Motivic Cohomology, *K*-Theory and Topological Cyclic Homology*

II.1

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Introduction 1.1

We give a survey on motivic cohomology, higher algebraic K-theory, and topological cyclic homology. We concentrate on results which are relevant for applications in arithmetic algebraic geometry (in particular, we do not discuss non-commutative rings), and our main focus lies on sheaf theoretic results for smooth schemes, which then lead to global results using local-to-global methods.

In the first half of the paper we discuss properties of motivic cohomology for smooth varieties over a field or Dedekind ring. The construction of motivic cohomology by Suslin and Voevodsky has several technical advantages over Bloch's construction, in particular it gives the correct theory for singular schemes. But because it is only well understood for varieties over fields, and does not give well-behaved étale motivic cohomology groups, we discuss Bloch's higher Chow groups. We give a list of basic properties together with the identification of the motivic cohomology sheaves with finite coefficients.

In the second half of the paper, we discuss algebraic K-theory, étale K-theory and topological cyclic homology. We sketch the definition, and give a list of basic properties of algebraic K-theory, sketch Thomason's hyper-cohomology construction of étale K-theory, and the construction of topological cyclic homology. We then give a short overview of the sheaf theoretic properties, and relationships between the three theories (in many situations, étale K-theory with p-adic coefficients and topological cyclic homology agree).

In an appendix we collect some facts on intersection theory which are needed to work with higher Chow groups. The results can be found in the literature, but we thought it would be useful to find them concentrated in one article.

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Motivic Cohomology 1.2

The existence of a complex of sheaves whose cohomology groups are related to special values of L-functions was first conjectured by Beilinson [1], [2, §5] (for the Zariski topology) and Lichtenbaum [62, 63](for the étale topology). Consequently, the conjectural relationship between these complexes of sheaves is called the Beilinson–Lichtenbaum conjecture. The most commonly used constructions of motivic cohomology are the ones of Bloch [5] and Suslin–Voevodsky [89, 91, 99, 100]. Bloch's higher Chow groups are defined for any scheme, but they have properties analogous to a Borel–Moore homology theory in topology. In particular, they behave like a cohomology theory only for smooth schemes over a field. Voevodsky's motivic cohomology groups has good properties for non-smooth schemes, but their basic properties are only established for schemes of finite type over a field, and they do not give a good étale theory (étale hypercohomology vanishes with mod p-coefficients over a field of characteristic p). By a theorem of Voevodsky [102], his motivic cohomology groups agree with Bloch's higher Chow groups for smooth varieties over a field. Since we want to include varieties over Dedekind rings into our discussion, we discuss Bloch's higher Chow groups.

1.2.1 Definition

Let X be separated scheme, which is essentially of finite type over a quotient of a regular ring of finite Krull dimension (we need this condition in order to have a well-behaved concept of dimension, see the appendix). We also assume for simplicity that X is equi-dimensional, i.e. every irreducible component has the same dimension (otherwise one has to replace codimension by dimension in the following discussion). Then *Bloch's higher Chow groups* are defined as the cohomology of the following complex of abelian groups. Let Δ^r be the algebraic r-simplex $\operatorname{Spec}\mathbb{Z}[t_0,\dots,t_r]/(1-\sum_j t_j)$. It is non-canonically isomorphic to the affine space $\mathbb{A}^r_{\mathbb{Z}}$, but has distinguished subvarieties of codimension s, given by $t_{i_1}=t_{i_2}=\dots=t_{i_s}=0$, $0\le i_1<\dots<i_s\le r$. These subvarieties are called faces of Δ^r, and they are isomorphic to Δ^{r-s}. The group $z^n(X,i)$ is the free abelian group generated by closed integral subschemes $Z\subset\Delta^i\times X$ of codimension n, such that for every face F of codimension s of Δ^i, every irreducible component of the intersection $Z\cap(F\times X)$ has codimension s in $F\times X$. This ensures that the intersection with a face of codimension 1 gives an element of $z^n(X,i-1)$, and we show in the appendix that taking the alternating sum of these intersections makes $z^n(X,*)$ a chain complex. Replacing a cycle in $z^n(X,*)$ by another cycle which differs by a boundary is called *moving the cycle*. We let $H^i(X,\mathbb{Z}(n))$ be the cohomology of the cochain complex with $z^n(X,2n-i)$ in degree i. For dimension reasons it is clear from the definition that $H^i(X,\mathbb{Z}(n))=0$ for $i>\min\{2n,n+\dim X\}$. In particular, if X is the spectrum of a field F, then $H^i(F,\mathbb{Z}(n))=0$ for $i>n$. It is a conjecture of Beilinson and Soulé that $H^i(X,\mathbb{Z}(n))=0$ for $i<0$.

The motivic cohomology with coefficients in an abelian group A is defined as the cohomology of the complex $A(n):=\mathbb{Z}(n)\otimes A$. In particular, motivic cohomology groups with finite coefficients fit into a long exact sequences

$$\cdots\to H^i(X,\mathbb{Z}(n))\stackrel{\times m}{\to} H^i\big(X,\mathbb{Z}(n)\big)\to H^i\big(X,\mathbb{Z}/m(n)\big)\to\cdots .$$

1.2.2 Hyper-cohomology

By varying X, one can view $\mathbb{Z}(n):=z^n(-,2n-*)$ as a complex of presheaves on X. It turns out that this is in fact a complex of sheaves for the Zariski, Nisnevich and étale topology on X [5] [28, Lemma 3.1]. It is clear from the definition that there is a canonical quasi-isomorphism $\mathbb{Z}(0)\simeq\mathbb{Z}$ of complexes of Zariski-sheaves if X is integral. Since étale covers of a normal scheme are normal, the same quasi-isomorphism holds for the Nisnevich and étale topology if X is normal. For X

smooth of finite type over a field or Dedekind ring, there is a quasi-isomorphism $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ for all three topologies. This has been shown by Bloch [5] for quasi-projective X over a field, and follows for general X by the localization below. We define $H^i(X_{\text{Zar}}, \mathbb{Z}(n))$, $H^i(X_{\text{Nis}}, \mathbb{Z}(n))$ and $H^i(X_{\text{ét}}, \mathbb{Z}(n))$ as the hyper-cohomology with coefficients in $\mathbb{Z}(n)$ for the Zariski, Nisnevich and étale topology of X, respectively. One defines motivic cohomology with coefficients in the abelian group A as the hyper-cohomology groups of $\mathbb{Z}(n) \otimes A$.

If X is of finite type over a finite field $\mathbb{F}_q$, then it is worthwhile to consider motivic cohomology groups $H^i(X_W, \mathbb{Z}(n))$ for the Weil-étale topology [29]. This is a topology introduced by Lichtenbaum [64], which is finer than the étale topology. Weil-étale motivic cohomology groups are related to étale motivic cohomology groups via the long exact sequence

$$\to H^i\left(X_{\text{ét}}, \mathbb{Z}(n)\right) \to H^i\left(X_W, \mathbb{Z}(n)\right) \to H^{i-1}\left(X_{\text{ét}}, \mathbb{Q}(n)\right) \xrightarrow{\delta} H^{i+1}\left(X_{\text{ét}}, \mathbb{Z}(n)\right) \to \ .$$

The map δ is the composition

$$H^{i-1}\left(X_{\text{ét}}, \mathbb{Q}(n)\right) \to H^{i-1}\left(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(n)\right) \xrightarrow{\cup e} H^i\left(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(n)\right) \xrightarrow{\beta} H^{i+1}\left(X_{\text{ét}}, \mathbb{Z}(n)\right) ,$$

where $e \in H^1(\mathbb{F}_q, \mathbb{Z}) \cong \text{Ext}^1_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ is a generator, and β the Bockstein homomorphism. Consequently, the sequence breaks up into short exact sequences upon tensoring with $\mathbb{Q}$. Weil-étale motivic cohomology groups are expected to be an integral model for l-adic cohomology, and are expected to be finitely generated for smooth and projective varieties over finite fields [29].

Most of the properties of motivic cohomology which follow are due to Bloch [5] for varieties over fields, and to Levine [59] for varieties over discrete valuation rings.

Functoriality 1.2.3

We show in the appendix that a flat, equidimensional map $f : X \to Y$ induces a map of complexes $z^n(Y, *) \to z^n(X, *)$, hence a map $f^* : H^i(Y, \mathbb{Z}(n)) \to H^i(X, \mathbb{Z}(n))$. The resulting map of complexes of sheaves $f^*\mathbb{Z}(n)_Y \to \mathbb{Z}(n)_X$ induces a map on hyper-cohomology groups.

We also show in the appendix that there is a map of cycle complexes $z^n(X, *) \to z^{n-c}(Y, *)$ for a proper map $f : X \to Y$ of relative dimension c of schemes of finite type over an excellent ring (for example, over a Dedekind ring of characteristic 0, or a field). This induces a map $f_* : H^i(X, \mathbb{Z}(n)) \to H^{i-2c}(Y, \mathbb{Z}(n-c))$ and a map of sheaves $f_*\mathbb{Z}(n)_X \to \mathbb{Z}(n-c)_Y[-2c]$, which induces a map of hyper-cohomology groups if $f_* = Rf_*$.

Motivic cohomology groups are contravariantly functorial for arbitrary maps between smooth schemes over a field [5, §4], [57, II §3.5] or discrete valuation ring [59]. This requires a moving lemma, because the pull-back of cycles may not meet faces properly. Hence one needs to show that every cycle c is equivalent to a cycle c' whose pull-back does meet faces properly, and that the pull-back does not depend on the choice of c'. This moving lemma is known for affine schemes, and one can employ the Mayer–Vietoris property below to reduce to this situation by a covering of X.

1.2.4 Localization

Let X be a scheme of finite type over a discrete valuation ring, and let $i : Z \to X$ be a closed subscheme of pure codimension c with open complement $j : U \to X$. Then the exact sequence of complexes

$$0 \to z^{n-c}(Z,*) \xrightarrow{i_*} z^n(X,*) \xrightarrow{j^*} z^n(U,*)$$

gives rise to a distinguished triangle in the derived category of abelian groups [8,58], i.e. the cokernel of j^* is acyclic. In particular, there are long exact *localization sequences* of cohomology groups

$$\cdots \to H^{i-2c}\left(Z, \mathbb{Z}(n-c)\right) \to H^i\left(X, \mathbb{Z}(n)\right) \to H^i\left(U, \mathbb{Z}(n)\right) \to \cdots .$$

and the complexes $\mathbb{Z}(n)$ satisfy the *Mayer–Vietoris property*, i.e. if $X = U \cup V$ is a covering of X by two Zariski open subsets, then there is a long exact sequence

$$\cdots \to H^i\left(X, \mathbb{Z}(n)\right) \to H^i\left(U, \mathbb{Z}(n)\right) \oplus H^i\left(V, \mathbb{Z}(n)\right) \to H^i\left(U \cap V, \mathbb{Z}(n)\right) \to \cdots .$$

If X is a separated noetherian scheme of finite Krull dimension, then the argument of Brown–Gersten [15, 17, 95] shows that whenever the cohomology groups $H^i(C^\cdot(-))$ of a complex of presheaves $C^\cdot$ satisfies the Mayer–Vietoris property, then the cohomology groups $H^i(C^\cdot(X))$ and hyper-cohomology groups $H^i(X_{\mathrm{Zar}}, \widetilde{C}^\cdot)$ of the associated complex of sheaves agree. Note that hyper-cohomology, i.e. an injective resolution of $C^\cdot$, always satisfies the Mayer–Vietoris property, but the Mayer–Vietoris property is not preserved by quasi-isomorphisms. For example, $\mathbb{Z}(1)$ satisfies the Mayer–Vietoris property, but the quasi-isomorphic sheaf $\mathbb{G}_m$ does not. As a consequence of the theorem of Brown–Gersten, cohomology and hyper-cohomology of $\mathbb{Z}(n)$ agree, $H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\mathrm{Zar}}, \mathbb{Z}(n))$, and the spectral sequence for the hyper-cohomology of a complex gives

$$E_2^{s,t} = H^s\left(X_{\mathrm{Zar}}, \mathcal{H}^t\left(\mathbb{Z}(n)\right)\right) \Rightarrow H^{s+t}\left(X, \mathbb{Z}(n)\right) . \qquad (1.1)$$

The argument of Brown–Gersten has been generalized by Nisnevich [75], see also [17, Thm. 7.5.2], replacing the Mayer–Vietoris property by the étale excision property. This property is satisfyed by Bloch's higher Chow groups in view of localization, hence motivic cohomology agrees with its Nisnevish hyper-cohomology $H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\mathrm{Nis}}, \mathbb{Z}(n))$, and we get a spectral sequence

$$E_2^{s,t} = H^s\left(X_{\mathrm{Nis}}, \mathcal{H}^t(\mathbb{Z}(n))\right) \Rightarrow H^{s+t}\left(X, \mathbb{Z}(n)\right) . \qquad (1.2)$$

For smooth schemes, the spectral sequences (1.1) and (1.2) are isomorphic [17]. On the other hand, if X is the node $\mathrm{Spec}\, k[x,y]/(y^2 - x(x+1))$, then $H^1(X_{\mathrm{Nis}}, \mathbb{Z}) \cong \mathbb{Z}$, but localization shows that $H^1(X, \mathbb{Z}(0)) = 0$. Hence $\mathbb{Z}(0)$ is not quasi-isomorphic to $\mathbb{Z}$ even for integral schemes.

Gersten Resolution 1.2.5

In order to study the sheaf $\mathcal{H}^t(\mathbb{Z}(n))$, one considers the spectral sequence coming from the filtration of $z^n(X,*)$ by *coniveau* [5, §10]. The complex $F^s z^n(X,*)$ is the subcomplex generated by closed integral subschemes such that the projection to X has codimension at least s. The localization property implies that the pull-back map $gr^s z^n(X,*) \to \oplus_{x\in X^{(s)}} z^n(k(x),*)$ is a quasi-isomorphism, where $X^{(s)}$ denotes the set of points $x \in X$ such that the closure of x has codimension s, and $k(x)$ is the residue field of x. The spectral sequence for a filtration of a complex then takes the form:

$$E_1^{s,t} = \bigoplus_{x\in X^{(s)}} H^{t-s}\left(k(x),\mathbb{Z}(n-s)\right) \Rightarrow H^{s+t}\left(X,\mathbb{Z}(n)\right) . \tag{1.3}$$

The spectral sequence degenerates at E_2 for an essentially smooth semi-local ring over a field. Hence, for X a smooth scheme over a field, the E_1-terms and differentials gives rise to an exact sequence of Zariski sheaves, the *Gersten resolution* [5, Thm. 10.1]

$$\begin{aligned} 0 \to \mathcal{H}^t(\mathbb{Z}(n)) \to & \bigoplus_{x\in X^{(0)}} (i_x)_* H^t\left(k(x),\mathbb{Z}(n)\right) \\ & \to \bigoplus_{x\in X^{(1)}} (i_x)_* H^{t-1}\left(k(x),\mathbb{Z}(n-1)\right) \to \cdots . \end{aligned} \tag{1.4}$$

Here $(i_x)_*G$ is the skyscraper sheaf with group G at the point x. The same argument works for motivic cohomology with coefficients. Since skyscraper sheaves are flabby, one can calculate the cohomology of $\mathcal{H}^t(\mathbb{Z}(n))$ with the complex (1.4), and gets $E_2^{s,t} = H^s(X_{\mathrm{Zar}},\mathcal{H}^t(\mathbb{Z}(n)))$.

If X is smooth over a discrete valuation ring V, there is a conditional result. Assume that for any discrete valuation ring R, essentially of finite type over V, with quotient field K of R, the map $H^t(R,\mathbb{Z}(n)) \to H^t(K,\mathbb{Z}(n))$ is injective (this is a special case of (1.4)). Then the sequence (1.4) is exact on X [28]. The analogous statement holds with arbitrary coefficients. Since the hypothesis is satisfied with mod p-coefficients if p is the residue characteristic of V (see below), we get a Gersten resolution for $\mathcal{H}^t(\mathbb{Z}/p^r(n))$. As a corollary of the proof of (1.4), one can show [28] that the complex $\mathbb{Z}(n)$ is acyclic in degrees above n.

Products 1.2.6

For X and Y varieties over a field, there is an external product structure, see [34, Appendix] [5, Section 5] [107].

$$z^n(X,*) \otimes z^m(Y,*) \to z^{n+m}(X\times Y,*) ,$$

which induces an associative and (graded) commutative product on cohomology. If $Z_1 \subseteq X\times\Delta^i$ and $Z_2 \subseteq Y\times\Delta^j$ are generators of $z^n(X,i)$ and $z^m(Y,j)$, respectively, then the map sends $Z_1 \otimes Z_2$ to $Z_1 \times Z_2 \subseteq X \times Y \times \Delta^i \times \Delta^j$. One then *triangulates* $\Delta^p \times \Delta^q$, i.e. covers it with a union of copies of Δ^{p+q}. The complication is to move cycles such that the pull-back along the maps $\Delta^{p+q} \to \Delta^p \times \Delta^q$ intersects faces properly. Sheafifying the above construction on $X \times Y$, we get a pairing of complexes of sheaves

$$p_X^* \mathbb{Z}(n)^X \otimes p_Y^* \mathbb{Z}(m)^Y \to \mathbb{Z}(n+m)^{X \times Y} .$$

which in turn induces a pairing of hyper-cohomology groups. For smooth X, the external product induces, via pull-back along the diagonal, an internal product

$$H^i\left(X, \mathbb{Z}(n)\right) \otimes H^j\left(X, \mathbb{Z}(m)\right) \to H^{i+j}\left(X, \mathbb{Z}(n+m)\right) .$$

Over a discrete valuation ring, we do not know how to construct a product structure in general. The problem is that the product of cycles lying in the closed fiber will not have the correct codimension, and we don't know how to move a cycle in the closed fiber to a cycle which is flat over the base. Sometimes one can get by with the following construction of Levine [59]. Let B be the spectrum of a discrete valuation ring, let Y be flat over B, and consider the subcomplex $z^m(Y/B, *) \subseteq z^m(Y, *)$ generated by cycles whose intersections with all faces are equidimensional over B. A similar construction as over fields gives a product structure

$$z^n(X, *) \otimes z^m(Y/B, *) \to z^{n+m}(X \times_B Y, *) .$$

This is helpful because often the cohomology classes one wants to multiply with can be represented by cycles in $z^m(Y/B, *)$.

Levine conjectures that the inclusion $z^m(Y/B, *) \subseteq z^m(Y, *)$ induces a quasi-isomorphism of Zariski sheaves. If B is the spectrum of a Dedekind ring, it would be interesting to study the cohomology groups of the complex $z^m(Y/B, *)$.

1.2.7 Affine and Projective Bundles, Blow-ups

Let X be of finite type over a field or discrete valuation ring, and let $p : E \to X$ be a flat map such that for each point $x \in X$ the fiber is isomorphic to $\mathbb{A}^n_{k(x)}$. Then the pull back map induced by the projection $E \to X$ induces a quasi-isomorphism $z^n(X, *) \to z^n(E, *)$. This was first proved by Bloch [5, Thm. 2.1] over a field, and can be generalized using localization. Note that the analogous statement for étale hyper-cohomology of the motivic complex is wrong. For example, one can see with Artin–Schreier theory that $H^2(\mathbb{A}^1_{\mathbb{F}_p, \text{ét}}, \mathbb{Z}(0))$ has a very large p-part. The localization sequence

$$\cdots \to H^{i-2}\left(\mathbb{P}^{n-1}_X, \mathbb{Z}(n-1)\right) \to H^i\left(\mathbb{P}^n_X, \mathbb{Z}(n)\right) \to H^i\left(\mathbb{A}^n_X, \mathbb{Z}(n)\right) \to \cdots$$

is split by the following composition of pull-back maps

$$H^i\left(\mathbb{A}^n_X, \mathbb{Z}(n)\right) \xleftarrow{\sim} H^i\left(X, \mathbb{Z}(n)\right) \to H^i\left(\mathbb{P}^n_X, \mathbb{Z}(n)\right) .$$

This, together with induction, gives a canonical isomorphism

$$H^i\left(\mathbb{P}^m_X, \mathbb{Z}(n)\right) \cong \bigoplus_{j=0}^{m} H^{i-2j}\left(X, \mathbb{Z}(n-j)\right) . \tag{1.5}$$

If X' is the blow-up of the smooth scheme X along the smooth subscheme Z of codimension c, then we have the blow-up formula

$$H^i\left(X', \mathbb{Z}(n)\right) \cong H^i\left(X, \mathbb{Z}(n)\right) \oplus \bigoplus_{j=1}^{c-1} H^{i-2j}\left(Z, \mathbb{Z}(i-j)\right) . \tag{1.6}$$

The case X over a field is treated in [57, Lemma IV 3.1.1] and carries over to X over a Dedekind ring using the localization sequence.

Milnor K-Theory 1.2.8

The *Milnor K-groups* [73] of a field F are defined as the quotient of the tensor algebra of the multiplicative group of units $F^\times$ by the ideal generated by the Steinberg relation $a \otimes (1-a) = 0$,

$$K^M_*(F) = T^*_{\mathbb{Z}}(F^\times)/\left(a \otimes (1-a) | a \in F - \{0,1\}\right) .$$

For R a regular semi-local ring over a field k, we define the Milnor K-theory of R as the kernel

$$K^M_n(R) = \ker\left(\bigoplus_{x \in R^{(0)}} K^M_n\left(k(x)\right) \xrightarrow{\delta} \bigoplus_{y \in R^{(1)}} K^M_{n-1}\left(k(y)\right)\right) . \tag{1.7}$$

Here δ_y is defined as follows [73, Lemma 2.1]. The localization V_y of R at the prime corresponding to y is a discrete valuation ring with quotient field $k(x)$ for some $x \in R^{(0)}$ and residue field $k(y)$. Choose a uniformizer $\pi \in V_y$, and if $u_j \in V_y^\times$ has reduction $\overline{u}_j \in k(y)$, set $\delta_y(\{u_1, \dots u_n\}) = 0$ and $\delta_y(\{u_1, \dots, u_{n-1}, \pi\}) = \{\overline{u}_1, \dots, \overline{u}_{n-1}\}$. By multilinearity of symbols this determines δ_y. Since there exists a universally exact Gersten resolution for Milnor K-theory [17, Example 7.3(5)], the equation (1.7) still holds after tensoring all terms with an abelian group.

For any ring, one can define a graded ring $\overline{K}^M_*(R)$ by generators and relations as above (including the extra relation $a \otimes (-a) = 0$, which follows from the Steinberg relation if R is a field). If R is a regular semi-local ring over a field, there is a canonical map $\overline{K}^M_i(R) \to K^M_i(R)$, and Gabber proved that this map is surjective, provided the base field is infinite, see also [19].

For a field F, $H^i(F, \mathbb{Z}(n)) = 0$ for $i > n$, and in the highest degree we have the isomorphism of Nesterenko–Suslin [74, Thm. 4.9] and Totaro [98]:

$$K^M_n(F) \xrightarrow{\sim} H^n\left(F, \mathbb{Z}(n)\right) . \tag{1.8}$$

The map is given by

$$\{u_1, \dots, u_n\} \mapsto \left(\frac{-u_1}{1 - \sum u_i}, \dots, \frac{-u_n}{1 - \sum u_i}, \frac{1}{1 - \sum u_i} \right) \in (\Delta_F^n)^{(n)} . \qquad (1.9)$$

If m is relatively prime to the characteristic of F, it follows from Kummer theory that $K_1^M(F)/m \cong F^\times/(F^\times)^m \cong H^1(F_{\text{ét}}, \mu_m)$. Since the cup-product on Galois cohomology satisfies the Steinberg relation [93, Thm. 3.1], we get the symbol map from Milnor K-theory to Galois cohomology

$$K_n^M(F)/m \to H^n\left(F_{\text{ét}}, \mu_m^{\otimes n}\right) . \qquad (1.10)$$

The *Bloch–Kato conjecture* [9] states that the symbol map is an isomorphism. Voevodsky [101] proved the conjecture for m a power of 2, and has announced a proof for general m in [103].

1.2.9 Beilinson–Lichtenbaum Conjecture

Motivic cohomology groups for the étale and Zariski topology are different. For example, for a field F, we have $H^3(F, \mathbb{Z}(1)) = 0$, but $H^3(F_{\text{ét}}, \mathbb{Z}(1)) \cong H^2(F_{\text{ét}}, \mathbb{G}_m) \cong \mathrm{Br}\, F$. The *Beilinson–Lichtenbaum conjecture* states that this phenomenon only occurs in higher degrees. More precisely, let X be a smooth scheme over a field, and let $\varepsilon : X_{\text{ét}} \to X_{\text{Zar}}$ be the canonical map of sites. Then the Beilinson–Lichtenbaum conjecture states that the canonical map

$$\mathbb{Z}(n) \xrightarrow{\sim} \tau_{\leq n+1} R\varepsilon_* \mathbb{Z}(n) . \qquad (1.11)$$

is a quasi-isomorphism, or more concretely, that for every smooth scheme X over a field,

$$H^i\left(X, \mathbb{Z}(n)\right) \cong H^i\left(X_{\text{ét}}, \mathbb{Z}(n)\right) \quad \text{for } i \leq n+1 .$$

In [91], Suslin and Voevodsky show that, assuming resolution of singularities, the Bloch–Kato conjecture (1.10) implies the Beilinson–Lichtenbaum conjecture (1.11) with mod m-coefficients; in [34] the hypothesis on resolution of singularities is removed.

1.2.10 Cycle Map

Let $H^i(X, n)$ be a bigraded cohomology theory which is the hyper-cohomology of a complex of sheaves $C(n)$; important examples are étale cohomology $H^i(X, \mu_m^{\otimes n})$ and Deligne cohomology $H^i_{\mathcal{D}}(X, \mathbb{Z}(n))$. Assume that $C(n)$ is contravariantly functorial, i.e. for $f : X \to Y$ there exists a map $f^* C(n)_X \to C(n)_Y$ in the derived category, compatible with composition. Assume furthermore that $C(n)$ admits a cycle class map $\mathrm{CH}^n(X) \to H^{2n}(X, n)$, is homotopy invariant, and satisfies a weak form of purity. Then Bloch constructs in [7] (see also [34]) a natural map

$$H^i\left(X,\mathbb{Z}(n)\right) \to H^i(X,n)\ . \tag{1.12}$$

Unfortunately, this construction does not work for cohomology theories which satisfy the projective bundle formula, but are not homotopy invariant, like crystalline cohomology, de Rham cohomology or syntomic cohomology.

Rational Coefficients 1.2.11

If X is a separated noetherian scheme of finite Krull dimension, then

$$H^i\left(X,\mathbb{Q}(n)\right) \cong H^i\left(X_{\text{ét}},\mathbb{Q}(n)\right)\ . \tag{1.13}$$

Indeed, in view of $H^i(X,\mathbb{Z}(n)) \cong H^i(X_{\text{Nis}},\mathbb{Z}(n))$, it suffices to observe that for any sheaf $\mathcal{F}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces, $H^i(X_{\text{Nis}},\mathcal{F}_{\mathbb{Q}}) \cong H^i(X_{\text{ét}},\mathcal{F}_{\mathbb{Q}})$. But for any henselian local ring R with residue field k, and $i > 0$, $H^i(R_{\text{ét}},\mathcal{F}_{\mathbb{Q}}) \cong H^i(k_{\text{ét}},\mathcal{F}_{\mathbb{Q}}) = 0$ because higher Galois cohomology is torsion. Hence $R^i\alpha_* = 0$ for $\alpha : X_{et} \to X_{\text{N}}$ is the canonical morphism of sites.

Parshin conjectured that for X smooth and projective over a finite field,

$$H^i\left(X,\mathbb{Q}(n)\right) = 0 \quad \text{for } i \neq 2n\ .$$

Using (1.3) and induction, this implies that for any field F of characteristic p, $H^i(F,\mathbb{Q}(n)) = 0$ for $i \neq n$. From the sequence (1.4), it follows then that for any smooth X over a field k of characteristic p, $H^i(X,\mathbb{Q}(n)) = 0$ unless $n \leq i \leq \min\{2n, n+d\}$. To give some credibility to Parshin's conjecture, one can show [26] that it is a consequence of the conjunction of the strong form of Tate's conjecture, and a conjecture of Beilinson stating that over finite fields numerical and rational equivalence agree up to torsion. The argument is inspired by Soulé [85] and goes as follows. Since the category of motives for numerical equivalence is semi-simple by Jannsen [52], we can break up X into simple motives M. By Beilinson's conjecture, Grothendieck motives for rational and numerical equivalence agree, and we can break up $H^i(X,\mathbb{Q}(n))$ correspondingly into a direct sum of $H^i(M,\mathbb{Q}(n))$. By results of Milne [72], Tate's conjecture implies that a simple motive M is characterized by the eigenvalue e_M of the Frobenius endomorphism φ_M of M. By Soule [85], φ_M acts on $H^i(M,\mathbb{Q}(n))$ as p^n, so this group can only be non-zero if $e_M = p^n$, which implies that $M \subseteq \mathbb{P}^n$. But the projective space satisfies Parshin's conjecture by the projective bundle formula (1.5).

In contrast, if K is a number field, then

$$H^1\left(K,\mathbb{Q}(n)\right) = \begin{cases} \mathbb{Q}^{r_1+r_2} & 1 < n \equiv 1 \mod 4 \\ \mathbb{Q}^{r_2} & n \equiv 3 \mod 4\ , \end{cases}$$

where r_1 and r_2 are the number of real and complex embeddings of K, respectively.

1.3 Sheaf Theoretic Properties

In this section we discuss sheaf theoretic properties of the motivic complex.

1.3.1 Invertible Coefficients

Let X be a smooth scheme over k, and m prime to the characteristic k. Then there is a quasi-isomorphism

$$\mathbb{Z}/m(n)_{\text{ét}} \simeq \mu_m^{\otimes n} \tag{1.14}$$

of the étale motivic complex with mod m-coefficients. The map is the map of (1.12), and it is an isomorphism, because for a henselian local ring R of X with residue field κ, there is a commutative diagram

$$\begin{array}{ccc} H^i\left(R_{\text{ét}}, \mathbb{Z}/m(n)\right) & \to & H^i\left(R_{\text{ét}}, \mu_m^{\otimes n}\right) \\ \sim\downarrow & & \sim\downarrow \\ H^i\left(\kappa_{\text{ét}}, \mathbb{Z}/m(n)\right) & \to & H^i\left(\kappa_{\text{ét}}, \mu_m^{\otimes n}\right) . \end{array}$$

The left vertical map is an isomorphism by Bloch [5, Lemma 11.1] (he assumes, but does not use, that R is strictly henselian), the right horizontal map by rigidity for étale cohomology (Gabber [22]). Finally, the lower horizontal map is an isomorphism for separably closed κ by Suslin [87].

In view of (1.14), the Beilinson–Lichtenbaum conjecture with mod m-coefficients takes the more familiar form

$$\mathbb{Z}/m(n) \xrightarrow{\sim} \tau_{\leq n} R\varepsilon_* \mu_m^{\otimes n} ,$$

or more concretely,

$$H^i\left(X, \mathbb{Z}/m(n)\right) \xrightarrow{\sim} H^i\left(X_{\text{ét}}, \mu_m^{\otimes n}\right) \quad \text{for } i \leq n .$$

One could say that motivic cohomology is completely determined by étale cohomology for $i \leq n$, whereas for $i > n$ the difference encodes deep arithmetic properties of X. For example, the above map for $i = 2n$ is the cycle map.

1.3.2 Characteristic Coefficients

We now consider the motivic complex mod p^r, where $p = \text{char } k$. For simplicity we assume that k is perfect. This is no serious restriction because the functors we study commute with filtered colimits of rings. Let $W(k)$ be the ring of Witt vectors of k, and K the field of quotients of $W(k)$. For a smooth variety X over k, Illusie [50], based on ideas of Bloch [4] and Deligne, defines the *de Rham–Witt pro-complex* $W.\Omega^*_X$. It generalizes Witt vectors $W.\mathcal{O}_X$, and the de Rham complex Ω^*_X, and comes equipped with operators $F : W_r\Omega^*_X \to W_{r-1}\Omega^*_X$ and $V : W_r\Omega^*_X \to W_{r+1}\Omega^*_X$, which generalize the Frobenius and Verschiebung maps on Witt vectors. The hypercohomology of the de Rham–Witt complex calculates the crystalline cohomology $H^*_{crys}(X/W(k))$ of X, hence it can be used to analyze crystalline cohomology using the *slope spectral sequence*

$$E_1^{s,t} = H^t\left(X, W.\Omega^s\right) \Rightarrow H^{s+t}_{crys}\left(X/W(k)\right) .$$

This spectral sequence degenerates at E_1 up to torsion [50, II Thm. 3.2], and if we denote by $H^j_{crys}(X/W(k))_K^{[s,s+1[}$ the part of the F-crystal $H^j_{crys}(X/W(k)) \otimes_{W(k)} K$ with slopes in the interval $[s, s+1[$, then [50, II (3.5.4)]

$$H^t_{crys}\left(X/W(k)\right)_K^{[s,s+1[} = H^{t-s}\left(X, W.\Omega^s_X\right) \otimes_{W(k)} K .$$

The (étale) *logarithmic de Rham–Witt sheaf* $\nu_r^n = W_r\Omega^n_{X,\log}$ is defined as the subsheaf of $W_r\Omega^n_X$ generated locally for the étale topology by $d\log\overline{x}_1 \wedge \cdots \wedge d\log\overline{x}_n$, where $\overline{x} \in W_r\mathcal{O}_X$ are Teichmüller lifts of units. See [40,69] for basic properties. For example, $\nu_r^0 \cong \mathbb{Z}/p^r$, and there is a short exact sequence of étale sheaves

$$0 \to \mathbb{G}_m \xrightarrow{p^r} \mathbb{G}_m \to \nu_r^1 \to 0 .$$

There are short exact sequences of pro-sheaves on the small étale site of X, [50, Théorème 5.7.2]

$$0 \to \nu_.^n \to W.\Omega^n_X \xrightarrow{F-1} W.\Omega^n_X \to 0 . \qquad (1.15)$$

For a quasi-coherent sheaf (or pro-sheaf) of $\mathcal{O}_X$-modules such as $W.\Omega^n_X$, the higher direct images $R^i\varepsilon_* W.\Omega^i_X$ of its associated étale sheaf are zero for $i > 0$. Thus we get an exact sequence of pro-Zariski-sheaves

$$0 \to \nu_.^n \to W.\Omega^n_X \xrightarrow{F-1} W.\Omega^n_X \to R^1\varepsilon_*\nu_.^n \to 0 .$$

and $R^i\varepsilon_*\nu_.^n = 0$ for $i \geq 2$. By Gros and Suwa [41], the sheaves ν_r^n have a Gersten resolution on smooth schemes X over k,

$$0 \to \nu_r^n \to \bigoplus_{x\in X^{(0)}} (i_x)_*\nu_r^n\left(k(x)\right) \to \bigoplus_{x\in X^{(1)}} (i_x)_*\nu_r^{n-1}\left(k(x)\right) \to \cdots .$$

In particular, $H^i(X_{\mathrm{Zar}}, \nu_r^n) = 0$ for $i > n$. Milnor K-theory and logarithmic de Rham–Witt sheaves of a field F of characteristic p are isomorphic by the theorem of Bloch–Gabber–Kato [9,55]

$$d\log : K_n^M(F)/p^r \xrightarrow{\sim} \nu_r^n(F) . \qquad (1.16)$$

Since motivic cohomology, Milnor K-theory and logarithmic de Rham Witt cohomology all admit Gersten resolutions, we get as a corollary of (1.16) and (1.8) that for a semi-localization R of a regular k-algebra of finite type, there are isomorphisms

$$H^n\left(R, \mathbb{Z}/p^r(n)\right) \xleftarrow{\sim} K_n^M(R)/p^r \xrightarrow{\sim} \nu_r^n(R) .$$

Using this as a base step for induction, one can show [33] that for any field K of characteristic p,

$$H^i\left(K, \mathbb{Z}/p^r(n)\right) = 0 \quad \text{for } i \neq n\ . \tag{1.17}$$

Consequently, for a smooth variety X over k, there is a quasi-isomorphism of complexes of sheaves for the Zariski (hence also the étale) topology,

$$\mathbb{Z}/p^r(n) \cong \nu_r^n[-n]\ . \tag{1.18}$$

so that

$$H^{s+n}\left(X, \mathbb{Z}/p^r(n)\right) \cong H^s\left(X_{\text{Zar}}, \nu_r^n\right),$$
$$H^{s+n}\left(X_{\text{ét}}, \mathbb{Z}/p^r(n)\right) \cong H^s\left(X_{\text{ét}}, \nu_r^n\right)\ .$$

Since $dF = pFd$ on the de Rham–Witt complex, we can define a map of truncated complexes $F : W.\Omega_X^{*\geq n} \to W.\Omega_X^{*\geq n}$ by letting $F = p^{j-n}F$ on $W.\Omega_X^j$. Since $p^jF - \text{id}$ is an automorphism on $W_r\Omega_X^i$ for every $j \geq 1$ [50, Lemma I 3.30], the sequence (1.15) gives rise to an exact sequence of pro-complexes of étale sheaves

$$0 \to \nu_\cdot^n[-n] \to W.\Omega_X^{*\geq n} \overset{F-\text{id}}{\to} W.\Omega_X^{*\geq n} \to 0\ .$$

The Frobenius endomorphism φ of X induces the map p^iF on $W_r\Omega_X^i$ [50, I 2.19], hence composing with the inclusion $W.\Omega_X^{*\geq n} \to W.\Omega_X^*$ and using (1.18), we get a map to crystalline cohomology [69]

$$H^s\left(X, \mathbb{Z}(n)\right) \to H^{s-n}\left(X_{\text{Zar}}, \nu_r^n\right) \to H^s_{crys}\left(X/W_r(k)\right)^{\varphi-p^n}\ .$$

This generalizes the crystalline cycle map of Gros [40].

1.3.3 Projective Bundle and Blow-up

If X is smooth over a field of characteristic p, let $\mathbb{Q}/\mathbb{Z}(n)' = \operatorname{colim}_{p\nmid m} \mu_m^{\otimes n}$, and for $n < 0$ define negative étale motivic cohomology to be $H^i(X_{\text{ét}}, \mathbb{Z}(n)) = H^{i-1}(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(n)')$. Then the formula (1.5) has the analog

$$H^i\left((\mathbb{P}_X^m)_{\text{ét}}, \mathbb{Z}(n)\right) \cong \bigoplus_{j=0}^{m} H^{i-2j}\left(X_{\text{ét}}, \mathbb{Z}(n-j)\right)\ . \tag{1.19}$$

Indeed, it suffices to show this after tensoring with $\mathbb{Q}$, and with finite coefficients $\mathbb{Z}/l^r$ for all primes l. Rationally, the formula holds by (1.5) and (1.13). With $\mathbb{Z}/l^r$-coefficients, it follows by (1.14) and (1.18) from the projective bundle formula for étale cohomology [71, Prop. VI 10.1] and logarithmic de Rham–Witt cohomology [40].

Similarly, the formula (1.6) for the blow-up X' of a smooth variety X in a smooth subscheme Z of codimension c has the analog

$$H^i\left(X'_{\text{ét}}, \mathbb{Z}(n)\right) \cong H^i\left(X_{\text{ét}}, \mathbb{Z}(n)\right) \oplus \bigoplus_{j=1}^{d-1} H^{i-2j}\left(Z_{\text{ét}}, \mathbb{Z}(i-j)\right)\ . \tag{1.20}$$

This follows rationally from (1.6), and with finite coefficients by the proper base-change for $\mu_m^{\otimes n}$ and [40, Cor. IV 1.3.6].

Mixed Characteristic 1.3.4

If X is an essentially smooth scheme over the spectrum B of a a Dedekind ring, one can show that the Bloch–Kato conjecture (1.10) implies the following sheaf theoretic properties of the motivic complex [28].

Purity: Let $i : Y \to X$ be the inclusion of one of the closed fibers. Then the map induced by adjointness from the natural inclusion map is a quasi-isomorphism

$$\mathbb{Z}(n-1)^{\text{ét}}[-2] \to \tau_{\leq n+1} Ri^! \mathbb{Z}(n)^{\text{ét}} . \tag{1.21}$$

Beilinson–Lichtenbaum: The canonical map is a quasi-isomorphism

$$\mathbb{Z}(n)^{\text{Zar}} \xrightarrow{\sim} \tau_{\leq n+1} R\varepsilon_* \mathbb{Z}(n)^{\text{ét}} .$$

Rigidity: For an essentially smooth henselian local ring R over B with residue field k and $m \in k^\times$, the canonical map is a quasi-isomorphism

$$H^i\left(R, \mathbb{Z}/m(n)\right) \xrightarrow{\sim} H^i\left(k, \mathbb{Z}/m(n)\right) .$$

Étale sheaf: There is a quasi-isomorphism of complexes of étale sheaves on $X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{m}]$

$$\mathbb{Z}/m(n)^{\text{ét}} \cong \mu_m^{\otimes n} .$$

Gersten resolution: For any m, there is an exact sequence

$$0 \to \mathcal{H}^s(\mathbb{Z}/m(n)^{\text{Zar}}) \to \bigoplus_{x \in X^{(0)}} (i_x)_* H^s\left(k(x), \mathbb{Z}/m(n)\right)$$
$$\to \bigoplus_{x \in X^{(1)}} (i_x)_* H^{s-1}\left(k(x), \mathbb{Z}/m(n-1)\right) \to \cdots .$$

Combining the above, one gets a Gersten resolution for the sheaf $R^s\varepsilon_*\mu_m^{\otimes n}$ for $s \leq n$, m invertible on X, and $\varepsilon : X_{\text{ét}} \to X_{\text{Zar}}$ the canonical map. This extends the result of Bloch and Ogus [11], who consider smooth schemes over a field.

If X is a smooth scheme over a discrete valuation ring V of mixed characteristic $(0, p)$ with closed fiber $i : Z \to X$ and generic fiber $j : U \to X$, and if (1.21) is a quasi-isomorphism, then the (truncated) decomposition triangle $Ri^! \to i^* \to i^*Rj_*j^*$ gives a distinguished triangle

$$\cdots \to i^*\mathbb{Z}/p^r(n)^{\text{ét}} \to \tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n} \to \nu_r^{n-1}[-n] \to \cdots . \tag{1.22}$$

By a result of Kato and Kurihara [56], for $n < p-1$ the *syntomic complex* $S_r(n)$ of Fontaine–Messing [20] fits into a similar triangle

$$\cdots \to S_r(n) \to \tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n} \xrightarrow{\kappa} \nu_r^{n-1}[-n] \to \cdots . \tag{1.23}$$

Here κ is the composition of the projection $\tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n} \to i^* R^n j_* \mu_{p^r}^{\otimes n}[-n]$ with the symbol map of [9, §6.6]. More precisely, $i^* R^n j_* \mu_{p^r}^{\otimes n}[-n]$ is locally generated by symbols $\{f_1, \ldots, f_n\}$, for $f_i \in i^* j_* \mathcal{O}_U^\times$ by [9, Cor. 6.1.1]. By multilinearity, each such symbol can be written as a sum of symbols of the form $\{f_1, \ldots, f_n\}$ and $\{f_1, \ldots, f_{n-1}, \pi\}$, for $f_i \in i^* \mathcal{O}_X^\times$ and π a uniformizer of V. Then κ sends the former to zero, and the latter to $d \log \overline{f}_1 \wedge \ldots \wedge d \log \overline{f}_{n-1}$, where $\overline{f}_i$ is the reduction of f_i to $\mathcal{O}_Y^\times$.

For $n \geq p-1$, we extend the definition of the syntomic complex $S_r(n)$ by defining it as the cone of the map κ. This cone has been studied by Sato [81]. Comparing the triangles (1.22) and (1.23), one can show [28] that there is a unique map

$$i^* \mathbb{Z}/p^r(n)^{\text{ét}} \to S_r(n)$$

in the derived category of sheaves on $Y_{\text{ét}}$, which is compatible with the maps of both complexes to $\tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n}$. The map is a quasi-isomorphism provided that the Bloch–Kato conjecture with mod p-coefficients holds. Thus motivic cohomology can be thought of as a generalization of syntomic cohomology, as anticipated by Milne [70, Remark 2.7] and Schneider [82]. As a special case, we get for smooth and projective X over V the *syntomic cycle map*

$$H^i(X, \mathbb{Z}(n)) \to H^i(X_{\text{Zar}}, S_r(n)) .$$

1.4 *K*-Theory

The first satisfactory construction of algebraic K-groups of schemes was the *Q-construction* of Quillen [79]. Given a scheme X, one starts with the category $\mathcal{P}$ of locally free $\mathcal{O}_X$-modules of finite rank on X, and defines an intermediate category $Q\mathcal{P}$ with the same objects, and where a morphism $P \to P'$ is defined to be an isomorphism of P with a sub-quotient of P'. Any (small) category $\mathcal{C}$ gives rise to a simplicial set, the *nerve* $N\mathcal{C}$. An n-simplex of the nerve is a sequence of maps $C_0 \to C_1 \to \cdots \to C_n$ in $\mathcal{C}$, and the degeneracy and face maps are defined by including an identity $C_i \stackrel{\text{id}}{\to} C_i$, and contracting $C_i \stackrel{f_i}{\to} C_{i+1} \stackrel{f_{i+1}}{\to} C_{i+2}$ to $C_i \stackrel{f_{i+1} \circ f_i}{\to} C_{i+2}$, respectively. The K-groups $K_i(X)$ of X are the homotopy groups $\pi_{i+1} BQ\mathcal{P}$ of the geometric realization $BQ\mathcal{P}$ of the nerve $NQ\mathcal{P}$ of $Q\mathcal{P}$. Algebraic K'-groups are defined similarly using the category of coherent $\mathcal{O}_X$-modules on X. If X is regular, then $K(X)$ and $K'(X)$ are homotopy equivalent, because every coherent $\mathcal{O}_X$-module has a finite resolution by finitely generated locally free $\mathcal{O}_X$-modules, hence one can apply the resolution theorem of Quillen [79, §4, Cor. 2]. The functor K' has properties analogous to the properties of Bloch's higher Chow groups.

For a ring R, a different construction of $K_i(R)$ is the *+-construction* [78]. It is defined by modifying the classifying space $BGL(R)$ of the infinite general linear group $GL(R) = \operatorname{colim}_i GL_i(R)$ to a space $BGL^+(R)$, which has the same homology groups as $BGL(R)$, but abelian fundamental group. The K-groups of R

are $K_i(R) = \pi_i BGL(R)^+$, and by [38] they agree with the groups defined using the Q-construction for $X = \operatorname{Spec} R$. The Q-construction has better functorial properties, whereas the $+$-construction is more accessible to calculations. For example, Quillen calculates the K-theory of finite fields in [78] using the $+$-construction. This was the only type of ring for which the K-theory was completely known, until 25 years later the K-theory of truncated polynomial algebras over finite fields was calculated [47]. It takes deep results on topological cyclic homology and 25 pages of calculations to determine $K_3(\mathbb{Z}/9\mathbb{Z})$ using the $+$-construction [27].

Waldhausen [105] gave an improved version of the Q-construction, called the *S-construction*, which gives a symmetric spectrum in the sense of [49], see [30, Appendix]. It also allows categories with more general weak equivalences than isomorphisms as input, for example categories of complexes and quasi-isomorphisms. Using this and ideas from [3], Thomason [97, §3] gave the following, better behaved definition of K-theory. For simplicity we assume that the scheme X is noetherian. The K'-theory of X is the Waldhausen K-theory of the category of complexes, which are quasi-isomorphic to a bounded complex of coherent $\mathcal{O}_X$-modules. This definition gives the same homotopy groups as Quillen's construction. The K-theory of X is the Waldhausen K-theory of the category of perfect complexes, i.e. complexes quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank. If X has an ample line bundle, which holds for example if X is quasi-projective over an affine scheme, or separated, regular and noetherian [97, §2], then this agrees with the definition of Quillen.

The *K-groups with coefficients* are the homotopy groups of the smash product $K/m(X) := K(X) \wedge M_m$ of the K-theory spectrum and the Moore spectrum. There is a long exact sequence

$$\cdots \to K_i(X) \xrightarrow{\times m} K_i(X) \to K_i(X, \mathbb{Z}/m) \to K_{i-1}(X) \to \cdots .$$

and similarly for K'-theory. We let $K_i(X, \mathbb{Z}_p)$ be the homotopy groups of the homotopy limit $\operatorname{holim}_n K/p^n(X)$. Then the homotopy groups are related by the Milnor exact sequence [14]

$$0 \to \lim_n{}^1 K_{i+1}(X, \mathbb{Z}/p^n) \to K_i(X, \mathbb{Z}_p) \to \lim_n K_i(X, \mathbb{Z}/p^n) \to 0 . \tag{1.24}$$

The K-groups with coefficients satisfy all of the properties given below for K-groups, except the product structure in case that m is divisible by 2 but not by 4, or by 3 but not by 9.

Basic Properties 1.4.1

By [79, §7.2] and [97, §3], the functor K is contravariantly functorial, and the functor K' is contravariantly functorial for maps $f : X \to Y$ of finite Tor-dimension, i.e. $\mathcal{O}_X$ is of finite Tor-dimension as a module over $f^{-1}\mathcal{O}_Y$. The functor K' is covariant functorial for proper maps, and K is covariant for proper maps of finite Tor-dimension.

Waldhausen S-construction gives maps of symmetric spectra [30, 104, §9]

$$K(X) \wedge K(Y) \to K(X \times Y)$$

$$K(X) \wedge K'(Y) \to K'(X \times Y),$$

which induces a product structure on algebraic K-theory, and an action of K-theory on K'-theory, respectively. If we want to define a product using the Q-construction, then because $K_i(X) = \pi_{i+1} BQ\mathcal{P}_X$, one needs a map from $BQ\mathcal{P}_X \times BQ\mathcal{P}_Y$ to a space $\mathcal{C}$ such that $K_i(X \times Y) = \pi_{i+2}\mathcal{C}$, i.e. a delooping $\mathcal{C}$ of $BQ\mathcal{P}_{X\times Y}$. Thus products can be defined more easily with the Thomason–Waldhausen construction. There is a product formula [97, §3]: If $f : X \to Y$ is proper, $y \in K_i(Y)$ and $x \in K'_j(X)$, then $f_*(f^* y \cdot x) = y \cdot f_* x$. The analogous result holds for $x \in K_j(X)$, if f is proper and of finite Tor-dimension.

The functor K' is *homotopy invariant* [79, §7, Pro. 4.1], i.e. for a flat map $f : E \to X$ whose fibers are affine spaces, the pull-back map induces an isomorphism $f^* : K'_i(X) \xrightarrow{\sim} K'_i(E)$. The projective bundle formula holds for K' and K: If $\mathcal{E}$ is a vector bundle of rank n on a noetherian separated scheme X, and $\mathbb{P}\mathcal{E} \xrightarrow{p} X$ the corresponding projective space, then there is an isomorphism [79, §7, Prop. 4.3]

$$K'_i(X)^n \xrightarrow{\sim} K'_i(\mathbb{P}\mathcal{E})$$

$$(x_i) \mapsto \sum_{i=0}^{n-1} p^*(x_i)[\mathcal{O}(-i)].$$

If X is a quasi-compact scheme then the analog formula holds for K-theory [79, §8, Thm. 2.1].

If $i : Z \to X$ is a regular embedding of codimension c (see the appendix for a definition), and X' is the blow-up of X along Z and $Z' = Z \times_X X'$, then we have the blow-up formula [96]

$$K_n(X') \cong K_n(X) \oplus K_n(Z)^{\oplus c-1} .$$

1.4.2 Localization

For Y a closed subscheme of X with open complement U, there is a localization sequence for K'-theory [79, §7, Prop. 3.2]

$$\cdots \to K'_{i+1}(U) \to K'_i(Y) \to K'_i(X) \to K'_i(U) \to \cdots .$$

In particular, K'-theory satisfies the Mayer–Vietoris property. If X is a noetherian and finite dimensional scheme, the construction of Brown and Gersten [15] then gives a spectral sequence

$$E_2^{s,t} = H^s(X_{\mathrm{Zar}}, \mathcal{K}'_{-t}) \Rightarrow K'_{-s-t}(X) . \tag{1.25}$$

Here $\mathcal{K}'_i$ is the sheaf associated to the presheaf $U \mapsto K'_i(U)$. A consequence of the main result of Thomason [97, Thm. 8.1] is that the modified K-groups K^B also satisfy the Mayer–Vietoris property, hence there is a spectral sequence analogous to (1.25). Here K^B is Bass-K-theory, which can have negative homotopy groups, but satisfies $K_i(X) \cong K_i^B(X)$ for $i \geq 0$. See Carlson's article [16] in this handbook for more on negative K-groups.

Gersten Resolution 1.4.3

As in (1.3), filtration by coniveau gives a spectral sequence [79, §7, Thm. 5.4]

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} K_{-s-t}\left(k(x)\right) \Rightarrow K'_{-s-t}(X) \,. \tag{1.26}$$

If X is smooth over a field, then as in (1.4), the spectral sequence (1.26) degenerates at E_2 for every semi-local ring of X, and we get the Gersten resolution [79, §7, Prop. 5.8, Thm. 5.11]

$$0 \to \mathcal{K}_i \to \bigoplus_{x \in X^{(0)}} i_* K_i\left(k(x)\right) \to \bigoplus_{x \in X^{(1)}} i_* K_{i-1}\left(k(x)\right) \to \cdots \,. \tag{1.27}$$

Because skyscraper sheaves are flabby, one can calculate cohomology of $\mathcal{K}_i$ with (1.27) and gets $E_2^{s,t} = H^s(X_{\mathrm{Zar}}, \mathcal{K}_{-t})$. By [37, Thm. 2, 4 (iv)] there is always a map from the spectral sequence (1.25) to the spectral sequence (1.26), and for smooth X the two spectral sequences agree from E_2 on. If X is essentially smooth over a discrete valuation ring V of mixed characteristic $(0, p)$, then the Gersten resolution exists with finite coefficients. The case $p \nmid m$ was treated by Gillet and Levine [35, 36], and the case $m = p^r$ in [33]. The correponding result is unknown for rational, hence integral coefficients.

The product structure of K-theory induces a canonical map from Milnor K-theory of fields to Quillen K-theory. Via the Gersten resolutions, this gives rise to a map $K_n^M(R) \to K_n(R)$ for any regular semi-local ring essentially of finite type over a field.

Motivic Cohomology and K-Theory 1.4.4

If X is of finite type over a discrete valuation ring, then Bloch's higher Chow groups and algebraic K'-theory are related by a spectral sequence

$$E_2^{s,t} = H^{s-t}\left(X, \mathbb{Z}(-t)\right) \Rightarrow K'_{-s-t}(X) \,. \tag{1.28}$$

This is an analog of the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K-theory. Consequences of the existence of the spectral sequence for K-theory had been observed previous to the definition of motivic cohomology. By [37, Thm. 7], there are *Adams* operators acting on the E_r-terms of the spectral sequence compatible with the action on the abutment. In particular, the spectral sequence degenerates after tensoring with $\mathbb{Q}$ by the argument of [86], and the induced filtration agrees with the γ-filtration. The resulting graded pieces have been used as a substitute for motivic cohomology before its definition.

The existence of the spectral sequence has been conjectured by Beilinson [2], and first been proved by Bloch and Lichtenbaum for fields [10]. Friedlander–Suslin [21] and Levine [59] used their result to generalize this to varieties over fields and discrete valuation rings, respectively. There are different methods of constructing the spectral by Grayson–Suslin [39, 90] and Levine [60], which do not use the theorem of Bloch and Lichtenbaum.

Using the spectral sequence (1.28), we can translate results on motivic cohomology into results on K-theory. For example, Parshin's conjecture states that for a smooth projective variety over a finite field, $K_i(X)$ is torsion for $i > 0$, and this implies that for a field F of characteristic p, $K_i^M(F) \otimes \mathbb{Q} \cong K_i(F) \otimes \mathbb{Q}$.

1.4.5 Sheaf Theoretic Properties

It is a result of Gabber [24] and Suslin [88] that for every henselian pair (A, I) with m invertible in A, and for all $i \geq 0$,

$$K_i(A, \mathbb{Z}/m) \xrightarrow{\sim} K_i(A/I, \mathbb{Z}/m) \ . \tag{1.29}$$

Together with Suslin's calculation of the K-theory of an algebraically closed field [88], this implies that if m is invertible on X, then the étale K-theory sheaf with coefficients in the sheaf $(\mathcal{K}/m)_n$, associated to the presheaf $U \mapsto K_m(U, \mathbb{Z}/m)$ can be described as follows:

$$(\mathcal{K}/m)_n = \begin{cases} \mu_m^{\otimes \frac{n}{2}} & n \geq 0 \text{ even,} \\ 0 & n \text{ odd.} \end{cases} \tag{1.30}$$

One can use the spectral sequence (1.28) to deduce this from (1.14), but historically the results for K-theory were proved first, and then the analogous results for motivic cohomology followed. However, for mod p-coefficients, Theorem (1.17) was proved first, and using the spectral sequence (1.28), it has the following consequences for K-theory: For any field F of characteristic p, the groups $K_n^M(F)$ and $K_n(F)$ are p-torsion free. The natural map $K_n^M(F)/p^r \to K_n(F, \mathbb{Z}/p^r)$ is an isomorphism, and the natural map $K_n^M(F) \to K_n(F)$ is an isomorphism up to uniquely p-divisible groups. Finally, for a smooth variety X over a perfect field of characteristic p, the K-theory sheaf for the Zariski or étale topology is given by

$$(\mathcal{K}/p^r)_n \cong \nu_r^n \ . \tag{1.31}$$

In particular, the spectral sequence (1.25) takes the form

$$E_2^{s,t} = H^s(X_{\text{Zar}}, \nu_r^{-t}) \Rightarrow K_{-s-t}(X, \mathbb{Z}/p^r) \ .$$

and $K_n(X, \mathbb{Z}/p^r) = 0$ for $n > \dim X$.

In a generalization of (1.29) to the case where m is not invertible, Suslin and Panin [76, 88] show that for a henselian valuation ring V of mixed characteristic $(0,p)$ with maximal ideal I, one has an isomorphism of pro-abelian groups

$$K_i\left(V,\mathbb{Z}/p^r\right) \cong K_i\left(\lim_s V/I^s,\mathbb{Z}/p^r\right) \cong \left\{K_i\left(V/I^s,\mathbb{Z}/p^r\right)\right\}_s .$$

In [31], the method of Suslin has been used in the following situation. Let R be a local ring, such that (R,pR) is a henselian pair, and such that p is not a zero divisor. Then the reduction map

$$K_i\left(R,\mathbb{Z}/p^r\right) \to \left\{K_i\left(R/p^s,\mathbb{Z}/p^r\right)\right\}_s \tag{1.32}$$

is an isomorphism of pro-abelian groups.

Etale *K*-Theory and Topological Cyclic Homology 1.5

We give a short survey of Thomason's construction of *hyper-cohomology spectra* [95, §1], which is a generalization of Godement's construction of hyper-cohomology of a complex of sheaves. Let X_τ be a site with Grothendieck topology τ on the scheme X, $X_\tau^\sim$ the category of sheaves of sets on X_τ, and $\mathfrak{S}ets$ the category of sets. A *point* of X consists of a pair of adjoint functors $p^* : X_\tau^\sim \to \mathfrak{S}ets$ and $p_* : \mathfrak{S}ets \to X_\tau^\sim$, such that the left adjoint p^* commutes with finite limits. We say X_{tau} has enough points if we can find a set of points $\mathcal{P}$ such that a morphism α of sheaves is an isomorphism provided that $p^*\alpha$ is an isomorphism for all points $p \in \mathcal{P}$. For example, the Zariski site X_{Zar} and étale site $X_{\text{ét}}$ on a scheme X have enough points (points of X_{Zar} are points of X and points of $X_{\text{ét}}$ are geometric points of X, p^* is the pull-back and p_* the push-forward along the inclusion map $p : \operatorname{Spec} k \to X$ of the residue field).

Let $\mathcal{F}$ be a presheaf of spectra on X_τ. Given $p \in \mathcal{P}$, we can consider the presheaf of spectra $p_*p^*\mathcal{F}$, and the endo-functor on the category of presheaves of spectra $T\mathcal{F} = \prod_{p\in\mathcal{P}} p_*p^*\mathcal{F}$. The adjunction morphisms $\eta : \mathrm{id} \to p_*p^*$ and $\varepsilon : p^*p_* \to \mathrm{id}$ induce natural transformation $\eta : \mathrm{id} \to T$ and $\mu = p_*\varepsilon p^* : TT \to T$. Thomason defines $T^\cdot\mathcal{F}$ as the cosimplicial presheaf of spectra $n \mapsto T^{n+1}\mathcal{F}$, where the coface maps are $d_n^i = T^i\eta T^{n+1-i}$ and the codegeneracy maps are $s_n^i = T^i\mu T^{n-i}$. The map η induces an augmentation $\eta : \mathcal{F} \to T^\cdot\mathcal{F}$, and since $T\mathcal{F}$ only depends on the stalks of $\mathcal{F}$, $T^\cdot\mathcal{F}$ for a presheaf $\mathcal{F}$ only depends on the sheafification of $\mathcal{F}$.

The hyper-cohomology spectrum of $\mathcal{F}$ is defined to be the homotopy limit of the simplicial spectrum $T^\cdot\mathcal{F}(X)$,

$$\mathbb{H}^\cdot(X_\tau,\mathcal{F}) := \operatorname{holim} T^\cdot\mathcal{F}(X) .$$

It comes equipped with a natural augmentation $\eta : \mathcal{F}(X) \to \mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$, and if $\mathcal{F}$ is contravariant in X_τ, then so is $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$. One important feature of $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$ is that it admits a spectral sequence [95, Prop. 1.36]

$$E_2^{s,t} = H^s(X_\tau, \widetilde{\pi}_{-t}\mathcal{F}) \Rightarrow \pi_{-s-t}\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}) , \qquad (1.33)$$

where $\widetilde{\pi}_i\mathcal{F}$ is the sheaf associated to the presheaf of homotopy groups $U \mapsto \pi_i\mathcal{F}(U)$. The spectral sequence converges strongly, if, for example, X_τ has finite cohomological dimension.

1.5.1 Continuous Hyper-cohomology

Let $\mathcal{A}$ be the category of (complexes of) sheaves of abelian groups on X_τ and consider the category $\mathcal{A}^{\mathbb{N}}$ of pro-sheaves. A pro-sheaf on the site X_τ is the same as a sheaf on the site $X_\tau \times \mathbb{N}$, where $\mathbb{N}$ is the category with objects $[n]$, a unique map $[n] \to [m]$ if $n \leq m$, and identity maps as coverings. If $\mathcal{A}$ has enough injectives, then so does $\mathcal{A}^{\mathbb{N}}$, and Jannsen [51] defines the *continuous cohomology group* $H^j_{cont}(X_\tau, A^{\cdot})$ to be the j-th derived functor of $A^{\cdot} \mapsto \lim_r \Gamma(X, A^r)$. There are exact sequences

$$0 \to \lim_r{}^1 H^{j-1}(X_\tau, A^r) \to H^j_{cont}(X_\tau, A^{\cdot}) \to \lim_r H^j(X_\tau, A^r) \to 0 . \qquad (1.34)$$

If $\mathbb{Z}/l^r(n)$ is the (étale) motivic complex mod l^r, then we abbreviate

$$H^i(X_{\text{ét}}, \mathbb{Z}_l(n)) := H^i_{cont}\left(X_{\text{ét}}, \mathbb{Z}/l^{\cdot}(n)\right) . \qquad (1.35)$$

In view of (1.14) and (1.18), this is consistent with the usual definition of the left hand side.

Given a pro-presheaf $\mathcal{F}^{\cdot}$ of spectra on X_τ, one gets the hyper-cohomology spectrum $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^{\cdot}) := \operatorname{holim}_r \mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^r)$. The corresponding spectral sequence takes the form [30]

$$E_2^{s,t} = H^s_{cont}(X_\tau, \widetilde{\pi}_{-t}\mathcal{F}^{\cdot}) \Rightarrow \pi_{-s-t}\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^{\cdot}) . \qquad (1.36)$$

1.5.2 Hyper-cohomology of K-Theory

If X_{Zar} is the Zariski site of a noetherian scheme of finite dimension, then by Thomason [95, 2.4] [97, Thm. 10.3], the augmentation maps

$$\eta : K'(X) \to \mathbb{H}^{\cdot}(X_{\text{Zar}}, K')$$
$$\eta : K^B(X) \to \mathbb{H}^{\cdot}(X_{\text{Zar}}, K^B)$$

are homotopy equivalences, and the spectral sequence (1.25) and (1.33) agree. By Nisnevich [75], the analogous result holds for the Nisnevich topology.

If $X_{\text{ét}}$ is the small étale site of the scheme X, then we write $K^{\text{ét}}(X)$ for $\mathbb{H}^{\cdot}(X_{\text{ét}}, K)$ and $K^{\text{ét}}(X, \mathbb{Z}_p)$ for $\operatorname{holim}_r \mathbb{H}^{\cdot}(X_{\text{ét}}, K/p^r)$. A different construction of étale K-theory was given by Dwyer and Friedlander [18], but by Thomason [95, Thm. 4.11] the two theories agree for a separated, noetherian, and regular scheme X of finite Krull dimension with l invertible on X. If m and l are invertible on X, then in view of (1.30), the spectral sequences (1.33) and (1.36) take the form

$$E_2^{s,t} = H^s\left(X_{\text{ét}}, \mu_m^{\otimes -(t/2)}\right) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbb{Z}/m) \tag{1.37}$$

$$E_2^{s,t} = H^s\left(X_{\text{ét}}, \mathbb{Z}_l(-\tfrac{t}{2})\right) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbb{Z}_l). \tag{1.38}$$

Similarly, if X is smooth over a field of characteristic p, then by (1.31) and (1.35) there are spectral sequences

$$E_2^{s,t} = H^s\left(X_{\text{ét}}, \nu_r^{-t}\right) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbb{Z}/p^r), \tag{1.39}$$

$$E_2^{s,t} = H^{s-t}\left(X_{\text{ét}}, \mathbb{Z}_p(-t)\right) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbb{Z}_p). \tag{1.40}$$

The Lichtenbaum–Quillen Conjecture 1.5.3

The *Lichtenbaum–Quillen conjecture* (although never published by either of them in this generality) is the K-theory version of the Beilinson–Lichtenbaum conjecture, and predates it by more than 20 years. It states that on a regular scheme X, the canonical map from K-theory to étale K-theory

$$K_i(X) \to K_i^{\text{ét}}(X)$$

is an isomorphism for sufficiently large i (the cohomological dimension of X is expected to suffice). Since rationally, K-theory and étale K-theory agree [95, Thm. 2.15], one can restrict oneself to finite coefficients. If X is smooth over a field k of characteristic p, then with mod p^r-coefficents, the conjecture is true by (1.31) for $i > \text{cd}_p\, k + \dim X$, because both sides vanish. Here $\text{cd}_p\, k$ is the p-cohomological dimension of k, which is the cardinality of a p-base of k (plus one in certain cases). For example, $\text{cd}_p\, k = 0$ if k is perfect.

Levine has announced a proof of an étale analog of the spectral sequence from Bloch's higher Chow groups to algebraic K'-theory for a smooth scheme X over a discrete valuation ring (and similarly with coefficients)

$$E_2^{s,t} = H^{s-t}\left(X_{\text{ét}}, \mathbb{Z}(-t)\right) \Rightarrow K^{\text{ét}}_{-s-t}(X)\ . \tag{1.41}$$

Comparing the mod m-version of the spectral sequences (1.28) and (1.41), one can deduce that the Beilinson–Lichtenbaum conjecture implies the Lichtenbaum–Quillen conjecture with mod m-coefficients for $i \geq \text{cd}_m X_{\text{ét}}$.

If one observes that in the spectral sequence (1.37) $E_2^{s,t} = E_3^{s,t}$ and reindexes $s = p - q, t = 2q$, then we get a spectral sequence with the same E_2-term as (1.41) with mod m-coefficients, but we don't know if the spectral sequences agree. Similarly, the E_2-terms of the spectral sequences (1.39) and (1.41) with mod p^r-coefficients agree, but we don't know if the spectral sequences agree.

1.5.4 Topological Cyclic Homology

Bökstedt, Hsiang and Madsen [12] define *topological cyclic homology* for a ring A. Bökstedt first defines topological Hochschild homology $\mathrm{TH}(A)$, which can be thought of as a topological analog of Hochschild homology, see [30] [45, §1]. It is the realization of a *cyclic spectrum* (i.e. a simplicial spectrum together with maps $\tau : [n] \to [n]$ satisfying certain compatibility conditions with respect to the face and degeneracy maps), hence $\mathrm{TH}(A)$ comes equipped with an action of the circle group S^1 [65]. Using Thomason's hyper-cohomology construction, one can extend this definition to schemes [30]: On a site X_τ, one considers the presheaf of spectra

$$\mathrm{TH} : U \mapsto \mathrm{TH}\left(\Gamma(U, \mathcal{O}_U)\right) .$$

and defines

$$\mathrm{TH}(X_\tau) = \mathbb{H}^{\cdot}(X_\tau, \mathrm{TH}) . \tag{1.42}$$

If the Grothendieck topology τ on the scheme X is coarser than or equal to the étale topology, then $\mathrm{TH}(X_\tau)$ is independent of the topology [30, Cor. 3.3.3], and accordingly we drop τ from the notation. If X is the spectrum of a ring A, then $\mathrm{TH}(A) \xrightarrow{\sim} \mathrm{TH}(X)$, [30, Cor. 3.2.2].

To define topological cyclic homology [45, §6], we let $\mathrm{TR}^m(X;p)$ be the fixed point spectrum under the cyclic subgroup of roots of unity $\mu_{p^{m-1}} \subseteq S^1$ acting on $\mathrm{TH}(X)$. If X is the spectrum of a ring A, the group of components $\pi_0 \,\mathrm{TR}^m(A;p)$ is isomorphic to the Witt vectors $W_m(A)$ of length m of A [46, Thm. F]. The maps F, V, R on $W_m(A) = \pi_0 \,\mathrm{TR}^m(A;p)$ are induced by maps of spectra: The inclusion of fixed points induces the map

$$F : \mathrm{TR}^m(X;p) \to \mathrm{TR}^{m-1}(X;p)$$

called *Frobenius*, and one can construct the *restriction map*

$$R : \mathrm{TR}^m(X;p) \to \mathrm{TR}^{m-1}(X;p)$$

and the *Verschiebung map*

$$V : \mathrm{TR}^m(X;p) \to \mathrm{TR}^{m+1}(X;p) .$$

Note that F, V exist for all cyclic spectra, whereas the existence of R is particular to the topological Hochschild spectrum. The two composities FV and VF induce multiplication by p and $V(1) \in \pi_0 TR^m(X;p)$, respectively, on homotopy groups. *Topological cyclic homology* $\mathrm{TC}^m(X;p)$ is the homotopy equalizer of the maps

$$F, R : \mathrm{TR}^m(X;p) \to \mathrm{TR}^{m-1}(X;p) .$$

We will be mainly interested in the version with coefficients

$$\mathrm{TR}^m(X;p,\mathbb{Z}/p^r) = \mathrm{TR}^m(X;p) \wedge M_{p^r},$$

$$\mathrm{TC}^m(X;p,\mathbb{Z}/p^r) = \mathrm{TC}^m(X;p) \wedge M_{p^r} ,$$

and its homotopy groups

$$\mathrm{TR}^m_i(X;p,\mathbb{Z}/p^r) = \pi_i \, \mathrm{TR}^m(X;p,\mathbb{Z}/p^r) ,$$

$$\mathrm{TC}^m_i(X;p,\mathbb{Z}/p^r) = \pi_i \, \mathrm{TC}^m(X;p,\mathbb{Z}/p^r) .$$

We view $\mathrm{TR}^{\cdot}(X;p,\mathbb{Z}/p^r)$ and $\mathrm{TC}^{\cdot}(X;p,\mathbb{Z}/p^r)$ as pro-spectra with R as the structure map. Then F and V induce endomorphisms of the pro-spectrum $\mathrm{TR}^{\cdot}(X;p,\mathbb{Z}/p^r)$. The homotopy groups fit into a long exact sequence of pro-abelian groups

$$\cdots \to \mathrm{TC}^{\cdot}_i(X;p,\mathbb{Z}/p^r) \to \mathrm{TR}^{\cdot}_i(X;p,\mathbb{Z}/p^r) \xrightarrow{F-1} \mathrm{TR}^{\cdot}_i(X;p,\mathbb{Z}/p^r) \to \cdots .$$

Finally, one can take the homotopy limit and define

$$\mathrm{TR}(X;p,\mathbb{Z}_p) = \operatorname*{holim}_{m,r} \mathrm{TR}^m(X;p,\mathbb{Z}/p^r) ,$$

$$\mathrm{TC}(X;p,\mathbb{Z}_p) = \operatorname*{holim}_{m,r} \mathrm{TC}^m(X;p,\mathbb{Z}/p^r) .$$

The corresponding homotopy groups again fit into a long exact sequence. If we let $(\mathrm{TC}^m/p^r)_i$ be the sheaf associated to the presheaf $U \mapsto \mathrm{TC}^m_i(U;p,\mathbb{Z}/p^r)$, then (1.33) and (1.36) take the form

$$E_2^{s,t} = H^s(X_\tau, (\mathrm{TC}^m/p^r)_{-t}) \Rightarrow \mathrm{TC}^m_{-s-t}(X;p,\mathbb{Z}/p^r) , \tag{1.43}$$

$$E_2^{s,t} = H^s_{cont}(X_\tau, (\mathrm{TC}^{\cdot}/p^{\cdot})_{-t}) \Rightarrow \mathrm{TC}_{-s-t}(X;p,\mathbb{Z}_p) , \tag{1.44}$$

and similarly for TR. The spectral sequences differ for different Grothendieck topologies τ on X, even though the abutment does not [30].

Topological cyclic homology comes equipped with the *cyclotomic trace map*

$$tr : K(X,\mathbb{Z}_p) \to \mathrm{TC}(X;p,\mathbb{Z}_p) ,$$

which factors through étale K-theory because topological cyclic homology for the Zariski and the étale topology agree. The map induces an isomorphism of homotopy groups in many cases, a fact which is useful to calculate K-groups. To show that the trace map is an isomorphism, the following result of McCarthy is the starting point [67]. If R is a ring and I a nilpotent ideal, then the following diagram is homotopy cartesian

$$\begin{array}{ccc} K(R,\mathbb{Z}_p) & \xrightarrow{tr} & \mathrm{TC}(R;p,\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ K(R/I,\mathbb{Z}_p) & \xrightarrow{tr} & \mathrm{TC}(R/I;p,\mathbb{Z}_p) . \end{array} \tag{1.45}$$

In particular, if the lower map is a homotopy equivalence, then so is the upper map. In [46], Hesselholt and Madsen use this to show that the trace map is an isomorphism in non-negative degrees for a finite algebra over the Witt ring of a perfect field. They apply this in [47] to calculate the K-theory with mod p coefficients of truncated polynomial algebras $k[t]/(t^i)$ over perfect fields k of characteristic p. In [48], they calculate the topological cyclic homology of a local number ring ($p \neq 2$), and verify the Lichtenbaum–Quillen conjecture for its quotient field with p-adic coefficients (prime to p-coefficients were treated in [9]). This has been generalized to certain discrete valuation rings with non-perfect residue fields in [32]. See [44] for a survey of these results.

1.5.5 Comparison

Hesselholt has shown in [43] that for a regular $\mathbb{F}_p$-algebra A, there is an isomorphism of pro-abelian groups

$$W.\Omega^i_A \xrightarrow{\sim} \mathrm{TR}^{\cdot}_i(A;p) ,$$

which is compatible with the Frobenius endomorphism on both sides. In [43] the result is stated for a smooth $\mathbb{F}_p$-algebra, but any regular $\mathbb{F}_p$-algebra is a filtered colimit of smooth ones [77, 92], and the functors on both sides are compatible with filtered colimits (at this point it is essential to work with pro-sheaves and not take the inverse limit). In particular, for a smooth scheme X over a field of characteristic p, we get from (1.15) the following diagram of pro-sheaves for the étale topology,

$$\begin{array}{ccccccccc} 0 & \to & \nu^i_{\cdot} & \to & W.\Omega^i_X & \xrightarrow{F-1} & W.\Omega^i_X & \to & 0 \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow & & \\ \cdots & \xrightarrow{\delta} & (\mathrm{TC}^{\cdot}/p^{\cdot})_i & \to & (\mathrm{TR}^{\cdot}/p^{\cdot})_i & \xrightarrow{F-1} & (\mathrm{TR}^{\cdot}/p^{\cdot})_i & \xrightarrow{\delta} & \cdots . \end{array}$$

This shows that $\delta = 0$, and that there is an isomorphism of pro-étale sheaves

$$\left\{\nu^i_r\right\}_r \cong \left\{(\mathrm{TC}^m/p^r)_i\right\}_{m,r} . \tag{1.46}$$

The spectral sequence (1.43) becomes

$$E_2^{s,t} = H^s_{cont}(X, \nu^{-t}_{\cdot}) \Rightarrow \mathrm{TC}_{-s-t}(X;p,\mathbb{Z}_p) . \tag{1.47}$$

and the cyclotomic trace map from étale K-theory to topological cyclic homology

$$K_i^{\text{ét}}(X,\mathbb{Z}_p) \xrightarrow{tr} \mathrm{TC}_i(X;p,\mathbb{Z}_p) \tag{1.48}$$

is an isomorphism, because the induced map on the strongly converging spectral sequences (1.40) and (1.47) is an isomorphism on E_2-terms.

If X is not smooth, then the above method does not work because we cannot identify the K-theory and TC-sheaves. However, we do not know of any example of a finitely generated algebra over a perfect field of characteristic p where the trace map (1.48) is not an isomorphism.

One can extend the isomorphism (1.48) to smooth, proper schemes X over a henselian discrete valuation ring V of mixed characteristic $(0, p)$, [31]. Comparing the hyper-cohomology spectral sequences (1.36), it suffices to show that the cyclotomic trace map induces an isomorphism on E_2-terms. If $i : Y \to X$ is the embedding of the closed fiber, then because V is henselian, the proper base change theorem together with (1.34) implies that for every pro-sheaf $\mathcal{F}^{\cdot}$ on X, $H^i_{cont}(X_{\text{ét}}, \mathcal{F}^{\cdot}) \cong H^i_{cont}(Y_{\text{ét}}, i^*\mathcal{F}^{\cdot})$. Hence it suffices to show that the cyclotomic trace map induces a pro-isomorphism of homotopy groups for an essentially smooth, strictly henselian local ring R over V. If π is a uniformizer of V, then from the characteristic p case we know that the trace map is an isomorphism for R/π. Using (a pro-version of) the method of McCarthy (1.45), this implies that the trace map is an isomorphism for R/π^s, $s \geq 1$. By (1.32), the p-adic K-theory of R is determined by the K-theory of the system $\{R/\pi^s\}_s$. The analogous statement for topological cyclic homology holds more generally: If R is a ring such that p is not a zero-divisor, then the reduction map

$$\mathrm{TC}_i^m(R; p, \mathbb{Z}/p^r) \to \left\{ \mathrm{TC}_i^m(R/p^s; p, \mathbb{Z}/p^r) \right\}_s$$

is an isomorphism of pro-abelian groups.

If the trace map (1.48) is an isomorphism for every finitely generated algebra over a perfect field of characteristic p (or any normal crossing scheme), then the same argument shows that (1.48) is an isomorphism for every proper V-scheme of finite type (or semi-stable scheme).

Appendix: Basic Intersection Theory A

In this appendix we collect some facts of intersection theory needed to work with higher Chow groups. When considering higher Chow groups, one always assumes that all intersections are proper intersections; furthermore one does not have to deal with rational equivalence. Thus the intersection theory used for higher Chow groups becomes simpler, and can be concentrated on a few pages. We hope that our treatment will allow a beginner to start working with higher Chow groups more quickly. For a comprehensive treatment, the reader should refer to the books of Fulton [23], Roberts [80], and Serre [83].

Proper Intersection A.1

To define intersections on a noetherian, separated scheme, we can always reduce ourselves to an open, affine neighborhood of the generic points of irreducible components of the intersection, hence assume that we deal with the spectrum of a noetherian ring A.

If A is a finitely generated algebra over the quotient of a regular ring R of finite Krull dimension (this is certainly true if we consider rings of finite type over a field or Dedekind ring), then A is catenary [66, Thm. 17.9]. This implies that the length of a maximal chain of prime ideals between two primes $\mathfrak{p} \subseteq \mathcal{P}$ does not depend on the chain. In particular, if A is local and equidimensional, i.e. $\dim A/\mathfrak{q}$ is equal for all minimal prime ideals $\mathfrak{q}$, then for every prime $\mathfrak{p}$ of A, the dimension equality holds

$$\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A\ . \tag{1.49}$$

In order for (1.49) to extend to rings A that are *localizations* of finitely generated algebras over the quotient of a regular ring R of finite Krull dimension d, it is necessary to modify the definition of dimension as follows, see [80, §4.3]:

$$\dim A/\mathfrak{p} := \operatorname{trdeg}\big(k(\mathfrak{p})/k(\mathfrak{q})\big) - \operatorname{ht}_R \mathfrak{q} + d\ . \tag{1.50}$$

Here $\mathfrak{q}$ is the inverse image in R of the prime ideal $\mathfrak{p}$ of A. With this modified definition, (1.49) still holds. If A is a quotient of a regular ring, or is of finite type over a field, then (1.50) agrees with the Krull dimension of $A/\mathfrak{p}$.

1 **Definition 1** We say that two closed subschemes V and W of X intersect properly, if for every irreducible component C of $V \times_X W$,

$$\dim C \le \dim V + \dim W - \dim X\ .$$

If $\operatorname{Spec} A \subseteq X$ is a neighborhood of the generic point of C, and if $V = \operatorname{Spec} A/\mathfrak{a}$ and $W = \operatorname{Spec} A/\mathfrak{b}$, then V and W meet properly, if for every minimal prime ideal $\mathcal{P} \supseteq \mathfrak{a} + \mathfrak{b}$

$$\dim A/\mathcal{P} \le \dim A/\mathfrak{a} + \dim A/\mathfrak{b} - \dim A\ . \tag{1.51}$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are primes in an equidimensional local ring, then by (1.49) this means $\operatorname{ht}\mathcal{P} \ge \operatorname{ht}\mathfrak{a} + \operatorname{ht}\mathfrak{b}$, i.e. the codimension of the intersection is at least the sum of the codimensions of the two irreducible subvarieties.

If A is regular, then it is a theorem of Serre [83, Thm. V.B.3] that the left hand side of (1.51) is always greater than or equal to the right hand side. We will see below (1.53) that the same is true if $\mathfrak{a}$ or $\mathfrak{b}$ can be generated by a regular sequence. On the other hand, in the 3-dimensional ring $k[X, Y, Z, W]/(XY - ZW)$, the subschemes defined by (X, Z) and (Y, W) both have dimension 2, but their intersection has dimension 0.

A.2 Intersection with Divisors

If $a \in A$ is neither a zero divisor nor a unit, then the divisor $D = \operatorname{Spec} A/aA$ has codimension 1 by Krull's principal ideal theorem. If $V = \operatorname{Spec} A/\mathfrak{p}$ is an irreducible subscheme of dimension r, then V and D meet properly if and only if $a \notin \mathfrak{p}$, if and only if V is not contained in D, if and only if $V \times_X D$ is empty or has dimension $r-1$.

Definition 2 If V and D intersect properly, then we define the intersection to be the cycle 2

$$[A/\mathfrak{p}]\cdot(a) = \sum_{\mathfrak{q}} \operatorname{lgth}_{A_\mathfrak{q}} \left(A_\mathfrak{q}/(\mathfrak{p}_\mathfrak{q} + aA_\mathfrak{q})\right) [A/\mathfrak{q}] \,.$$

Here $\mathfrak{q}$ runs through the prime ideals such that $\dim A/\mathfrak{q} = r-1$.

Note that only prime ideals $\mathfrak{q}$ containing $\mathfrak{p} + aA$ can have non-zero coefficient, and the definition makes sense because of [80, Cor. 2.3.3]:

Lemma 3 If $(A, \mathfrak{m})$ is a local ring, then $A/\mathfrak{a}$ has finite length if and only if $\mathfrak{m}$ is the only prime ideal of A containing $\mathfrak{a}$. 3

A sequence of elements $a_1, \ldots, a_n$ in a ring A is called a *regular sequence*, if the ideal $(a_1, \ldots, a_n)$ is a proper ideal of A, and if a_i is not a zero-divisor in $A/(a_1, \ldots, a_{i-1})$. In particular, a_i is not contained in any minimal prime ideal of $A/(a_1, \ldots, a_{i-1})$, and $\dim A/(a_1, \ldots, a_i) = \dim A - i$. If A is local, then the regularity of the sequence is independent of the order [66, Cor. 16.3]. Our principal example is the regular sequence $(t_1, \ldots, t_i)$ in the ring $A[t_0, \ldots, t_i]/(1 - \sum t_i)$.

Given a regular sequence $a_1, \ldots, a_n$, assume that the closed subscheme $V = \operatorname{Spec} A/\mathfrak{p}$ of dimension r meets all subschemes $\operatorname{Spec} A/(a_{i_1}, \ldots, a_{i_j})$ properly. This amounts to saying that the dimension of every irreducible component of $A/(\mathfrak{p}, a_{i_1}, \ldots, a_{i_j})$ has dimension $r-j$ if $r \geq j$, and is empty otherwise. In this case, we can inductively define the intersection $[V]\cdot(a_1)\cdots(a_n)$, a cycle of dimension $r-n$, using Definition 2.

Proposition 4 Let $a_1, \ldots, a_n$ be a regular sequence in A. Then the cycle $[V]\cdot(a_1)\cdots(a_n)$ is independent of the order of the a_i. 4

Proof See also [23, Thm. 2.4]. Clearly it suffices to show $[V]\cdot(a)\cdot(b) = [V]\cdot(b)\cdot(a)$. Let $\mathfrak{m}$ is a minimal ideal containing $\mathfrak{p} + aA + bA$. To calculate the multiplicity of $\mathfrak{m}$ on both sides, we can localize at $\mathfrak{m}$ and divide out $\mathfrak{p}$, which amounts to replacing X by V. Thus we can assume that A is a two-dimensional local integral domain, and (a, b) is not contained in any prime ideal except $\mathfrak{m}$. If $\mathcal{P}$ runs through the minimal prime ideals of A containing a, then the cycle $[V] \cap (a)$ is $\sum_\mathcal{P} \operatorname{lgth}_{A_\mathcal{P}}(A_\mathcal{P}/aA_\mathcal{P})[A/\mathcal{P}]$, and if we intersect this with (b) we get the multiplicity

$$\sum_{\mathcal{P}} \operatorname{lgth}_{A_\mathcal{P}}(A_\mathcal{P}/aA_\mathcal{P}) \cdot \operatorname{lgth}_{A/\mathcal{P}}(A/(\mathcal{P} + bA)) \,.$$

The following lemma applied to $A/(a)$ shows that this is $\mathrm{lgth}_A(A/(a,b)) - \mathrm{lgth}_A({}_b(A/aA))$. This is symmetric in a and b, because since A has no zero-divisors, we have the bijection ${}_b(A/aA) \to {}_a(A/bA)$, sending a b-torsion element x of A/aA to the unique $y \in A$ with $bx = ay$. The map is well-defined because $x + za$ maps to $y + zb$.

5 **Lemma 5** Let $(B, \mathfrak{m})$ be a one-dimensional local ring with minimal primes $\mathcal{P}_1, \dots, \mathcal{P}_r$, and let b be an element of B not contained in any of the $\mathcal{P}_i$. Then for every finitely generated B-module M, we have

$$\mathrm{lgth}_B(M/bM) - \mathrm{lgth}_B({}_bM) = \sum_i \mathrm{lgth}_{B_{\mathcal{P}_i}}\left(M_{\mathcal{P}_i}\right) \cdot \mathrm{lgth}_B\left(B/(\mathcal{P}_i + bB)\right) .$$

Proof See also [23, Lemma A.2.7]. Every finitely generated module M over a noetherian ring admits a finite filtration by submodules M_i such that the quotients M_i/M_{i+1} are isomorphic to $B/\mathfrak{p}$ for prime ideals $\mathfrak{p}$ of B. Since both sides are additive on short exact sequence of B-modules, we can consider the case $M = B/\mathcal{P}_i$ or $M = B/\mathfrak{m}$. If $M = B/\mathfrak{m}$, then $\mathrm{lgth}_B(M/bM) = \mathrm{lgth}_B({}_bM) = 0$ or 1 for $b \notin \mathfrak{m}$ and $b \in \mathfrak{m}$, respectively, and $M_{\mathcal{P}_i} = 0$ for all i.

For $M = B/\mathcal{P}_i$, we have $\mathrm{lgth}_{B_{\mathcal{P}_j}}(M_{\mathcal{P}_j}) = 0$ or 1 for $j \neq i$ and $j = i$, respectively, and $b \notin \mathcal{P}_i$ implies ${}_b(B/\mathcal{P}) = 0$, so that $\mathrm{lgth}_B(M/bM) - \mathrm{lgth}_B({}_bM) = \mathrm{lgth}_B(B/(\mathcal{P}_i + bB))$.

6 **Corollary 6** The cycle complex $z^n(X, *)$ is a complex.

Proof Recall that the differentials are alternating sums of intersection with face maps. Since intersecting with two faces does not depend on the order of intersection, it follows from the simplicial identities that the composition of two differentials is the zero map.

A.3 Pull-back Along a Regular Embedding

A closed embedding $i : Z \to X$ of schemes is called a *regular embedding of codimension c*, if every point of Z has an affine neighborhood $\mathrm{Spec}\, A$ in X such that the ideal $\mathfrak{a}$ of A defining Z is generated by a regular sequence of length c. If Z is smooth over a base S, then i is regular if and only if X is smooth over S in some neighborhood of Z by EGA IV.17.12.1 [42]. In particular, if Y is smooth over S of relative dimension n, then for any morphism $X \to Y$, the graph map $X \to X \times_S Y$ is a regular embedding of codimension n. Indeed, the graph map is a closed embedding of smooth schemes over X. If Y is smooth and X is flat over S, this allows us to factor $X \to Y$ into a regular embedding followed by a flat map.

Let $i : Z \to X$ be a closed embedding and V a closed irreducible subscheme of X which intersects Z properly. (In practice, the difficult part is to find cycles which intersects Z properly.) In order to define the pull-back i^*V of V along i, we need to determine the multiplicity m_W of each irreducible component W of $Z \times_X V$. If we localize at the point corresponding to W, we can assume that X is the spectrum of a local ring $(A, \mathfrak{m})$, Z is defined by an ideal $\mathfrak{a}$ generated by a regular sequence $a_1, \dots, a_c$, and V is defined by a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{m}$ is the only prime containing $\mathfrak{a} + \mathfrak{p}$. The intersection multiplicity m_W at W is defined as the multiplicity of the iterated intersection with divisors

$$m_W \cdot [A/\mathfrak{m}] = [A/\mathfrak{p}] \cdot (a_1) \cdots (a_c) \ .$$

Corollary 7 If V and Z intersect properly, then the pull-back of V along the regular embedding $i : Z \to X$ is compatible with the boundary maps in the cycle complex. **7**

Proof It is easy to see that if $(a_1, \dots, a_c)$ is a regular sequence in A, then $(a_1, \dots, a_c, t_{i_1}, \dots, t_{i_j})$ is a regular sequence in $A[t_0, \dots, t_s]$ for any $j \leq s$ and pairwise different indices i_l. Hence the corollary follows from Proposition 4.

The Koszul complex $K(x_1, \dots, x_n)$ for the elements $x_1, \dots, x_n$ in a ring A is defined to be the total complex of the tensor product of chain complexes

$$K(x_1, \dots, x_n) := \bigotimes_{i=1}^{n} \left(A \overset{\times x_i}{\to} A\right) .$$

Here the source and target are in degrees 1 and 0, respectively.

Proposition 8 Let $(A, \mathfrak{m})$ be a local ring, $\mathfrak{a} \subset A$ be an ideal generated by the regular sequence $a_1, \dots, a_c$. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{m}$ is minimal over $\mathfrak{a} + \mathfrak{p}$, and such that $A/\mathfrak{a}$ and $A/\mathfrak{p}$ meet properly. Then **8**

$$[A/\mathfrak{p}] \cdot (a_1) \cdots (a_c) = \sum_{i=0}^{c} (-1)^i \operatorname{lgth}_A \left(H_i \left(K(a_1, \dots, a_c) \otimes A/\mathfrak{p}\right)\right) \cdot [A/\mathfrak{m}] \ . \qquad (1.52)$$

Proof By Krull's principal ideal theorem, any minimal prime divisor $\mathfrak{q}$ of $(a_1, \dots, a_i) + \mathfrak{p}$ in $A/\mathfrak{p}$ has height at most i, so that using (1.49) we get $\dim A/(\mathfrak{p}, a_1, \dots, a_i) = \dim A/\mathfrak{p} - \operatorname{ht} \mathfrak{q} \geq \dim A/\mathfrak{p} - i$. On the other hand, since $A/\mathfrak{a}$ and $A/\mathfrak{p}$ meet properly, $\dim A/(\mathfrak{p} + \mathfrak{a}) \leq \dim A/\mathfrak{p} + \dim A/\mathfrak{a} - \dim A = \dim A/\mathfrak{p} - c$. This can only happen if we have equality everywhere, i.e.

$$\dim A/(\mathfrak{p}, a_1, \dots, a_i) = \dim A/\mathfrak{p} - \operatorname{ht} \mathfrak{q} = \dim A/\mathfrak{p} - i \ . \qquad (1.53)$$

We now proceed by induction on c. For $c = 1$, by regularity, ${}_{a_1}(A/\mathfrak{p}) = 0$, hence both sides equal $\mathrm{lgth}_A(A/(\mathfrak{p} + a_1A))\cdot[A/\mathfrak{m}]$. If we denote $H_i(K(a_1, \dots, a_l) \otimes A/\mathfrak{p})$ by $H_i(l)$, then by definition of the Koszul complex, there is a short exact sequence of A-modules

$$0 \to H_i(c-1)/a_c \to H_i(c) \to {}_{a_c}H_{i-1}(c-1) \to 0\,.$$

Hence we get for the multiplicity of the right hand side of (1.52)

$$\begin{aligned}\sum_{i=0}^{c}(-1)^i \mathrm{lgth}_A\left(H_i(c)\right) &= \sum_{i=0}^{c}(-1)^i\Big(\mathrm{lgth}_A\left(H_i(c-1)/a_c\right) + \mathrm{lgth}_A\left({}_{a_c}H_{i-1}(c-1)\right)\Big)\\ &= \sum_{i=0}^{c-1}(-1)^i\Big(\mathrm{lgth}_A\left(H_i(c-1)/a_c\right) - \mathrm{lgth}_A\left({}_{a_c}H_i(c-1)\right)\Big)\,.\end{aligned}$$

If $\mathfrak{q}$ runs through the minimal ideals of $A/(\mathfrak{p}, a_1, \dots, a_{c-1})$, then by Lemma 5, this can be rewritten as

$$\sum_{i=0}^{c-1}(-1)^i \sum_{\mathfrak{q}} \mathrm{lgth}_{A_\mathfrak{q}}\left(H_i(c-1)_\mathfrak{q}\right) \cdot \mathrm{lgth}_A\left(A/(\mathfrak{q} + a_cA)\right)\,.$$

By induction, we can assume that the multiplicity $[A/\mathfrak{p}]\cdot(a_1)\cdots(a_{c-1})$ at $\mathfrak{q}$ is $\sum_{i=0}^{c-1}(-1)^i \mathrm{lgth}_{A_\mathfrak{q}}(H_i(c-1)_\mathfrak{q})$, hence we get for the left hand side of (1.52)

$$\begin{aligned}&[A/\mathfrak{p}]\cdot(a_1)\cdots(a_{c-1})\cdot(a_c)\\ &\quad= \Big(\sum_{\mathfrak{q}}\sum_{i=0}^{c-1}(-1)^i \mathrm{lgth}_{A_\mathfrak{q}}\left(H_i(c-1)_\mathfrak{q}\right)\cdot[A/\mathfrak{q}]\Big)\cdot(a_c)\\ &\quad= \sum_{\mathfrak{q}}\sum_{i=0}^{c-1}(-1)^i \mathrm{lgth}_{A_\mathfrak{q}}\left(H_i(c-1)_\mathfrak{q}\right)\cdot \mathrm{lgth}_A\left(A/(\mathfrak{q} + a_cA)\right)\cdot[A/\mathfrak{m}].\end{aligned}$$

9 **Corollary 9** The pull-back of V along the regular embedding $i : Z \to X$ agrees with Serre's intersection multiplicity

$$[A/\mathfrak{p}]\cdot(a_1)\cdots(a_c) = \sum_{i=0}^{c}(-1)^i \mathrm{lgth}_A\left(\mathrm{Tor}_i^A(A/\mathfrak{a}, A/\mathfrak{p})\right)\cdot[A/\mathfrak{m}]\,.$$

In particular, it is independent of the choice of the regular sequence.

Proof If $(x_1, \dots, x_n)$ are elements in a local ring A, then by [80, Thm. 3.3.4] the Koszul-complex $K(x_1, \dots, x_n)$ is acyclic above degree 0, if and only if $(x_1, \dots, x_n)$ is a regular sequence. Hence $K(a_1, \dots, a_c)$ is a free resolution of $A/\mathfrak{a}$. Tensoring the Koszul complex with $A/\mathfrak{p}$ and taking cohomology gives

$$H_i\left(K(a_1, \dots, a_c) \otimes A/\mathfrak{p}\right) \cong \mathrm{Tor}_i^A(A/\mathfrak{a}, A/\mathfrak{p}) .$$

Flat Pull-back A.4

Let $f : X \to Y$ be a flat morphism. We assume that f is of relative dimension n, i.e. for each subvariety V of Y and every irreducible component W of $X \times_Y V$, $\dim W = \dim V + n$. This is satisfied if for example Y is irreducible and every irreducible component of X has dimension equal to $\dim Y + n$, see EGA IV.14.2. In particular, the hypothesis implies that the pull-back of subschemes which intersect properly also intersects properly. For every closed integral subscheme V of Y, we define the pull-back

$$f^*[V] = \sum_W \mathrm{lgth}_{\mathcal{O}_{X,W}}\left(\mathcal{O}_{X,W} \otimes_{\mathcal{O}_{Y,V}} k(V)\right) \cdot [W] ,$$

where W runs through the irreducible components of $V \times_Y X$.

Given $f : X \to Y$ and a subscheme D of Y locally defined by $a \in A$ on $\mathrm{Spec}\, A \subseteq Y$, then we can define the subscheme $f^{-1}D$ of X on any $\mathrm{Spec}\, B \subseteq X$ mapping to $\mathrm{Spec}\, A$ by $f^*a \in B$. If f is flat and D is a divisor, then so is $f^{-1}D$, because f^* sends non-zero divisors to non-zero divisors.

Proposition 10 Let $f : X \to Y$ be a flat map. Then intersection with an effective principal divisor D of Y is compatible with flat pull-back, i.e. if V is a closed subscheme of Y not contained in D, then $f^*[V] \cdot f^{-1}D = f^*[V \cdot D]$ as cycles on X. 10

Proof Intersection with a divisor was defined in Definition 2. It suffices to compare the multiplicities of $f^*[V] \cdot f^{-1}D$ and $f^*[V \cdot D]$ at each irreducible component $\mathcal{Q}$ of $V \times_Y D \times_Y X$. Let $\mathrm{Spec}\, A \subseteq Y$ be an affine neighborhood of the generic point $\mathfrak{p}$ of V and $\mathrm{Spec}\, B \subseteq X$ an affine neighborhood of $\mathcal{Q}$ mapping to $\mathrm{Spec}\, A$. We denote the induced flat map $A \to B$ by g. We can replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}$; this corresponds to replacing Y by V and X by $f^{-1}(V)$. We can also localize B at $\mathcal{Q}$ and A at $\mathfrak{q} = g^{-1}\mathcal{Q}$ ($\mathfrak{q}$ is the generic point of the irreducible component of $V \times_X D$ to which $\mathcal{Q}$ maps). If $a \in A$ defines D, then a is non-zero because V was not contained in D. Let $\mathcal{P}_1, \dots, \mathcal{P}_r$ be the finitely many minimal primes of B corresponding to the irreducible components of $f^{-1}(V)$ passing through $\mathcal{Q}$; for all i, $\mathcal{Q}$ is minimal among the primes of B containing $\mathcal{P}_i + aB$. We are thus in the following situation:

$(A, \mathfrak{q})$ is a one-dimensional local integral domain, $(B, \mathcal{Q})$ a one-dimensional local ring with minimal prime ideals $\mathcal{P}_1, \dots, \mathcal{P}_r$, a is a non-zero divisor in A, and because B is flat over A, a is also a non-zero divisor in B.

The pull-back $f^*[V]$ of V is given by the cycle $\sum_i \mathrm{lgth}_{B_{\mathcal{P}_i}}(B_{\mathcal{P}_i})\cdot[B/\mathcal{P}_i]$. The multiplicity of the intersection of $B/\mathcal{P}_i$ with $f^{-1}D$ (at its only point $\mathcal{Q}$) is $\mathrm{lgth}_B(B/(\mathcal{P}_i+aB))$, hence the multiplicity of $f^*[V]\cdot f^{-1}D$ at $\mathcal{Q}$ is:

$$\sum_i \mathrm{lgth}_{B_{\mathcal{P}_i}}\left(B_{\mathcal{P}_i}\right)\cdot \mathrm{lgth}_B\left(B/(\mathcal{P}_i + aB)\right) \ . \qquad (1.54)$$

The multiplicity of the intersection of V with D (at its only point $\mathfrak{q}$) is $\mathrm{lgth}_A(A/aA)$, and the pull-back of the point $\mathfrak{q}$ of A has multiplicity $\mathrm{lgth}_B(B/\mathfrak{q})$, hence the multiplicity of $f^*[V\cdot D]$ is $\mathrm{lgth}_A(A/aA)\cdot \mathrm{lgth}_B(B/\mathfrak{q})$, which by the following lemma, applied to A/a and B/aB, agrees with $\mathrm{lgth}_B(B/aB)$. Noting that a is not a zero-divisor in B, the latter agrees with (1.54) by Lemma 5.

11 **Lemma 11** Let $A \to B$ be a flat homomorphism of zero-dimensional artinian local rings, then $\mathrm{lgth}_B(B) = \mathrm{lgth}_A(A)\cdot \mathrm{lgth}_B(B/\mathfrak{m}_A B)$.

Proof See also [23, Lemma A.4.1]. There is a finite sequence of ideals I_i of A, say of length r, such that the quotients I_i/I_{i+1} are isomorphic to $A/\mathfrak{m}_A$. Then $r = \mathrm{lgth}_A A$, and tensoring with B, we get a chain of ideals $B\otimes_A I_i$ of B with quotients $B\otimes_A A/\mathfrak{m}_A \cong B/\mathfrak{m}_A B$. Thus $\mathrm{lgth}_B(B) = r\cdot \mathrm{lgth}_B(B/\mathfrak{m}_A B)$.

A.5 Proper Push-forward

In this section we suppose that our schemes are of finite type over an excellent base. This holds for example if the base is the spectrum of a Dedekind ring of characteristic 0, or a field. Given a proper map $f: X \to Y$ and a cycle V in X, we define the proper push-forward to be

$$f_*[V] = \begin{cases} [k(V) : k(f(V))]\cdot[f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{if } \dim V > \dim f(V)\ . \end{cases}$$

12 **Proposition 12** Let $f: X \to Y$ be a proper map. Then intersection with an effective principal divisor D of Y is compatible with push-forward, i.e. if V is a closed subscheme of X not contained in $f^{-1}D$, then $f_*[V\cdot f^{-1}D] = f_*[V]\cdot D$ as cycles on Y.

Proof Let $\mathrm{Spec}\, A \subseteq Y$ be an affine neighborhood of the generic point of $f(V)$, $\mathrm{Spec}\, B \subseteq X$ an affine neighborhood of the generic point of V mapping to $\mathrm{Spec}\, A$, and let $a \in A$ be an equation for D. Let $\mathcal{P} \subseteq B$ and $\mathfrak{p} \subseteq A$ be the prime ideals corresponding to V and $f(V)$, respectively. If we denote the map $A \to B$ by g, then $\mathfrak{p} = g^{-1}\mathcal{P}$.

First assume that $f_*[V] = 0$, i.e. $\dim B/\mathcal{P} > \dim A/\mathfrak{p}$. Since V is not contained in $f^{-1}D$, we have $a \notin \mathcal{P}$, hence by Krull's principal ideal theorem, we get for any minimal ideal $\mathcal{Q}$ of B containing $\mathcal{P} + aB$ (corresponding to a component of $V \cap f^{-1}D$) with inverse image $\mathfrak{q} = g^{-1}\mathcal{Q}$ in A,

$$\dim B/\mathcal{Q} \geq \dim B/\mathcal{P} - 1 > \dim A/\mathfrak{p} - 1 \geq \dim A/\mathfrak{q} .$$

This implies $f_*[V \cdot f^{-1}D] = 0$.

In general, we can divide out $\mathfrak{p}$ and $\mathcal{P}$ from A and B, respectively (which amounts to replacing X by V, and Y by the closed subscheme $f(V)$), and assume that A and B are integral domains. It suffices to consider the multiplicities at each irreducible component of $f(V) \cap D$, i.e. we can localize A and B at a minimal ideal $\mathfrak{q}$ of A containing a. Then A is a one-dimensional local integral domain and B a one-dimensional semi-local integral domain with maximal ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ (corresponding to the irreducible components of $V \cap f^{-1}D$ which map to the component of $f(V) \cap D$ corresponding to $\mathfrak{q}$). Let K and L be the fields of quotients of A and B, respectively. We need to show that

$$\mathrm{lgth}_A(A/aA) \cdot [L : K] = \sum_{\mathcal{Q}} \mathrm{lgth}_{B_{\mathcal{Q}}}(B_{\mathcal{Q}}/aB_{\mathcal{Q}}) \cdot [B_{\mathcal{Q}}/\mathcal{Q} : A/\mathfrak{q}] . \tag{1.55}$$

Let $\widetilde{A}$ and $\widetilde{B}$ be the integral closures of A and B in K and L, respectively. Note that $\widetilde{A}$ and $\widetilde{B}$ are finitely generated over A because the base is supposed to be excellent, hence a Nagata ring. Let $\mathfrak{m}$ and $\mathfrak{n}$ run through the maximal ideals of $\widetilde{A}$ and $\widetilde{B}$, respectively. The localizations $\widetilde{A}_{\mathfrak{m}}$ and $\widetilde{B}_{\mathfrak{n}}$ are discrete valuation rings. The following lemma, applied to A and K gives

$$\mathrm{lgth}_A(A/aA) = \sum_{\mathfrak{m}} \mathrm{lgth}_{\widetilde{A}_{\mathfrak{m}}} \left(\widetilde{A}_{\mathfrak{m}}/a\widetilde{A}_{\mathfrak{m}}\right) \cdot [\widetilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{q}] .$$

Applying the lemma to $\widetilde{A}_{\mathfrak{m}}$ and L, we get

$$[L : K] \cdot \mathrm{lgth}_{\widetilde{A}_{\mathfrak{m}}} \left(\widetilde{A}_{\mathfrak{m}}/a\widetilde{A}_{\mathfrak{m}}\right) = \sum_{\mathfrak{n}|\mathfrak{m}} \mathrm{lgth}_{\widetilde{B}_{\mathfrak{n}}} \left(\widetilde{B}_{\mathfrak{n}}/a\widetilde{B}_{\mathfrak{n}}\right) \cdot \left[\widetilde{B}_{\mathfrak{n}}/\mathfrak{n} : \widetilde{A}_{\mathfrak{m}}/\mathfrak{m}\right] .$$

By multiplicativity of the degree of field extensions, the left hand side of (1.55) is

$$\sum_{\mathfrak{m}} \sum_{\mathfrak{n}|\mathfrak{m}} \mathrm{lgth}_{\widetilde{B}_{\mathfrak{n}}} \left(\widetilde{B}_{\mathfrak{n}}/a\widetilde{B}_{\mathfrak{n}}\right) \cdot \left[\widetilde{B}_{\mathfrak{n}}/\mathfrak{n} : A/\mathfrak{q}\right] .$$

On the other hand, if we apply the lemma to $B_{\mathcal{Q}}$ and L, we get

$$\mathrm{lgth}_{B_{\mathcal{Q}}} \left(B_{\mathcal{Q}}/aB_{\mathcal{Q}}\right) = \sum_{\mathfrak{n}|\mathcal{Q}} \mathrm{lgth}_{\widetilde{B}_{\mathfrak{n}}} \left(\widetilde{B}_{\mathfrak{n}}/a\widetilde{B}_{\mathfrak{n}}\right) \cdot \left[\widetilde{B}_{\mathfrak{n}}/\mathfrak{n} : B/\mathcal{Q}\right] .$$

Hence the right hand side of (1.55) becomes

$$\sum_{\mathcal{Q}} \sum_{\mathfrak{n}|\mathcal{Q}} \mathrm{lgth}_{\widetilde{B}_{\mathfrak{n}}} \left(\widetilde{B}_{\mathfrak{n}}/a\widetilde{B}_{\mathfrak{n}}\right) \cdot \left[\widetilde{B}_{\mathfrak{n}}/\mathfrak{n} : A/\mathfrak{q}\right] .$$

We only need to show that for every maximal ideal $\mathfrak{n}$ of $\widetilde{B}$, there is a maximal ideal $\mathcal{Q}$ of B which it divides. This follows from the valuation property of properness. Indeed, consider the commutative square

$$\begin{array}{ccc} \operatorname{Spec} L & \to & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \widetilde{B}_{\mathfrak{n}} & \to & Y \end{array}$$

The image in X of the closed point $\mathfrak{n}$ of $\operatorname{Spec} \widetilde{B}_{\mathfrak{n}}$ under the unique lift $\operatorname{Spec} \widetilde{B}_{\mathfrak{n}} \to X$ provides an ideal $\mathcal{Q}$ in B with $\mathfrak{n}|\mathcal{Q}$.

13 **Lemma 13** Let A be a one-dimensional local integral domain with maximal ideal $\mathfrak{p}$ and quotient field K. Let L be an extension of K of degree n and $\widetilde{A}$ be the integral closure of A in L. Assume that $\widetilde{A}$ is finitely generated over A. Then for any $a \in A-\{0\}$,

$$n \cdot \operatorname{lgth}_A(A/aA) = \operatorname{lgth}_A(\widetilde{A}/a\widetilde{A}) = \sum_{\mathfrak{m}|\mathfrak{p}} \operatorname{lgth}_{\widetilde{A}_{\mathfrak{m}}}(\widetilde{A}_{\mathfrak{m}}/a\widetilde{A}_{\mathfrak{m}}) \cdot [\widetilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] \; .$$

Proof Choose a K-basis of L consisting of elements of $\widetilde{A}$. This basis generates a free A-submodule F of rank n of $\widetilde{A}$, with finitely generated torsion quotient $\widetilde{A}/F$. Any finitely generated torsion A-module M has a composition series with graded pieces $A/\mathfrak{p}$, hence is of finite length. Then the sequence

$$0 \to {}_aM \to M \to M \to M/a \to 0$$

shows that $\operatorname{lgth}_A({}_aM) = \operatorname{lgth}_A(M/a)$. Mapping the short exact sequence $0 \to F \to \widetilde{A} \to \widetilde{A}/F \to 0$ to itself by multiplication by a, we have $\operatorname{lgth}_A(\widetilde{A}/a\widetilde{A}) = \operatorname{lgth}_A(F/aF) = n \cdot \operatorname{lgth}_A(A/aA)$ by the snake lemma.

We now show more generally that for any $\widetilde{A}$-module M of finite length,

$$\operatorname{lgth}_A(M) = \sum_{\mathfrak{m}|\mathfrak{p}} \operatorname{lgth}_{\widetilde{A}_{\mathfrak{m}}}(M_{\mathfrak{m}}) \cdot [\widetilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] \; .$$

The statement is additive on exact sequences, so we can reduce to the case $M = \widetilde{A}/\mathfrak{m} \cong \widetilde{A}_{\mathfrak{m}}/\mathfrak{m}$ for one of the maximal ideals of $\widetilde{A}$. But in this case,

$$\operatorname{lgth}_A(\widetilde{A}/\mathfrak{m}) = \operatorname{lgth}_{A/\mathfrak{p}}(\widetilde{A}_{\mathfrak{m}}/\mathfrak{m}) = [\widetilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] \; .$$

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