{2n-s}(z^n(\tilde{X}_b, *)) \cong H^s_{\mathrm{Zar}}(\tilde{X}_b, \mathbb{Z}(n)) = R^s p_*\mathbb{Z}(n)_b.$$

□

Remark. Similarly, we get a quasi-isomorphism $(p\circ j)_*\mathbb{Z}(n)^U \cong R(p\circ j)_*\mathbb{Z}(n)^U$, and after applying p^*, a quasi-isomorphism

$$j_*\mathbb{Z}(n)^U_{\mathrm{Zar}} \xrightarrow{\sim} Rj_*\mathbb{Z}(n)^U_{\mathrm{Zar}}.$$

Consequently,

$$\mathbb{Z}(n-c)^Z_{\mathrm{Zar}}[-2c] \xrightarrow{\sim} i^!\mathbb{Z}(n)^X_{\mathrm{Zar}} \xrightarrow{\sim} Ri^!\mathbb{Z}(n)^X_{\mathrm{Zar}}, \tag{6}$$

because all complexes are quasi-isomorphic to the cone of $i^*\mathbb{Z}(n)^X \to i^*j_*\mathbb{Z}(n)^U$. In particular, we can identify the canonical maps

$$\begin{array}{ccc} \epsilon^* Rj_*\mathbb{Z}(n)_{\mathrm{Zar}} & \longrightarrow & Rj_*\epsilon^*\mathbb{Z}(n)_{\mathrm{Zar}} \\ \| & & \| \\ j_*\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & Rj_*\mathbb{Z}(n)_{\text{ét}} \end{array} \qquad \begin{array}{ccc} \epsilon^* Ri^!\mathbb{Z}(n)_{\mathrm{Zar}} & \longrightarrow & Ri^!\epsilon^*\mathbb{Z}(n)_{\mathrm{Zar}} \\ \| & & \| \\ i^!\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & Ri^!\mathbb{Z}(n)_{\text{ét}}. \end{array}$$

Corollary 3.4. *Let X be essentially of finite type over B. Then there is a distinguished triangle*

$$\cdots \to \bigoplus_{b\in B^{(1)}} i_{b*}\mathbb{Z}(n-1)_{\mathrm{Zar}} \to \mathbb{Z}(n)_{\mathrm{Zar}} \to Rj_*\mathbb{Z}(n)_{\mathrm{Zar}} \to \cdots. \tag{7}$$

Proof. This is (6) applied to Lemma 2.4. □

Corollary 3.5. *Let X be essentially of finite type over B and $\mathbb{A}^1_X \xrightarrow{f} X$ be the projection map. Then the pull back map*

$$H^s(X,\mathbb{Z}(n)) \xrightarrow{f^*} H^s(\mathbb{A}^1_X,\mathbb{Z}(n))$$

is an isomorphism.

Proof. Because Rf_* commutes with direct sums on a noetherian site [20, III 3.6(d)], the localization sequence (7) reduces the statement $\mathbb{Z}(n)^X \cong Rf_*\mathbb{Z}(n)^{\mathbb{A}^1_X}$ to the case where the base is a field, which was treated by Bloch [1]. □

Since proper maps are stable under base-change, a proper map $f : X \to Y$ of relative dimension c induces a map $f_*\mathbb{Z}(n)^X_t \to \mathbb{Z}(n-c)^Y_t[-2c]$. Composing with the push-forward along the structure morphism $q : Y \to B$ we get a map of complexes on B, $p_*\mathbb{Z}(n)^X_t \to q_*\mathbb{Z}(n-c)^Y_t[-2c]$, hence in view of Corollary 3.3 covariant functoriality

$$H^i(X,\mathbb{Z}(n)) \to H^{i-2c}(Y,\mathbb{Z}(n-c)).$$

For the étale topology we get covariant functoriality if $f_*\mathbb{Z}(n)_{\text{ét}} \cong Rf_*\mathbb{Z}(n)_{\text{ét}}$, for example if f_* is exact:

$$H^i_{\text{ét}}(X,\mathbb{Z}(n)) \xrightarrow{\sim} H^i_{\text{ét}}(Y,Rf_*\mathbb{Z}(n)) \xleftarrow{\sim} H^i_{\text{ét}}(Y,f_*\mathbb{Z}(n)) \to H^{i-2c}_{\text{ét}}(Y,\mathbb{Z}(n-c)).$$

We get push-forwards for the Nisnevich topology by the following

Proposition 3.6. *If X is essentially of finite type over B, then the canonical maps are quasi-isomorphism*

$$\mathbb{Z}(n)_{\mathrm{Zar}} \xrightarrow{\sim} R\beta_*\mathbb{Z}(n)_{\mathrm{Nis}},$$
$$\mathbb{Q}(n)_{\mathrm{Zar}} \xrightarrow{\sim} R\epsilon_*\mathbb{Q}(n)_{\mathrm{et}}.$$

Proof. Since $\mathbb{Z}(n)$ satisfies the localization property, it has étale excission. Thus the argument of Nisnevich [22], see also [5, Thm. 7.5.2], shows that $\mathbb{Z}(n)$ agrees with its Nisnevich hypercohomology.

For the second statement we only need to show that $\mathbb{Q}(n)_{\mathrm{Nis}} \cong R\alpha_*\mathbb{Q}(n)_{\text{ét}}$. But for any étale sheaf $\mathcal{F}$, the stalk of $R^i\alpha_*\mathcal{F}$ at x is $H^s_{\text{ét}}(\mathcal{O}^h_{X,x},\mathcal{F}) \cong H^s_{\text{ét}}(k(x), i^*\mathcal{F})$, and this is zero for $s > 0$ and $\mathcal{F}$ a $\mathbb{Q}$-vector space. Hence $\mathbb{Q}(n)_{\mathrm{Nis}} = \alpha_*\mathbb{Q}(n)_{\text{ét}} \cong R\alpha_*\mathbb{Q}(n)_{\text{ét}}$ by (4). □

4. The Gersten resolution

We use the methods of Bloch [1,2] and Gillet-Levine [11] to prove Theorem 1.1. Let $X \stackrel{p}{\to} B$ be the local ring at a point x of an essentially smooth scheme over B and $b = p(x)$. Consider the descending filtration by coniveau on cycles. Let

$$F^s z^n(X,*) = \operatorname*{colim}_{\substack{Z \subseteq X \\ \operatorname{codim}_X Z \geq s}} \operatorname{im}\Big(z^{n-s}(Z,*) \to z^n(X,*)\Big),$$

be the subcomplex of $z^n(X,*)$ generated by cycles whose projection to X has codimension at least s. The spectral sequence corresponding to this filtration is

$$E_1^{s,t} = H_{-s-t}(\operatorname{gr}^s z^n(X,*)) \Rightarrow H_{-s-t}(z^n(X,*)) = H^{2n+s+t}(X,\mathbb{Z}(n)).$$

Taking the colimit of the localization sequence (5) over pairs $Y \subseteq Z$ with Z of codimension s and Y of codimension $s+1$, we get isomorphisms

$$\operatorname{gr}^s z^n(X,*) \stackrel{\sim}{\to} \operatorname*{colim}_{(Z,Y)} z^{n-s}(Z-Y,*) \underset{\sim}{\stackrel{res}{\to}} \bigoplus_{x \in X^{(s)}} z^{n-s}(k(x),*).$$

Here the right hand map sends a cycle on a subscheme to its restriction to the generic points. Consequently,

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} H_{-s-t}(z^{n-s}(k(x),*)) = \bigoplus_{x \in X^{(s)}} H^{2n-s+t}(k(x),\mathbb{Z}(n-s)).$$

By construction of the spectral sequence, the E_1-terms form the column of the following diagram

$$\begin{array}{ccccc}
H_m(z^n(X,*)) & \longrightarrow & H_m(\operatorname{gr}^0 z^n(X,*)) & \longrightarrow & H_{m-1}(F^1 z^n(X,*)) \\
 & & \downarrow & & \\
H_{m-1}(F^1 z^n(X,*)) & \longrightarrow & H_{m-1}(\operatorname{gr}^1 z^n(X,*)) & \longrightarrow & H_{m-2}(F^2 z^n(X,*)) \\
 & & \downarrow & & \\
H_{m-2}(F^2 z^n(X,*)) & \longrightarrow & H_{m-2}(\operatorname{gr}^2 z^n(X,*)) & \longrightarrow & H_{m-3}(F^3 z^n(X,*)) \\
 & & \downarrow & &
\end{array} \tag{8}$$

We state the following easy Lemma for future reference:

Lemma 4.1. *Let X be a scheme of finite type over B, Z a closed irreducible subscheme of codimension* 1 *and U the complement of X. Let V be a closed irreducible subscheme of X not contained in Z, and W an arbitrary subscheme. If V_Z meets W_Z properly and V_U meets W_U properly, then V meets W properly.*

Proof. Let C be an irreducible component of the intersection. If C is not contained in Z, the claim follows from the proper intersection on U. If $C \subseteq Z$, then by hypothesis

$$\begin{aligned}\dim C \leq \dim V_Z + \dim W_Z - \dim Z \leq \dim V - 1 + \dim W_Z - (\dim X - 1)\\ \leq \dim V + \dim W - \dim X.\end{aligned}$$

□

The heart of the proof is the following theorem.

Theorem 4.2. *Let c be a cycle in $F^{s+1}z^n(X, *)$ which is in the image of $F^s z^{n-1}(Y, *)$, for Y a principal effective divisor, flat over the discrete valuation ring $\Lambda = B_b$. Then the image of c under the map $H_m(F^{s+1}z^n(X, *)) \to H_m(F^s z^n(X, *))$ is zero.*

Proof. Since X is the localization of $X' = \operatorname{Spec} S$ in x, for S a finitely generated algebra over Λ, we can (after decreasing X') assume that c is the restriction of an element in $F^{s+1}z^n(X', *)$ and comes from $Y' = \operatorname{Spec} S/tS$, where t is a non zero-divisor of S.

By [11, Lemma 1, Section 2], after replacing X' by a smaller affine neighborhood of x, we can find a morphism $\pi : X' \to \mathbb{A}^{d-1}_\Lambda$, such that π is smooth at x of relative dimension 1, and $\pi|_{Y'}$ is quasi-finite. Setting $T = S/t \otimes_{\mathbb{A}^{d-1}_\Lambda} S$ and $Z = \operatorname{Spec} T$, we get the cartesian square

$$\begin{array}{ccc} Z = \operatorname{Spec} T & \xrightarrow{p} & X' = \operatorname{Spec} S \\ {\scriptstyle q}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y' = \operatorname{Spec} S/t & \longrightarrow & \mathbb{A}^{d-1}_\Lambda. \end{array}$$

The inclusion $Y' \xrightarrow{j} X'$ induces a section $s : Y' \to Z$ of q, which is a closed immersion. Its image $s(Y')$ is finite over X'. Furthermore q is smooth at the points $p^{-1}(x)$, and p is quasi-finite. By Zariski's main theorem (EGA IV, 18.12.13) there is an affine scheme $\bar{Z} = \operatorname{Spec} \bar{T}$ such that p factors as $Z \xrightarrow{u} \bar{Z} \xrightarrow{\bar{p}} X'$ with u an open embedding and $\bar{p}$ finite. Let $\bar{s} = u \circ s : Y' \to \bar{Z}$. Since $j = \bar{p} \circ \bar{s}$ is closed, $\bar{s}$ is closed and $D = \bar{Z} - Z$ is disjoint from $\bar{s}(Y')$. Because $\bar{p}$ is proper, we can factor $j_* = \bar{p}_* \bar{s}_*$, and it suffices to show that $\bar{s}_* = 0$.

Since Z is smooth over Y' of relative dimension 1 at the finite set of points of $p^{-1}(x)$, the ideal $I \subset T$ defining $s(Y')$ is principal near these points (SGA 1, II 4.15), say $I = (g')$. Let $\bar{I} \subset \bar{T}$ be the ideal defining $\bar{s}(Y)$, and $J \subset \bar{T}$ the ideal defining D. Because $\bar{s}(Y') \cap D = \emptyset$, we can use the Chinese remainder theorem to find $g \in \bar{T}$ which maps to $(1, g') \in \bar{T}/J \oplus \bar{T}/\bar{I}^2$, where g' is viewed as an element of $\bar{T}/\bar{I}^2$ via the isomorphism $\bar{T}/\bar{I}^2 \xrightarrow{\sim} T/I^2$. Then $g = 1$ on D and g defines $\bar{s}(Y)$ in a neighborhood of $\bar{p}^{-1}(x)$ because $g \equiv g' \bmod I^2$.

Let C be the closed set of $\bar{Z}$ consisting of all irreducible components of the divisor $g = 0$ which are not components of $\bar{s}(Y')$. Since g defines $s(Y')$ near $p^{-1}(x)$

and is 1 on $\bar{p}^{-1}(x) - p^{-1}(x)$, C does not contain any point of $\bar{p}^{-1}(x)$. Hence its image $\bar{p}(C) \in X'$ is a closed subset not containing x. By removing $\bar{p}(C)$ from X', we can assume that the divisor $g = 0$ is $\bar{s}(Y')$.

Let $E \in z^{n-1}(Y', t)$, $V = q^*E \in z^{n-1}(Z, t)$ and $\bar{V}$ be the closure of V in $\bar{Z} \times \Delta^t$. Consider the closed embeddings

$$\gamma = (\mathrm{id}, \tfrac{g}{g-1}) : Z \to Z \times \mathbb{P}^1,$$
$$\bar{\gamma} = (\mathrm{id}, \tfrac{g}{g-1}) : \bar{Z} \to \bar{Z} \times \mathbb{P}^1,$$

and let $\gamma_* : z^{n-1}(Z, t) \to z^n(Z \times \mathbb{P}^1, t) \xrightarrow{res} z^n(Z \times \mathbb{A}^1, t)$ be the induced map on cycles. Since $\frac{g}{g-1}$ has a pole on D, $M = \gamma_*(V)$ agrees with the restriction of $\bar{\gamma}_*(\bar{V})$ to $\bar{Z} \times \mathbb{A}^1 \times \Delta^t$, hence is closed in $\bar{Z} \times \mathbb{A}^1 \times \Delta^t$, and can be viewed as an element of $z^n(\bar{Z} \times \mathbb{A}^1, t)$.

Since g has no poles, the image of $\bar{\gamma}_*$ does not meet $\bar{Z} \times \{1\} \times \Delta^t$, hence $M \cap (\bar{Z} \times \{1\} \times \Delta^t) = \emptyset$. Consider the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{s}} & \bar{Z} \\ {\scriptstyle \bar{s}}\downarrow & & \downarrow{\scriptstyle \bar{\gamma}} \\ \bar{Z} & \xrightarrow{i_0} & \bar{Z} \times \mathbb{P}^1. \end{array}$$

Because of $\bar{V} \cap \bar{s}(Y') = q^*E \cap s(Y')$, $\bar{V} \subset \bar{Z}$ meets Y' properly, and the base change for algebraic cycles gives $i_0^* \bar{\gamma}_* \bar{V} = \bar{s}_* \bar{s}^* \bar{V}$. As the divisor $\bar{s}(Y')$ defined by $g = 0$ is disjoint from D, we get $i_0^* M = \bar{s}_* \bar{s}^*(\bar{V}) = \bar{s}_* s^*(V) = \bar{s}_* s^* q^*(E) = \bar{s}_*(E)$.

We show that for some closed subscheme $i_a : \mathrm{Spec}\Lambda' \to \mathbb{A}^1_\Lambda$ with Λ' finite and flat over Λ, there exists a homotopy between $i_{0'}^*(M)$ and $i_a^*(M)$ as well as a homotopy between $i_{1'}^*(M)$ and $i_a^*(M)$. Here $0'$ and $1'$ are the zero- and one-section $\mathrm{Spec}\Lambda' \to \mathrm{Spec}\Lambda \xrightarrow{0,1} \mathbb{A}^1_\Lambda$, respectively. The strategy is to construct a homotopy by triangulating $\mathbb{A}^1 \times \Delta^t$ in the standard way, and we have to show that each face of the triangulation meets M in the correct codimension. Let W be the support of M.

Let $\theta_j : \Delta^{t+1} \to \mathbb{A}^1 \times \Delta^t$ be the unique linear map which sends the vertices $v_0, \ldots, v_{t+1}$ to the vertices $(0, v_0), \ldots, (0, v_j), (1, v_{j+1}), \ldots, (1, v_{t+1})$. A face F of $\mathbb{A}^1 \times \Delta^t$ is the image under some map θ_j of a face of Δ^{t+1}, in particular it is the linear span of a collection of points $(0, v_{i_1}), \ldots, (0, v_{i_s}), (1, v_{i_{s+1}}), \ldots, (1, v_{i_r})$ with $i_l \le i_{l+1}$. Let F' be the image of F under the projection $\mathbb{A}^1 \times \Delta^t \to \Delta^t$.

First assume that $\dim F = \dim F' + 1$, which happens if and only if F contains both $(0, v_l)$ and $(1, v_l)$ for some l. Then $F = \mathbb{A}^1 \times F'$, we have $W \cap (\bar{Z} \times F) = \gamma(\mathrm{Supp} V \cap (\bar{Z} \times F'))$, and $\mathrm{Supp}\, V \cap (\bar{Z} \times F')$ has the correct dimension because V meets faces properly.

If $\dim F = \dim F'$, the projection $\mathbb{A}^1 \times \Delta^t \to \Delta^t$ induces an isomorphism $F \xrightarrow{\sim} F'$. If $W \cap (\bar{Z} \times \mathbb{A}^1 \times F')$ meets F properly, then so does W:

$$\begin{aligned} \dim W \cap (\bar{Z} \times F) &= \dim (W \cap (\bar{Z} \times \mathbb{A}^1 \times F')) \cap F \\ &= \dim W \cap (\bar{Z} \times \mathbb{A}^1 \times F') - 1 = \dim V \cap (\bar{Z} \times F') - 1 \\ &= \dim W + \dim \bar{Z} \times F - \dim \bar{Z} \times \Delta^t - 1, \end{aligned}$$

the last equation holds because V meets $\bar{Z} \times F'$ properly. Hence replacing W by $W \cap (\bar{Z} \times \mathbb{A}^1 \times F')$, and possibly decreasing t, we can assume that $F' = \Delta^t$ and F is the image of one of the maps θ_j.

If $F = i_0(F')$, i.e. F contains no point $(1, v_l)$, then F meets W properly because $M \cap (\bar{Z} \times F) = i_0^*(M) \cap (\bar{Z} \times F') = \bar{s}_*(E) \cap (\bar{Z} \times F')$, and $\bar{s}_*(E)$ meets faces properly. Similarly, M meets $\bar{Z} \times (F \cap (\{0\} \times \Delta^t))$ properly because $M \cap (\bar{Z} \times (F \cap (\{0\} \times \Delta^t))) = \bar{s}_*(E) \cap (\bar{Z} \times F')$, hence by Lemma 4.1 we can consider our intersection problem on the complement $U = (\mathbb{A}^1 - \{0\}) \times \Delta^t$ of the zero-section. Let $\mu : \mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$ be the multiplication map $(a, x) \mapsto ax$. Consider the following cartesian diagram of integral schemes, where $\bar{\mu}$ is the restriction of $\mathrm{id}_{\bar{Z}} \times \mu \times \mathrm{id}_{\Delta^t}$

$$\begin{array}{ccccc} W' & \xrightarrow{q} & \mathbb{G}_m \times \bar{Z} \times (F \cap U) & \xrightarrow{p} & \mathbb{G}_m \\ {\scriptstyle \mu'}\downarrow & & {\scriptstyle \bar{\mu}}\downarrow & & \\ W \cap (\bar{Z} \times U) & \longrightarrow & \bar{Z} \times U. & & \end{array} \tag{9}$$

Because we removed the zero section, $\bar{\mu}$, hence μ' is flat, and a dimension count shows that it is of relative dimension 0. Let k be the residue field and L the quotient field of Λ. By [14, Lemma 1(i)] we can find dense open sets $A_k \subseteq \mathbb{G}_{m,k}$ and $A_L \subseteq \mathbb{G}_{m,L}$, such that for $a \in A_k$ or $a \in A_L$ the fiber $p^{-1}(a)$ meets $(W \cap (\bar{Z} \times U))_k$ or $(W \cap (\bar{Z} \times U))_L$ in the correct dimension, respectively. Let C be the closure of $\mathbb{G}_{m,L} - A_L$ in $\mathbb{G}_m$, and $A(F) = \mathbb{G}_m - (\mathbb{G}_{m,k} - A_k) - C$, a dense open set of $\mathbb{A}^1_\Lambda$ meeting the closed fiber. Then for every closed subscheme $a : \mathrm{Spec}\Lambda' \to A(F)$ such that Λ' is finite and flat over Λ, $p^{-1}(a)$ meets $W \cap (\bar{Z} \times S)$ properly. Indeed, the former is flat over Λ and we can calculate dimensions fiberwise by Lemma 4.1. Let $A = \bigcap_F A(F)$, where F runs through the finitely many faces involved.

For every closed subscheme $a : \mathrm{Spec}\Lambda' \to A \subseteq \mathbb{A}^1_\Lambda$, with Λ' finite and flat over Λ, let $\theta_j^a : \Delta^{t+1}_{\Lambda'} \to \mathbb{A}^1_{\Lambda'} \times_{\Lambda'} \Delta^t_{\Lambda'}$ be the unique linear map which sends the vertices $v_0, \ldots, v_{t+1}$ to the vertices $(0', v_0), \ldots, (0', v_j), (a, v_{j+1}), \ldots, (a, v_{t+1})$. Note that in (9), $\bar{\mu}$ embeds $p^{-1}(a)$ into $\bar{Z} \times U$ as the product of $\bar{Z}$ with the image of θ_j^a intersected with U.

Let $T_t^a = \sum_{j=0}^{t+1} (-1)^j (\theta_j^a)^*$ be the alternating sum of the pull-back maps along θ_j^a, giving the map $T_t^a : z^n(\bar{Z} \times \mathbb{A}^1_{\Lambda'}, t)^+ \to z^n(\bar{Z}_{\Lambda'}, t+1)$, where the $+$ indicates the subgroup generated by cycles whose image meets all faces properly. One easily verifies that $\partial T_t^a + T_{t-1}^a \partial = i_{0'}^* - i_a^*$. By definition of A, we have $M \in z^n(\bar{Z} \times \mathbb{A}^1_{\Lambda'}, t)^+$, hence $\partial T_t^a(M) = i_{0'}^*(M) - i_a^*(M)$. By the same argument, we can construct a dense open set B, such every closed subscheme $b : \mathrm{Spec}\Lambda' \to B$, finite and flat over Λ, gives a homotopy between $\emptyset = i_{1'}^*(M)$ and $i_b^*(M)$.

If the residue field k is infinite, we can always find a section $a : \Lambda \to A \cap B$ and the proof is finished. If k is finite, we fix a prime $l \neq p$ and consider the set of monic polynomials $P(T)$ with coefficients in Λ with degree a power of l. Then $\Lambda' = \Lambda[T]/(P(T)) \xrightarrow{p} \Lambda$ is finite (of degree a power of l) and flat. The obvious map $\Lambda[T] \to \Lambda'$ gives the closed subscheme $a_P : \mathrm{Spec}\Lambda' \to \mathbb{A}^1_\Lambda = \mathrm{Spec}\Lambda[T]$. Since $A \cap B \subseteq \mathbb{A}^1_\Lambda$ is open and dense (and meets the closed fiber), all but finitely many such closed subschemes will be contained in $A \cap B$. If a is one of them, then we get a homotopy between $i_{0'}^*(M)$ and $i_a^*(M)$ as well as a homotopy between $i_{1'}^*(M)$ and $i_a^*(M)$. Since $i_{0'}$ and $i_{1'}$ are the compositions Spec $\Lambda' \xrightarrow{0,1} \mathbb{A}^1_{\Lambda'} \xrightarrow{p} \mathbb{A}^1_{\Lambda}$, and since $p_* p^*$ is multiplication by a power of l, we see that $i_{0'}^*(M) - i_{1'}^*(M)$ is l-power torsion. Since this holds for any prime $l \neq p$, the two cycles are in fact equivalent. □

We can now prove the following stronger version of Theorem 1.1:

Corollary 4.3. *Let $X \xrightarrow{p} B$ be the local ring at a point x of an essentially smooth scheme over B. Then the following sequence is exact except at the first two terms,*

$$0 \to H^t(X, \mathbb{Z}(n)) \to \bigoplus_{x \in X^{(0)}} H^t(k(x), \mathbb{Z}(n)) \to \bigoplus_{x \in X^{(1)}} H^{t-1}(k(x), \mathbb{Z}(n-1)) \to \cdots. \tag{10}$$

If V is the local ring of X at the generic point of the fiber X_b, then the exactness at the first and second term follows from the injectivity of $H^j(V, \mathbb{Z}(n)) \to H^j(k(X), \mathbb{Z}(n))$ for $j = t$ and $j = t+1$, respectively.

Proof. If $s > 0$, then any cycle in $F^{s+1} z^n(X, *)$ satisfies the hypothesis of Theorem 4.2. It follows that all horizontal sequences in (8), except the first, break up into short exact sequences. The kernel of the first vertical map in (8) is the image of $H_m(z^n(X, *)) \xrightarrow{r} H_m(z^n(k(X), *))$, and an easy diagram chase shows that the cohomology of the column in (8) at $H_m(gr^1 z^n(X, *))$ is the kernel of $H_{m-1}(z^n(X, *)) \to H_{m-1}(z^n(k(X), *))$. Removing flat divisors by Theorem 4.2, we see that $H_j(z^n(X, *)) \to H_j(z^n(V, *))$ is injective, hence the kernel of r is contained in the kernel of $H_m(z^n(V, *)) \to H_m(z^n(k(X), *))$, and the cohomology at $H_m(gr^1 z^n(X, *))$ is contained in the kernel of $H_{m-1}(z^n(V, *)) \to H_{m-1}(z^n(k(X), *))$. □

Corollary 4.4. *Let X be an essentially smooth scheme over B. Then $\mathcal{H}^i(\mathbb{Z}(n)) = 0$ for $i > n$.*

Proof. We can assume that X is local. In the notation of Corollary 4.3, it suffices to show that $H^i(V, \mathbb{Z}(n)) = 0$ for $i > n$. If k is the residue field of V, the localization sequence

$$\cdots \to H^t(V, \mathbb{Z}(n)) \to H^t(k(X), \mathbb{Z}(n)) \xrightarrow{\delta} H^{t-1}(k, \mathbb{Z}(n-1)) \to \cdots,$$

implies that $H^t(V, \mathbb{Z}(n)) = 0$ for $t > n+1$. To show that $H^{n+1}(V, \mathbb{Z}(n)) = 0$, observe that by [8, Lemma 3.2], the map δ agrees with the symbol map $K_n^M(k(X)) \to K_{n-1}^M(k)$. The latter map is surjective, because we can lift the symbol $\{x_1, \ldots, x_{n-1}\}$ of $K_{n-1}^M(k)$ to the symbol $\{x_1, \ldots, x_{n-1}, \pi\} \in K_n^M(k(X))$. □

Corollary 4.5. *Let X be a smooth scheme over a discrete valuation ring with residue characteristic p. Then there is an exact sequence*

$$\begin{aligned} 0 \to \mathcal{H}^t(\mathbb{Z}/p^r(n)_{Zar}) &\to \bigoplus_{x \in X^{(0)}} (i_x)_* H^t(k(x), \mathbb{Z}/p^r(n)) \\ &\to \bigoplus_{x \in X^{(1)}} (i_x)_* H^{t-1}(k(x), \mathbb{Z}/p^r(n-1)) \to \cdots. \end{aligned}$$

Proof. As above, it suffices to show that $H^j(k(X), \mathbb{Z}/p^r(n)) \xrightarrow{\delta} H^{j-1}(k, \mathbb{Z}/p^r(n-1))$ is surjective for $j = t-1, t$. By the main theorem of [8] the right group is trivial for $j \neq n$, and in case $j = n$ we argue as above. □

5. Etale motivic cohomology

In this section we consider étale motivic cohomology. We have examined rational motivic cohomology in Corollary 3.6, so we focus on étale motivic cohomology with finite coefficients. If X is essentially smooth over the field k and $m \in k^\times$, then there is a quasi-isomorphism of étale sheaves on X, [9, Thm. 1.5]

$$\mathbb{Z}/m(n)_{\text{ét}} \simeq \mu_m^{\otimes n}[0]. \tag{11}$$

On the other hand, if k has characteristic p, then [8, Thm. 8.5]

$$\mathbb{Z}/p^r(n)_{\text{Zar}} \simeq \nu_r^n[-n]. \tag{12}$$

Here $\nu_r^n = \nu_{p^r}(n) = W\Omega^n_{r,log}$ is the logarithmic de Rham-Witt sheaf of Bloch-Illusie-Milne [13, 18]. Etale sheafification gives the analog statement for the étale topology. In [8], the quasi-isomorphism (12) is only stated for perfect k. But because both sides of (12) are compatible with filtered colimits of rings, we can extend this by writing k as the generic point of a colimit of smooth schemes over a perfect field contained in k.

The Beilinson-Lichtenbaum conjecture states that for X essentially smooth over the field k and $m \in \mathbb{N}$ invertible in k, the canonical map $\mathbb{Z}/m(n)_{\text{Zar}} \to \tau_{\leq n} R\epsilon_* \mathbb{Z}/m(n)_{\text{ét}}$ is a quasi-isomorphism, and that $R^{n+1}\epsilon_*\mathbb{Z}(n)_{\text{ét}}$ vanishes. The rational analog is known by Corollary 3.6, and the mod p^r-analog, p the characteristic of k, is known by [8, Thm. 8.5, 8.6], hence the conjecture is equivalent to

$$\mathbb{Z}(n)_{\text{Zar}} \xrightarrow{\sim} \tau_{\leq n+1} R\epsilon_* \mathbb{Z}(n)_{\text{ét}}.$$

We prove Theorem 1.2 of the introduction, assuming the Bloch-Kato conjecture.

Proof. 1) Consider the embeddings $X_b \xrightarrow{\iota} \tilde{X}_b \xrightarrow{j} X$. It is easy to check at stalks that $\iota^! j^* = i^!$ and that j^* maps injective sheaves to sheaves which are acyclic for $\iota^!$. Hence we can localize the base at b. Applying $\epsilon^* i^*$ to (7) and comparing to (3) via the adjoint map, we get the map of distinguished triangles

$$\begin{array}{ccccc} \epsilon^*\mathbb{Z}(n-1)_{\text{Zar}}[-2] & \longrightarrow & \epsilon^* i^* \mathbb{Z}(n)_{\text{Zar}} & \longrightarrow & \epsilon^* i^* Rj_* \mathbb{Z}(n)_{\text{Zar}} \\ \downarrow & & \| & & \downarrow \\ Ri^!\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & i^*\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & i^* Rj_* \mathbb{Z}(n)_{\text{ét}}. \end{array} \tag{13}$$

The right hand map is quasi-isomorphism up to degree $n+1$, because the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture over a field [8,9], hence

$$\epsilon^* Rj_* \mathbb{Z}(n)_{\text{Zar}} \xrightarrow{\sim} \tau_{\leq n+1} \epsilon^* Rj_* R\epsilon_* \mathbb{Z}(n)_{\text{ét}} \xrightarrow{\sim} \tau_{\leq n+1} Rj_* \mathbb{Z}(n)_{\text{ét}}.$$

Thus the left map of (13) is a quasi-isomorphism up to degree $n+1$.

Rationally, truncation is not necessary by (6) and Corollary 3.6. If m is invertible on the fiber over b and $n > 0$, then the right two terms of the lower sequence of (13) are acyclic with $\mathbb{Z}/m$-coefficients above degree n in view of absolute cohomological purity [7]:

$$R^s j_*\mathbb{Z}/m(n)_{\text{ét}} \cong R^s j_*\mu_m^{\otimes n} \cong \begin{cases} \mu_m^{\otimes n} & s = 0 \\ \mu_m^{\otimes n-1} & s = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $R^s i^! \mathbb{Z}\left[\frac{1}{p}\right](n)_{\text{ét}}$ is uniquely divisible and torsion, hence zero for $s > \max\{n+1, 2\}$, whereas $\mathbb{Z}(n-1)_{\text{ét}}[-2]$ is acyclic above degree $n+1$ by Corollary 4.4.

2) The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture over a field [8,9]. In general, we can consider stalks and assume that the base is local. We compare the distinguished triangle (7) to $R\epsilon_*$ applied to the distinguished triangle (3) via the adjoint map:

$$\begin{array}{ccccc} i_*\mathbb{Z}(n-1)[-2]_{\text{Zar}} & \longrightarrow & \mathbb{Z}(n)_{\text{Zar}} & \longrightarrow & Rj_*\mathbb{Z}(n)_{\text{Zar}} \\ \downarrow f & & \downarrow g & & \downarrow h \\ R\epsilon_* i_* Ri^! \mathbb{Z}(n)_{\text{ét}} & \longrightarrow & R\epsilon_*\mathbb{Z}(n)_{\text{ét}} & \longrightarrow & R\epsilon_* Rj_*\mathbb{Z}(n)_{\text{ét}} \end{array}$$

The maps f and h are quasi-isomorphisms up to degree $n+1$, because the conjecture holds over fields and because by part (1),

$$i_*\mathbb{Z}(n-1)_{\text{Zar}}[-2] \xrightarrow{\sim} \tau_{\leq n+1} i_* R\epsilon_*\mathbb{Z}(n-1)_{\text{ét}}[-2] \xrightarrow{\sim} \tau_{\leq n+1} i_* R\epsilon_* Ri^!\mathbb{Z}(n)_{\text{ét}}.$$

On the other hand, f is injective in degree $n+2$ because the domain is 0. By the five-lemma, g is a quasi-isomorphism up to degree $n+1$.

3) Consider the commutative diagram

$$\begin{array}{ccc} H^i(R, \mathbb{Z}/m(n)) & \longrightarrow & H^i(k, \mathbb{Z}/m(n)) \\ \downarrow & & \downarrow \\ H^i_{\text{ét}}(R, \mu_m^{\otimes n}) & \xrightarrow{\sim} & H^i_{\text{ét}}(k, \mu_m^{\otimes n}) \end{array}$$

The lower horizontal arrow is an isomorphism by the argument of Proposition 2.2(b), and the vertical maps are isomorphisms for $i \leq n$ by part (2). On the other hand, the upper terms are both zero for $i > n$ by Corollary 4.4.

4) Compare the following two distinguished triangles of the form (3) via the cycle class map $\mathbb{Z}/m(n)_{\text{ét}} \xrightarrow{cl_X^n} \mu_m^{\otimes n}$ of [17, §2.3] or [9]:

$$\begin{array}{ccccc} \bigoplus_{b\in B^{(1)}} i_{b*}Ri_b^!\mathbb{Z}/m(n)_{\text{ét}} & \longrightarrow & \mathbb{Z}/m(n)_{\text{ét}} & \longrightarrow & Rj_*\mathbb{Z}/m(n)_{\text{ét}} \\ \downarrow & & \downarrow & & \| \\ \bigoplus_{b\in B^{(1)}} i_{b*}Ri_b^!\mu_m^{\otimes n} & \longrightarrow & \mu_m^{\otimes n} & \longrightarrow & Rj_*\mu_m^{\otimes n}. \end{array}$$

The right hand map is an isomorphism by (11). By part (1) and purity for $\mu_m^{\otimes n}$, we get $Ri_b^!\mathbb{Z}/m(n)_{\text{ét}} \cong \mathbb{Z}/m(n-1)_{\text{ét}}[-2] \cong \mu_m^{\otimes n-1}[-2] \cong Ri_b^!\mu_m^{\otimes n}$ for every closed fiber, hence the middle map is a quasi-isomorphism.

5) It suffices to check this for a local ring R of X, and for m a power of some prime number l. If R has residue characteristic l, then the statement was proved in Corollary 4.5. If l is invertible in R, then by Corollary 4.3, and parts (2) and (4), it suffices to show that the map $H^s_{\text{ét}}(R, \mu_m^{\otimes n}) \to H^s_{\text{ét}}(k(R), \mu_m^{\otimes n})$ is injective. This is [5, Theorem B.2.1]. □

Remark. Using the product with equidimensional cycles defined by Levine [17], one can show under the same hypothesis that $\mu_m^{\otimes n}$ is a split direct summand of $\mathbb{Z}/m(n)_{\text{ét}}$. This is because the cycle class map is split by the composition

$$\mu_m^{\otimes n} \underset{\sim}{\overset{cl_X^1}{\longleftarrow}} \mathbb{Z}/m(1)^{\otimes n} \xrightarrow{\cup} \mathbb{Z}/m(n).$$

The cup product is defined because we are multiplying with one-codimensional cycles, and the composition with the cycle class map is the identity because the cycle class map is multiplicative [9, Prop. 4.7].

We define the p-cohomological dimension $\mathrm{cd}_p X_{\text{ét}}$ of the site $X_{\text{ét}}$ to be the smallest integer n such that $H^i_{\text{ét}}(V, F) = 0$ for all $i > n$, any scheme V étale over X, and any p-torsion sheaf F.

Proposition 5.1. *Let X be essentially smooth over B with a closed fiber $i : X_b \to X$ of residue characteristic p and generic fiber U. Then the Bloch-Kato conjecture implies*

$$\mathbb{Z}(n-1)_{\acute{e}t}[-2] \xrightarrow{\sim} Ri^!\mathbb{Z}(n)_{\acute{e}t} \qquad \textit{for } n \geq cd_p U_{\acute{e}t}.$$

Proof. In view of Proposition 3.6 and Corollary 3.3, the sheaves $R^s i^!\mathbb{Z}(n)_{\text{ét}}$ are torsion for $s > n+1$. By Theorem 1.2(4), they are m-divisible for $p \not| m$ and $s > n+1$. By a diagram chase, it suffices to show that $R^s i^!\mathbb{Z}/p(n)_{\text{ét}} = 0$ for $s > n+1 \geq \mathrm{cd}_p U + 1$.

By étale excision, we can assume that B is local. As $\mathcal{H}^s(\mathbb{Z}/p(n)_{\text{ét}}) = 0$ for $s > n$, the long exact decomposition sequence gives isomorphisms $i^*R^{s-1}j_*\mu_p^{\otimes n} \xrightarrow{\sim} R^s i^!\mathbb{Z}/p(n)_{\text{ét}}$, for $s > n+1$. The stalk on the left hand side at a point y of X_b is $H^{s-1}_{\text{ét}}(\mathrm{Spec}\mathcal{O}^{sh}_{X,y} \times_X U, \mu_p^{\otimes n})$, which vanishes for $s > \mathrm{cd}_p U + 1$ because étale cohomology commutes with inverse limits of affine schemes. □

6. Motivic cohomology and syntomic cohomology

In this section we assume that B is the spectrum of a complete discrete valuation ring of characteristic $(0, p)$ with uniformizer π and residue field with finite p-base. If $i : Y \to X$ is the embedding of the closed fiber, then (1) with $\mathbb{Z}/p^r$-coefficients takes the form

$$R^s i^!\mathbb{Z}/p^r(n)_{\text{ét}} \cong \begin{cases} 0 & s \leq n \\ \nu_r^{n-1} & s = n+1. \end{cases} \tag{14}$$

Let $S_r(n)$ be the syntomic complex of Fontaine-Messing [6], a complex of étale sheaves on Y. By a result of Kato and Kurihara [15], there is an distinguished triangle for $n < p-1$,

$$\cdots \to S_r(n) \to \tau_{\le n} i^* Rj_* \mu_{p^r}^{\otimes n} \xrightarrow{\kappa} \nu_r^{n-1}[-n] \to \cdots .$$

The map κ is the composition of the canonical map $\tau_{\le n} i^* Rj_* \mu_{p^r}^{\otimes n} \to i^* R^n j_* \mu_{p^r}^{\otimes n}[-n]$ with the symbol map of [3, §6.6]. More precisely, $i^* R^n j_* \mu_{p^r}^{\otimes n}$ is locally generated by symbols $\{f_1, \dots, f_n\}$, for $f_i \in i^* j_* \mathcal{O}_U^\times$ by [3, Cor. 6.1.1]. By multilinearity, each such symbol can be written as a sum of symbols of the form $\{f_1, \dots, f_n\}$ and $\{f_1, \dots, f_{n-1}, \pi\}$, for $f_i \in i^* \mathcal{O}_X^\times$. Then κ sends the former to zero, and the latter to $\{d \log \bar{f}_1, \dots, d \log \bar{f}_{n-1}\}$, where $\bar{f}_i$ is the reduction of f_i to $\mathcal{O}_Y^\times$. For $n \ge p-1$, we can define $S_r(n)$ to be the cone of the map κ. This cone has been studied by Sato [26] more generally for semi-stable schemes. We now prove Theorem 1.3 of the introduction.

Proof. In view of Corollary 4.5, we have $\mathcal{H}^s(\mathbb{Z}/p^r(n)) = 0$ for $s > n$, so that we can replace the motivic complex by its truncation $\tau_{\le n}\mathbb{Z}/p^r(n)_{\mathrm{Zar}}$. We then have the following diagram of distinguished triangles in $D^-(Shv_{\text{ét}}(Y))$:

$$\begin{array}{ccccc}
\tau_{\le n} i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}} & \xrightarrow{c} & \tau_{\le n} i^* \epsilon^* Rj_* \mathbb{Z}/p^r(n)_{\mathrm{Zar}} & \xrightarrow{\delta} & \tau_{\le n}(\epsilon^* Ri^! \mathbb{Z}/p^r(n)_{\mathrm{Zar}}[1]) \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta}\ \wr \\
S_r(n) & \longrightarrow & \tau_{\le n} i^* Rj_* \mu_{p^r}^{\otimes n} & \xrightarrow{\kappa} & \nu_r^{n-1}[-n]
\end{array}$$

Here α is the composition of the adjoint map $\epsilon^* Rj_* \to Rj_* \epsilon^*$ with the cycle map $\mathbb{Z}/p^r(n)_{\text{ét}} \to \mu_{p^r}^{\otimes n}$ of [9], and β is the quasi-isomorphism of (6) and (12)

$$\tau_{\le n}(\epsilon^* Ri^! \mathbb{Z}/p^r(n)_{\mathrm{Zar}}[1]) \xleftarrow{\sim} \tau_{\le n}(\epsilon^* \mathbb{Z}/p^r(n-1)_{\mathrm{Zar}}[-1]) \xrightarrow{\sim} \nu_r^{n-1}[-n].$$

Since $\mathrm{Hom}(\tau_{\le n} i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}, \nu_r^{n-1}[-n-1]) = 0$ for degree reasons, we get an exact sequence of groups of homomorphisms in the derived category

$$0 \to \mathrm{Hom}(\tau_{\le n} i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}, S_r(n)) \to$$

$$\mathrm{Hom}(\tau_{\le n} i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}, \tau_{\le n} i^* Rj_* \mu_{p^r}^{\otimes n}) \xrightarrow{\kappa \circ} \mathrm{Hom}(\tau_{\le n} i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}, \nu_r^{n-1}[-n]).$$

It follows that there is a unique map making the diagram commutative if and only if $\kappa \circ \alpha \circ c = 0$. Again for degree reasons, the last group injects into the group $\mathrm{Hom}(\mathcal{H}^n(i^* \epsilon^* \mathbb{Z}/p^r(n)_{\mathrm{Zar}}), \nu_r^{n-1}[-n])$, and we only have to check that $\kappa \circ \alpha \circ c = 0$ in degree n. Thus it suffices to show that for a strictly henselian local ring R of X in a point of the closed fiber, the composition

$$H^n(R, \mathbb{Z}/p^r(n)) \xrightarrow{c} H^n(R\left[\tfrac{1}{\pi}\right], \mathbb{Z}/p^r(n)) \xrightarrow{\alpha} H^n_{\text{ét}}(R\left[\tfrac{1}{\pi}\right], \mu_{p^r}^{\otimes n}) \xrightarrow{\kappa} \nu_r^{n-1}(R/\pi)$$

is the zero map. Let L be the field of quotients of R, and F be the field of quotients of R/π. The localization $D = R_{(\pi)}$ is a discrete valuation ring with quotient field L and residue field F. Let D^h be the henselization of D, L^h its quotient field, and consider the following commutative diagram, where all vertical maps are induced by inclusions of rings:

$$\begin{array}{ccccc}
H^n(R, \mathbb{Z}/p^r(n)) & & & & \\
\downarrow{\scriptstyle c} & & & & \\
H^n(R\left[\frac{1}{\pi}\right], \mathbb{Z}/p^r(n)) & \xrightarrow{\alpha} & H^n_{\text{ét}}(R\left[\frac{1}{\pi}\right], \mu_{p^r}^{\otimes n}) & \xrightarrow{\kappa} & \nu_r^{n-1}(R/\pi) \\
\downarrow & & \downarrow & & \downarrow \\
H^n(L, \mathbb{Z}/p^r(n)) & \longrightarrow & H^n_{\text{ét}}(L, \mu_{p^r}^{\otimes n}) & \longrightarrow & \nu_r^{n-1}(F) \\
\downarrow & & \downarrow & & \| \\
H^n(L^h, \mathbb{Z}/p^r(n)) & \xrightarrow{\alpha'} & H^n_{\text{ét}}(L^h, \mu_{p^r}^{\otimes n}) & \xrightarrow{\kappa'} & \nu_r^{n-1}(F)
\end{array}$$

If we write R/π as a filtered colimit of rings smooth over a perfect field, then one sees by Gros-Suwa [12, Cor. 1.6] that the right vertical map is injective. Hence it suffices to show that the total composition is the zero map. The composition

$$\begin{aligned}
H^n(R, \mathbb{Z}/p^r(n)) &\to H^n(D^h, \mathbb{Z}/p^r(n)) \\
&\to H^n(L^h, \mathbb{Z}/p^r(n)) \xrightarrow{\delta} H^{n-1}(F, \mathbb{Z}/p^r(n-1)),
\end{aligned}$$

where the last two maps are the maps of the localization sequence, is zero. Thus it suffices to show that the right square in following diagram commutes:

$$\begin{array}{ccccc}
K_n^M(L^h)/p^r & \underset{\sim}{\xrightarrow{\sigma}} & H^n(L^h, \mathbb{Z}/p^r(n)) & \underset{\sim}{\xrightarrow{\alpha'}} & H^n_{\text{ét}}(L^h, \mu_{p^r}^{\otimes n}) \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \kappa'} \\
K_{n-1}^M(F)/p^r & \underset{\sim}{\xrightarrow{\sigma}} & H^{n-1}(F, \mathbb{Z}/p^r(n-1)) & \underset{\sim}{\xrightarrow{\alpha''}} & \nu_r^{n-1}(F).
\end{array}$$

Here t is the tame symbol, and σ is the map of Nesterenkov and Suslin [21]. Since σ is multiplicative by [8, Prop. 3.3], and α' is multiplicative by [9, Prop. 4.7], $\alpha' \circ \sigma$ is the symbol map to étale cohomology. Thus the outer square commutes by the definition of κ'. The left square commutes by [8, Lemma 3.2], hence the right square commutes.

Clearly, $i^*\mathbb{Z}/p^r(n)_{\text{ét}} \to S_r(n)$ is a quasi-isomorphism if and only if α is a quasi-isomorphism, which follows from the Beilinson-Lichtenbaum conjecture with mod p^r-coefficients by applying $i^*\epsilon^*Rj_*$ and using (11). □

Remark. Without assuming the Bloch-Kato conjecture, one can still show that $\nu_r^{n-1}[-n-1]$ is a split direct summand of $\tau_{\le n+1}Ri^!\mathbb{Z}/p^r(n)_{\text{ét}}$. The inverse map (in the derived category) to the canonical map comes from the commutative diagram

$$\begin{array}{ccccc}
i^*\mathbb{Z}/p^r(n)_{\text{ét}} & \longrightarrow & \tau_{\le n} i^*Rj_*\mu_{p^r}^{\otimes n} & \longrightarrow & \tau_{\le n}(Ri^!\mathbb{Z}/p^r(n)_{\text{ét}}[1]) \\
\downarrow & & \| & & \\
S_r(n) & \longrightarrow & \tau_{\le n} i^*Rj_*\mu_{p^r}^{\otimes n} & \xrightarrow{\kappa} & \nu_r^{n-1}[-n].
\end{array}$$

There is a unique map s mapping to the map κ in the second group in the following exact sequence:

$$0 \to \mathrm{Hom}(\tau_{\leq n}(Ri^!\mathbb{Z}/p^r(n)_{\text{ét}}[1]), \nu_r^{n-1}[-n]) \to$$
$$\mathrm{Hom}(\tau_{\leq n} i^* Rj_*\mu_{p^r}^{\otimes n}, \nu_r^{n-1}[-n]) \to \mathrm{Hom}(i^*\mathbb{Z}/p^r(n)_{\text{ét}}, \nu_r^{n-1}[-n]).$$

It is easy to verify that s is a splitting of the natural map.

Remark. Theorem 1.3 sheds some light on the relationship of different proofs of the crystalline conjecture. Let $\bar{V}$ be the ring of integers of the algebraic closure of a p-adic field; $\bar{V}$ is not a discrete valuation ring, but it is the direct limit of discrete valuation rings, so our methods still apply. Analog to Niziol [23], we get a commutative diagram for $n < p - 1$ and X smooth and proper over B,

$$\begin{array}{ccc}
H^i(\bar{X}, \mathbb{Z}/p^r(n)) & \xrightarrow{\sim} & H^i(\bar{U}, \mathbb{Z}/p^r(n)) \\
\downarrow{\scriptstyle \epsilon^*} & & \downarrow{\scriptstyle \epsilon^*} \\
H^i_{\text{ét}}(\bar{X}, \mathbb{Z}/p^r(n)) & \longrightarrow & H^i_{\text{ét}}(\bar{U}, \mu_{p^r}^{\otimes n}) \\
\downarrow{\scriptstyle i^*} & & \\
H^i_{\text{ét}}(\bar{Y}, S_r(n)) & &
\end{array}$$

For $\dim U \leq n$, the right map ϵ^* is an isomorphism by Suslin [29]. Thus for $p - 2 \geq n \geq \dim U$ we get the map

$$H^i_{\text{ét}}(\bar{U}, \mu_{p^r}^{\otimes n}) \to H^i_{\text{ét}}(\bar{Y}, S_r(n))$$

needed to prove the crystalline conjecture. However, our result uses [15], which is the key ingredient of Kato's proof of the conjecture.

Remark. Theorem 1.3 gives a cycle map to syntomic cohomology, hence to crystalline cohomology. For X smooth and proper over B, this fits into the following diagram (the right column exists for $p > n + 1$):

$$\begin{array}{ccccc}
\mathrm{CH}^{n-1}(Y)/p^r & \xrightarrow{cl} & H^{2n}_Y(X, \mathbb{Z}/p^r(n)) & \longrightarrow & H^{n-1}_{\text{ét}}(Y, \nu_r^{n-1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{CH}^n(X)/p^r & \xrightarrow{cl} & H^{2n}_{\text{ét}}(X, \mathbb{Z}/p^r(n)) & \longrightarrow & H^{2n}_{\text{ét}}(Y, S_r(n)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{CH}^n(X_K)/p^r & \xrightarrow{cl} & H^{2n}_{\text{ét}}(X_K, \mathbb{Z}/p^r(n)) & = & H^{2n}_{\text{ét}}(X_K, \mu_{p^r}^{\otimes n}) \\
\downarrow & & \downarrow{\scriptstyle surj} & & \downarrow \\
0 & \longrightarrow & H^{2n+1}_Y(X, \mathbb{Z}/p^r(n)) & \longrightarrow & 0.
\end{array}$$

The left upper map factors through $H^{n-1}_{\text{ét}}(Y, \mathbb{Z}/p^r(n-1))$. Thus étale motivic cohomology is a good recipient for a cycle map, and we hope that our map can be useful in studying algebraic cycles on smooth schemes over discrete valuation rings.

References

1. Bloch, S.: Algebraic Cycles and Higher K-theory, Adv. Math. **61**, 267–304 (1986)
2. Bloch, S.: A note on Gersten's conjecture in the mixed characteristic case. Appl. Alg. K-theory Alg. Geometry and Number Theory, Cont. Math. **55** (Part I), 75–78 (1986)
3. Bloch, S., Kato, K.: p-adic étale cohomology. Publ. Math. IHES **63**, 147–164 (1986)
4. Brown, K., Gersten, S.: Algebraic K-theory as generalized sheaf cohomology. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 266–292, Lecture Notes in Math., **341**, Springer, Berlin, 1973
5. Colliot-Thélène, J.L., Hoobler, R., Kahn, B.: The Bloch-Ogus-Gabber theorem. Fields Inst. Com. **16**, 31–94 (1997)
6. Fontaine, J.M., Messing, W.: p-adic periods and p-adic étale cohomology. Cont. Math. **67**, 179–207 (1987)
7. Fujiwara, K.: A proof of the absolute purity conjecture (after Gabber). Algebraic geometry 2000, Azumino (Hotaka), pp. 153–183, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002
8. Geisser, T., Levine, M.: The p-part of K-theory of fields in characteristic p. Inv. Math. **139** , 459–494 (2000)
9. Geisser, T., Levine, M.: The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. J. Reine Angew. Math. **530**, 55–103 (2001)
10. Geisser, T.: Motivic cohomology, K-theory and topological cyclic homology. To appear in: K-theory handbook
11. Gillet, H., Levine, M.: The relative form of Gersten's conjecture over a discrete valuation ring: The smooth case. J. Pure Appl. Alg. **46**, 59–71 (1987)
12. Gros, M., Suwa, N.: La conjecture de Gersten pour les faisceaux de Hodge-Witt logarithmique. Duke Math. J. **57**, 615–628 (1988)
13. Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. Ecole Norm. Sup. **12** (4), 501–661 (1979)
14. Kleiman, S.: The transversality of a general translate. Compositio Math. **28**, 287–297 (1974)
15. Kurihara, M.: A Note on p-adic Etale Cohomology. Proc. Jap. Acad. **63A**, 275–278 (1987)
16. Levine, M.: Techniques of localization in the theory of algebraic cycles. J. Alg. Geom. **10**, 299–363 (2001)
17. Levine, M.: K-theory and Motivic Cohomology of Schemes. Preprint 1999
18. Milne, J.S.: Values of zeta functions of varieties over finite fields. Am. J. Math. **108** (2), 297–360 (1986)
19. Milne, J.S.: Motivic cohomology and values of zeta functions. Comp. Math. **68**, 59–102 (1988)
20. Milne, J.S.: Etale cohomology. Princeton Math. Series 33
21. Nesterenko, Y., Suslin, A.: Homology of the general linear group over a local ring, and Milnor's K-theory. Math. USSR-Izv. **34** (1), 121–145 (1990)
22. Nisnevich, Y.A.: The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **279**, 241–342 (1989)
23. Niziol, W.: Crystalline Conjecture via K-theory. Ann. Sci. Math. ENS **31**, 659–682 (1998)
24. Quillen, D.: Higher algebraic K-theory. In: Algebraic K-theory, I: Higher K-theories (Proc. Conf. Battelle Memorial Inst. Seattle, Wash. 1972), Lecture Notes in Math. **341**, Springer, 1973, pp. 85–147

25. Roberts, P.: Multiplicities and Chern classes in local algebra. Cambridge Tracts in Mathematics, 133. Cambridge University Press, Cambridge, 1998
26. Sato, K.: An étale Tate twist with finite coefficients and duality in mixed characteristic. Preprint 2002
27. Schneider, P.: p-adic points of motives. Proc. Symp. Pure Math. AMS **55** Part 2, 225–249 (1994)
28. Spaltenstein, N.: Resolutions of unbounded complexes. Compositio Math. **65**, 121–154 (1988)
29. Suslin, A.: Higher Chow group and étale cohomology. Ann. Math. Studies **132**, 239–554 (2000)
30. Voevodsky, V.: Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. Int. Math. Res. Not. **7**, 351–355 (2002)
31. Voevodsky, V.: The 2-part of motivic cohomology. To appear in Publ. Math. IHES
32. Voevodsky, V.: On motivic cohomology with $\mathbb{Z}/l$-coefficients. Preprint, 2003