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Tate's Conjecture, Algebraic Cycles and Rational K-Theory in Characteristic p

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Abstract. The purpose of this article is to discuss conjectures on motives, algebraic cycles and K-theory of smooth projective varieties over finite fields. We give a characterization of Tate's conjecture in terms of motives and their Frobenius endomorphism. This is used to prove that if Tate's conjecture holds and rational and numerical equivalence over finite fields agree, then higher rational K-groups of smooth projective varieties over finite fields vanish (Parshin's conjecture). Parshin's conjecture in turn implies a conjecture of Beilinson and Kahn giving bounds on rational K-groups of fields in finite characteristic. We derive further consequences from this result.

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1. Introduction

There are three results which allow to deduce properties of the category of motives for numerical equivalence: Deligne's proof of the Weil conjectures, Jannsen's semi-simplicity theorem, and the existence of the Frobenius automorphism π_M for motives M. If one further assumes Tate's conjecture, one can give a very precise description of this category (Milne [10]). For example, simple motives are determined by their Frobenius endomorphism, and one can recover the endomorphism algebra with the Frobenius. Our first result is a partial converse of this. Let π_M be the Frobenius endomorphism for a motive M for numerical equivalence (it can be identified with an algebraic number modulo the action of the Galois group of $\mathbb{Q}$).

THEOREM 1.1. *The following statements are equivalent for the finite field* $\mathbb{F}_q$*:*

(1) *Tate's conjecture holds for all smooth projective varieties over* $\mathbb{F}_q$.
(2) *For simple motives* M *and* M', $M \cong M' \iff \pi_M = \pi_{M'}$.
(3) *For a simple motive* M, $M \cong 1 \iff \pi_M = 1$.
(4) *For every motive* M, $\mathbb{Q}[\pi_M]$ *is the center of* $\operatorname{End}(M)$.

We use this result as a starting point to study K-theory in characteristic p.

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THEOREM 1.2. *If Tate's conjecture holds and numerical and rational equivalence over finite fields agree rationally, then for all smooth projective varieties X over $\mathbb{F}_q$ and $a > 0$, $K_a(X)_{\mathbb{Q}} = 0$ (Parshin's conjecture).*

The proof uses Jannsen's semi-simplicity theorem, the characterization of motives via their Frobenius endomorphism and an argument on eigenvalues of Frobenius, which was first used by Soulé [11]. We have to assume that numerical and rational equivalence agree in order for the Adams eigenspaces of K-theory to factor through motives for numerical equivalence.

We prove that Parshin's conjecture implies the following bounds on rational K-groups in characteristic p, which have been independently conjectured by Beilinson [2] and Kahn [7].

THEOREM 1.3. *Let k be a field of characteristic p and assume Parshin's conjecture. Then*

(i) $K_a(k)_{\mathbb{Q}} = 0$ *for* $a > \operatorname{tr\,deg} k/\mathbb{F}_p$,
(ii) $K_a(k)_{\mathbb{Q}} = K_a(k)_{\mathbb{Q}}^{(a)} = K_a^M(k)_{\mathbb{Q}}$.

The proof uses de Jong's theorem on alterations and an induction argument in the Gersten–Quillen spectral sequence. Kahn [8] proved the same statement assuming Bass's conjecture (saying that K-groups of schemes of finite type over $\mathbb{Z}$ are finitely generated).

Finally, this conjecture has various consequences for K-groups of varieties in characteristic p. The first corollary gives bounds on Adams operators for X a variety of dimension d over a field k of transcendence degree r:

$$K'_a(X)_{\mathbb{Q}} = \bigoplus_{j=a}^{\min(a+d,r+d)} K'_a(X)_{\mathbb{Q}}^{(j)}.$$

The theorem also implies that the Gersten–Quillen spectral sequence degenerates with split filtration at E_2, $K_a(X)_{\mathbb{Q}}^{(j)} = H^{j-a}(X, \mathcal{K}_j)_{\mathbb{Q}}$.

2. Motives and Tate's Conjecture

In this section we recall the definition of the category of (pure) motives. We derive consequences from Jannsen's theorem that the category of pure motives for numerical equivalence is a semi-simple Abelian category. Finally, we recall Tate's conjecture, give some consequences for the category of motives, and give a formulation of Tate's conjecture in terms of motives and their Frobenius endomorphism.

2.1. MOTIVES

Let k be a field and $\mathcal{V}(k)$ be the category of smooth projective varieties over k. Fix an adequate equivalence relation $\sim$ on the group of algebraic cycles tensored with

$\mathbb{Q}$, i.e. such that pull-back, push-forward and intersection are defined modulo the relation. For a variety X and an integer $j \geqslant 0$, let $A^j_\sim(X)$ be the group of $\mathbb{Q}$-linear algebraic cycles of codimension j on X modulo $\sim$. For equi-dimensional varieties, one can define a composition law

$$A_\sim^{\dim X_1+r}(X_1 \times X_2) \times A_\sim^{\dim X_2+s}(X_2 \times X_3) \longrightarrow A_\sim^{\dim X_1+r+s}(X_1 \times X_3)$$

by sending (f, g) to

$$g \circ f := p_{13*}(p_{12}^* f \cdot p_{23}^* g)$$

and extend this to arbitrary varieties. In particular, $A_\sim^{\dim X}(X \times X)$ is a ring.

The category $\mathcal{M}_\sim(k)$ of (pure) motives with respect to $\sim$ is defined as follows [4]:

Objects of $\mathcal{M}_\sim(k)$ are triples (X, p, m), where X is a variety, $p \in A_\sim^{\dim X}(X \times X)$ is a projector, and m an integer.

Morphisms are defined to be

$$\mathrm{Hom}_\sim((X, p, m), (Y, q, n)) = q \circ A_\sim^{\dim X-m+n}(X \times Y) \circ p$$

and the composition of morphisms is induced by the composition law above.

Denote by 1 the unit motive $(k, \mathrm{id}, 0)$ and by $\mathcal{L}^i$ the Lefschetz motive $(k, \mathrm{id}, -i)$ such that $\mathbb{P}^n = \oplus_{i=0}^n \mathcal{L}^i$. Note that by definition $A^i_\sim(X) = \mathrm{Hom}_\sim(\mathcal{L}^i, X)$.

We consider the following equivalence relations: rational, homological (for a fixed Weil cohomology theory), and numerical. For rational equivalence we have by definition $A^i_{\mathrm{rat}}(X) = \mathrm{CH}^i(X)_\mathbb{Q}$. Consequently, we call motives for rational equivalence Chow motives. As rational equivalence implies homological equivalence implies numerical equivalence, we get functors

$$\mathcal{V}(k) \longrightarrow \mathcal{M}_{\mathrm{rat}} \longrightarrow \mathcal{M}_{\mathrm{hom}} \longrightarrow \mathcal{M}_{\mathrm{num}}.$$

It is conjectured that homological equivalence agrees with numerical equivalence and, in particular, is independent of the chosen Weil cohomology theory. This would follow, for example, from the standard conjectures.

If $k = \mathbb{F}_q$ is a finite field, then as a consequence of Deligne's proof of the Weil conjectures, the Künneth components of the diagonal are algebraic (for homological equivalence), and so there is a decomposition $h(X) = \bigoplus_{i=0}^{2d} h^i(X)$.

We have the following theorems of Jannsen [4, Theorem 1, Corollary 2]

THEOREM 2.1. *The category $\mathcal{M}_{\mathrm{num}}$ is a semi-simple Abelian category. For every object X of this category,* $\mathrm{End}_{\mathrm{num}}(X)$ *is a finite-dimensional semi-simple $\mathbb{Q}$-algebra.*

COROLLARY 2.2. *If the Künneth components of the diagonal are algebraic, the kernel of the surjective ring homomorphism*

$$\mathrm{End}(X)_{\mathrm{hom}} \longrightarrow \mathrm{End}(X)_{\mathrm{num}}$$

is the Jacobsen radical and is a nilpotent ideal.

2.2. THE FROBENIUS MAP

For a variety X over $\mathbb{F}_q$, we denote by π_X the geometric Frobenius map of X, i.e. the map which is the identity on the topological space and the qth power map on its structure sheaf. We have the following proposition of Soulé:

PROPOSITION 2.3 [11, Prop. 2.ii]. *Let $c: X \to Y$ be a correspondence (for rational equivalence), then $c \circ \pi_X = \pi_Y \circ c$.*

In particular, we can (for an arbitrary adequate equivalence relation) define the Frobenius endomorphism π_M for the motive $M = (X, p, m)$ to be $\pi_X \circ p \cdot q^{-m}$ (the factor q^{-m} is due to the fact that $\pi_{\mathcal{L}} = q$). Obviously, there is an inclusion $\mathbb{Q}[\pi_M] \subseteq \mathrm{End}(M)$.

PROPOSITION 2.4. *If $k = \mathbb{F}_q$ is a finite field, then for any simple motive M, the algebra $\mathbb{Q}[\pi_M]$ generated by π_M in $\mathrm{End}_{\mathrm{num}}(M)$ is a finite-field extension of $\mathbb{Q}$. Consequently, we can identify π_M with an algebraic number up to conjugation by* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. By Jannsen's theorem, $\mathbb{Q}[\pi_M]$ is a commutative subalgebra of the finite dimensional $\mathbb{Q}$-division algebra $\mathrm{End}(M)_{\mathrm{num}}$. □

2.3. TATE'S CONJECTURE

Let X be a smooth projective variety over $\mathbb{F}_q$, let $\bar{X} = X \times_{\mathbb{F}_q} \bar{F}_q$ and $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Consider the l-adic cycle map

$$c_r: A^r(X) \longrightarrow H^{2r}(\bar{X}, \mathbb{Q}_l(r))^{\Gamma}.$$

By definition, its image is isomorphic to $A^r_{\mathrm{hom}}(X)$. Denote the $\mathbb{Q}_l$-subspace generated by the image by $A^r_{\mathrm{hom}}(X) \cdot \mathbb{Q}_l$. The strong form of Tate's conjecture is

CONJECTURE 2.5. *Let X be a smooth projective variety over $\mathbb{F}_q$. Then*

$$\dim_{\mathbb{Q}} A^r_{\mathrm{num}}(X) = \mathrm{ord}_{s=r}\zeta(X, s).$$

This can be formulated in terms of the following conjectures, see Tate [14]:

- E : numerical and homological equivalence agree,
- I : the cycle map $c_r \otimes \mathbb{Q}_l$ is injective,
- T : the cycle map $c_r \otimes \mathbb{Q}_l$ is surjective,
- S : the eigenvalue 1 of the Frobenius endomorphism on $H^{2r}(\bar{X}, \mathbb{Q}_l(r))$ has multiplicity 1.

According to [14, Prop. 2.8], we get the following diagram of inequalities, where a letter indicates that the corresponding conjecture implies equality:

$$\begin{array}{ccccccc}\dim_{\mathbb{Q}} A^r_{\text{num}}(X) & \overset{E}{\leqslant} & \dim_{\mathbb{Q}_l} A^r_{\text{hom}}(X)\cdot\mathbb{Q}_l & \overset{T}{\leqslant} & \dim_{\mathbb{Q}_l} H^{2r}(\bar{X},\mathbb{Q}_l(r))^{\Gamma} & \overset{S}{\leqslant} & \operatorname{ord}_{s=r}\zeta(X,s), \\ E\| & & I\| & & & & \\ \dim_{\mathbb{Q}} A^r_{\text{hom}}(X) & = & \dim_{\mathbb{Q}_l} A^r_{\text{hom}}(X)\otimes\mathbb{Q}_l & & & & \end{array}$$

If we denote by E^* the conjecture dual to E (i.e. for $\dim X - r$ instead of r), then we get the following implications between the conjectures:

PROPOSITION 2.6 [14, Prop. 2.6, Cor. 2.7].

$$E + T \Rightarrow T^* + S \Rightarrow E, \qquad S + T \Rightarrow E^*$$

2.4. CONSEQUENCES OF TATE'S CONJECTURE

Let $W(q)$ be the set of Weil q-numbers, i.e. algebraic numbers $\pi \in \bar{\mathbb{Q}}$ such that there is an n with $q^n\pi$ an algebraic integer, and a w such that for all embeddings $\rho\colon \mathbb{Q}[\pi] \to \mathbb{C}$ we have $|\rho\pi| = q^{w/2}$.

In [10], Milne gives a description of the category of motives over a finite field assuming Tate's conjecture. For example, the following statements are consequences of Tate's conjecture:

- [10, Cor. 1.16, Prop. 3.7] The étale cohomology functor

$$\omega_l\colon \mathcal{M}_{\text{num}}(\mathbb{F}_q)\otimes_{\mathbb{Q}}\mathbb{Q}_l \longrightarrow V_l(\mathbb{F}_q)$$

is a fully faithful tensor functor to the category $V_l(\mathbb{F}_q)$ of semi-simple, finite-dimensional, continuous representations of Γ over $\mathbb{Q}_l$.

It identifies $\mathcal{M}_{\text{num}}(\mathbb{F}_q)\otimes_{\mathbb{Q}}\mathbb{Q}_l$ with the full subcategory of $V_l(\mathbb{F}_q)$ consisting of semi-simple representations such that the eigenvalues of $\rho(\text{Frob}_q)$ are Weil-q-numbers.

- [10, Prop. 1.17, Prop. 3.8] The crystalline cohomology functor

$$\omega_p\colon \mathcal{M}_{\text{num}}(\mathbb{F}_q)\otimes_{\mathbb{Q}}\mathbb{Q}_p \longrightarrow V_p(\mathbb{F}_q)$$

is a fully faithful tensor functor to the category $V_p(\mathbb{F}_q)$ of semi-simple F-isocrystals over $\mathbb{F}_q$.

It identifies $\mathcal{M}_{\text{num}}(\mathbb{F}_q)\otimes_{\mathbb{Q}}\mathbb{Q}_p$ with the full subcategory of isocrystals (M, F_M) such that π_M acts semi-simply on M with eigenvalues that are Weil-q-numbers.

- [10, Prop. 2.4] For a simple motive M over $\mathbb{F}_q$, $\text{End}(M) = E$ is a central division algebra over $\mathbb{Q}[\pi_M]$ with invariants

$$\text{inv}_\nu(E) = \begin{cases} \frac{1}{2} & \nu \text{ real, } M \text{ of odd weight} \\ -\dfrac{\text{ord}_\nu(\pi_M)}{\text{ord}_\nu(q)}[\mathbb{Q}[\pi_M]_\nu\colon\mathbb{Q}_p] & \nu|p \\ 0 & \text{otherwise.} \end{cases}$$

• [10, Prop. 2.6] There exists a bijection between the simple objects of the category $\mathcal{M}_{\text{num}}$ and Weil q-numbers up to conjugation:

$$\begin{aligned} \Sigma\mathcal{M}_{\text{num}} &\xrightarrow{\sim} W(q)/\text{Gal}(\mathbb{Q}) \\ M &\mapsto [\pi_M]. \end{aligned}$$

2.5. REFORMULATION IN TERMS OF MOTIVES

Tate's conjecture amounts to an identity theorem for motives for numerical equivalence:

THEOREM 2.7. *The following statements are equivalent for a field* $\mathbb{F}_q$:

(1) *Tate's conjecture holds for all smooth projective varieties over* $\mathbb{F}_q$.
(2) *For simple motives* M *and* M', $M \cong M' \iff \pi_M = \pi_{M'}$.
(3) *For a simple motive* M, $M \cong 1 \iff \pi_M = 1$.
(4) *For every motive* M, $\mathbb{Q}[\pi_M]$ *is the center of* $\text{End}(M)$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and $(i) \Rightarrow (iv)$ follow from the last section.

(iii) $\Rightarrow$ (i) Let X be a smooth projective variety, then by Proposition 2.6 it suffices to show that the inequality

$$\dim_{\mathbb{Q}} A^r_{\text{num}}(X) \leqslant \dim_{\mathbb{Q}_l} H^{2i}(\bar{X}, \mathbb{Q}_l(i))^\Gamma$$

is an equality. Decompose $X = \oplus M_i$ into simple motives in $\mathcal{M}_{\text{num}}$. As a consequence of Corollary 2.2, one can lift orthogonal idempotents from $\text{End}_{\text{num}}(X)$ to orthogonal idempotents in $\text{End}_{\text{hom}}(X)$, and thus can define étale cohomology groups of motives in $\mathcal{M}_{\text{num}}$ up to nonunique isomorphism. This is not functorial, but the geometric Frobenius still acts on these groups because it is central in $\text{End}_{\text{hom}}(X)$. So we may assume that X is a simple motive.

Assume the right-hand side is nontrivial. Then the arithmetic Frobenius Frob_q is trivial on some subspace of $H^{2i}(\bar{X}, \mathbb{Q}_l(i))$. Consequently, the geometric Frobenius π_X acts like q^i on the same subspace. But then by hypothesis

$$\pi_X = q^i \quad \Rightarrow \quad \pi_{X\otimes\mathcal{L}^{-i}} = 1 \quad \Rightarrow \quad X \otimes \mathcal{L}^{-i} = 1 \quad \Rightarrow \quad X = \mathcal{L}^i.$$

In this case both sides are one-dimensional.

(iv) $\Rightarrow$ (ii) Assume there are two different simple motives M and N with $\pi_M = \pi_N$. Then $\text{End}(M \oplus N) = \text{End}(M) \times \text{End}(N)$ and the center of this algebra contains $\mathbb{Q}[\pi_M] \times \mathbb{Q}[\pi_N]$. But $\mathbb{Q}[\pi_{M\times N}]$ is strictly smaller than this algebra (in fact, it embeds diagonally into $\mathbb{Q}[\pi_M] \times \mathbb{Q}[\pi_N]$). □

Remark. It is an interesting question if it is possible to prove without using Tate's conjecture that, for a simple motive M, $\mathbb{Q}[\pi_M]$ is the center of $\text{End}(M)$.

3. Algebraic Cycles and Rational *K*-Theory

In this section, we recall some facts about K-theory. Then we give a criterion for rational and numerical equivalence to agree and show that this implies Parshin's conjecture [3, 12.2]. Parshin's conjecture in turn implies that rationally, Milnor and Quillen K-theory agree in characteristic p. From this we derive further corollaries.

3.1. ALGEBRAIC K-THEORY

Let $K_a(X)$ be Quillen's higher algebraic K-groups associated to the category of vector bundles on X. It is a contravariant functor on the category of schemes over a field. Similarly, let $K'_a(X)$ be the K-groups associated to coherent sheaves. It is a covariant functor for projective morphisms and contravariant for flat morphisms.

For regular X, the groups $K_a(X)$ and $K'_a(X)$ agree and we will tacitly identify them. This identification gives us a covariant and contravariant functoriality on the category of smooth projective schemes over a field. Let $K_a(X)_{\mathbb{Q}} = K_a(X) \otimes \mathbb{Q}$ and $K'_a(X)_{\mathbb{Q}} = K'_a(X) \otimes \mathbb{Q}$.

There are Adams operators ψ^k acting naturally on the K-groups, and the rational K-groups decompose into eigenspaces for this operation:

$$K_a(X)_{\mathbb{Q}} = \bigoplus_{j=0}^{\dim X+a} K_a(X)_{\mathbb{Q}}^{(j)},$$

where ψ^k acts like k^j on $K_a(X)_{\mathbb{Q}}^{(j)}$ for all k [12, Prop. 5].

Adams operators for $K'_a(X)$ are defined by choosing an embedding X into a smooth scheme M and using the identity $K'_a(X) = K_a^X(M)$ for K-theory with support in X [12].

For Z a closed subscheme of codimension r of X with complement U, there is a localization sequence

$$\cdots \longrightarrow K'_a(Z)_{\mathbb{Q}}^{(j-r)} \longrightarrow K'_a(X)_{\mathbb{Q}}^{(j)} \longrightarrow K'_a(U)_{\mathbb{Q}}^{(j)} \longrightarrow \mathrm{K}'_{a-1}(Z)_{\mathbb{Q}}^{(j-r)} \longrightarrow \cdots$$

It is a consequence of the Grothendieck–Riemann–Roch theorem that the Adams eigenspaces $K_a(X)_{\mathbb{Q}}^{(j)}$ factor through the category of Chow motives [11].

PROPOSITION 3.1 [12, Prop. 8.1]. The map induced by the absolute Frobenius map on K-theory agrees with the Adams operator ψ^p. Consequently, for X a scheme over $\mathbb{F}_q$, the geometric Frobenius of X acts like q^j on $K_a(X)_{\mathbb{Q}}^{(j)}$.

There is a fourth quadrant Gersten–Quillen spectral sequence induced by the coniveau filtration [12, Théorème 4]:

$$E_1^{st} = \bigoplus_{x \text{ codim } s} K_{-s-t}(k(x))_{\mathbb{Q}}^{(j-s)} \Rightarrow K'_{-s-t}(X)_{\mathbb{Q}}^{(j)}.$$

Let $\mathcal{K}_a$ be the Zariski sheaf associated to the presheaf $U \mapsto K_a(U)$. Then the E_2-term of the spectral sequence is $E_2^{st} = H^s(X, \mathcal{K}_{-t})$.

For example, there is a filtration of K_0 such that

$$\mathrm{gr}^s K_0(X) = H^s(X, \mathcal{K}_s) = \mathrm{CH}^s(X)$$

which splits rationally, i.e.

$$K_0(X)_{\mathbb{Q}}^{(s)} = H^s(X, \mathcal{K}_s)_{\mathbb{Q}} = A^s_{\mathrm{rat}}(X).$$

Denote by $K_a^M(F)$ Milnor's K-groups of fields. According to [12, Théorème 2],

$$K_a^M(F)_{\mathbb{Q}} = K_a(F)_{\mathbb{Q}}^{(a)}.$$

3.2. PARSHIN'S CONJECTURE

In order to factor the Adams eigenspaces of K-theory through the category of motives for numerical equivalence, we need numerical and rational equivalence to agree. This is in fact a conjecture of Beilinson. For arbitrary base fields, it is still expected that there exists a separated filtration on Chow groups such that the graded pieces factor through numerical equivalence, see [5, Remark 4.5 b].

PROPOSITION 3.2. *Assume that Tate's conjecture holds, that Chow groups of smooth projective varieties over finite fields are finite-dimensional, and that there is no Chow motive M with the following properties:*

(1) $\mathbb{Q}[\pi_M] = \mathbb{Q}[T]/(T - q^i)^n$ *as a subalgebra of* $\mathrm{End}_{\mathrm{rat}}(M)$;
(2) $\mathrm{CH}^i(M)_{\mathbb{Q}} \neq 0$;
(3) *M is trivial considered as a motive for numerical equivalence.*

Then rational and numerical equivalence over finite fields agree.

Proof. We have to show that $\mathrm{CH}^i(M)_{\mathbb{Q}} = A^i_{\mathrm{num}}(X)$ for any indecomposable Chow motive M. As the former group surjects onto the latter, we can assume $\mathrm{CH}^i(M)_{\mathbb{Q}} \neq 0$.

Since $\mathrm{CH}^i(M)_{\mathbb{Q}}$ is finite dimensional by assumption, the subalgebra generated by the Frobenius endomorphism $\mathbb{Q}[\pi_M]$ is isomorphic to $\mathbb{Q}[t]/(f(t))$ for some f. If f had two relatively prime factors P and Q, we could use the Chinese reminder theorem to find two polynomials R and S such that

$$R \equiv \begin{cases} 1 & \mathrm{mod}\ P \\ 0 & \mathrm{mod}\ Q \end{cases} \qquad S \equiv \begin{cases} 0 & \mathrm{mod}\ P \\ 1 & \mathrm{mod}\ Q. \end{cases}$$

Then $R(\pi) + S(\pi)$ would be a decomposition of 1 into orthogonal central idempotents, contrary to the assumption that M is indecomposable. We conclude $f = P^n$ for some irreducible polynomial P.

If P does not have a root q^i, then $0 = P(\pi)^n = P(q^i)^n \neq 0$ on $\mathrm{CH}^i(M)_{\mathbb{Q}}$ by Proposition 3.1, which shows that $\mathrm{CH}^i(M)_{\mathbb{Q}} = 0$. Thus $\mathbb{Q}[\pi_M] = \mathbb{Q}[T]/(T - q^i)^n$.

Consider the functor $F: \mathcal{M}_{\text{rat}} \to \mathcal{M}_{\text{num}}$. F maps $\text{End}_{\text{rat}}(M)$ onto $\text{End}_{\text{num}}(M)$, and as the latter is semi-simple and $F(M) \neq 0$ by assumption, it surjects $\mathbb{Q}[\pi] = \mathbb{Q}[T]/P^n$ onto $\mathbb{Q}[T]/P$. By Theorem 2.7 we conclude that $F(M)$ is a finite direct sum of motives of the form $\mathcal{L}^i$. It remains to show that $M = \mathcal{L}^i$, for then we have $\text{CH}^i(M)_\mathbb{Q} = A^i_{\text{num}}(M) = \mathbb{Q}$ by the known structure of cycles of the projective space.

Fix a summand $\mathcal{L}^i$ of $F(M)$. As F is surjective on endomorphisms and $\text{End}_{\text{rat}}(\mathcal{L}^i) = \text{End}_{\text{num}}(\mathcal{L}^i) = \mathbb{Q}$, the embedding and projection of $\mathcal{L}^i$ lift to maps f and g in $\text{Hom}_{\text{rat}}(\mathcal{L}^i, M)$, respectively $\text{Hom}_{\text{rat}}(M, \mathcal{L}^i)$, such that $g \circ f = id_{\mathcal{L}^i}$. But then the endomorphism $f \circ g \in \text{End}_{\text{rat}}(M)$ is a projector with image $\mathcal{L}^i$. As M was supposed to be indecomposable, $M = \mathcal{L}^i$. □

THEOREM 3.3. *If Tate's conjecture holds and numerical and rational equivalence over finite fields agree, then for all smooth projective varieties X over $\mathbb{F}_q$ and $a > 0$, $K_a(X)_\mathbb{Q} = 0$ (Parshin's conjecture).*

Proof. We show that the Adams eigenspaces $K_a(X)^{(j)}_\mathbb{Q}$ vanish. Decompose $X = \oplus M_i$ into simple motives. According to our assumptions, we can decompose $K_a(X)^{(j)}_\mathbb{Q}$ accordingly and thus assume that $X = M$ is a simple motive with Frobenius π_M. Let P_M be the minimal polynomial of π_M.

If $M = \mathcal{L}^j$, then $K_a(\mathcal{L}^j)^{(j)}_\mathbb{Q} \subseteq K_a(\mathbb{P}^j)^{(j)}_\mathbb{Q} = 0$.

If $M \neq \mathcal{L}^j$, then P_M does not have q^j as a root by Theorem 2.7. So $0 = P_M(\pi_M) = P_M(q^j) \neq 0$ on $K_a(M)^{(j)}_\mathbb{Q}$. □

Remark. Soulé uses a similar technique in [11] to prove unconditional results in cases where he can control the occurrence of the motivic factors $\mathcal{L}^j$. For example, he proves that $K_a(X)^{(j)}_\mathbb{Q} = 0$ for $a > 0$ and $j \geqslant \dim X - 1$ in case X is a smooth curve, an Abelian variety, a unirational variety of dimension at most 3 or a Fermat hypersurface of level m with $p \nmid m$, [11, théorème 4].

3.3. CONSEQUENCES FOR K-THEORY

In this section we derive consequences from Parshin's conjecture for rational K-theory of fields in characteristic p:

THEOREM 3.4. *Let k be a field of characteristic p and assume Parshin's conjecture.*

(i) $K_a(k)_\mathbb{Q} = 0$ *for* $a > \text{tr deg } k/\mathbb{F}_p$.
(ii) $K_a(k)_\mathbb{Q} = K_a(k)^{(a)}_\mathbb{Q} = K^M_a(k)_\mathbb{Q}$.

Proof. As K-theory (of rings) commutes with direct limits, we can write $k = \lim k_i$ with k_i finitely generated and assume that k is finitely generated of transcendence degree r over $\mathbb{F}_p$.

According to de Jong [6, Remark 3.2], there is a smooth projective variety X over $\mathbb{F}_p$ such that the function field $k(X)$ of X is a finite extension of k. Since the composition of the inclusion and the transfer map

$$K_a(k) \longrightarrow K_a(k(X)) \longrightarrow K_a(k)$$

is multiplication by the degree of the extension, we can assume $k = k(X)$.

Consider the (rational) Gersten–Quillen spectral sequence for X:

$$E_1^{st} = \bigoplus_{x \text{ codim } s} K_{-s-t}(k(x))_{\mathbb{Q}}^{(j-s)} \Rightarrow K_{-s-t}(X)_{\mathbb{Q}}^{(j)}.$$

It is a fourth quadrant spectral sequence with $K_a(k)_{\mathbb{Q}} = E_1^{0,-a}$ and all E_∞-terms vanish except on the diagonal because the higher rational K-groups of X vanish. The only terms in the spectral sequence to which there can be differentials coming from $E_1^{0,-a}$ are subquotients of

$$E_1^{s,-a-s+1} = \bigoplus_{x \text{ codim } s} K_{a-1}(k(x))_{\mathbb{Q}}^{(j-s)}.$$

(i) We proceed by induction on tr deg k. Let $a >$ tr deg k. The fields $k(x)$ occurring in $E_1^{s,-a-s+1}$ have smaller transcendence degree than k, so by induction hypothesis the groups $K_{a-1}(k(x))_{\mathbb{Q}}$ vanish. Thus $K_a(k)_{\mathbb{Q}} = E_1^{0,-a} = E_\infty^{0,-a}$ is a quotient of $K_a(X)_{\mathbb{Q}} = 0$.

(ii) We proceed by induction on a. We have $K_a(k)_{\mathbb{Q}}^{(j)} = 0$ for $j > a$ by [12, Cor. 1]. But for $j < a$ and $s \geqslant 1$ we have $j - s < a - 1$, hence $K_{a-1}(k(x))_{\mathbb{Q}}^{(j-s)} = 0$ by induction hypothesis. □

Remark. (1) The statement of the theorem is a conjecture of Beilinson [1, 8.3.3] and in greater generality by Kahn [7]. It was shown to be a consequence of Bass conjecture by Kahn [8].

(2) One should compare the result for global fields in characteristic p to the statement that for k a number field with r_1 real and r_2 complex embeddings we have

$$\dim_{\mathbb{Q}} K_{2a-1}(k)_{\mathbb{Q}} = \dim_{\mathbb{Q}} K_{2a-1}(k)_{\mathbb{Q}}^{(a)} = \begin{cases} r_2 & \text{a even} \\ r_1 + r_2 & \text{a}>1 \text{ odd.} \end{cases}$$

The following corollary gives a generalization of 3.3 to quasi-projective varieties over arbitrary fields of characteristic p:

COROLLARY 3.5. *Assume Parshin's conjecture and let X be a variety of dimension d over a field k of characteristic p with* tr deg $k/\mathbb{F}_p = r$. *Then*

$$K'_a(X)_{\mathbb{Q}} = \bigoplus_{j=a}^{\min(a+d,r+d)} K'_a(X)_{\mathbb{Q}}^{(j)}.$$

In particular $K'_a(X)_{\mathbb{Q}} = 0$ for $a > d + r$.

Proof. From the Gersten–Quillen spectral sequence for X, $K'_a(X)^{(j)}_{\mathbb{Q}}$ has a filtration such that the graded pieces are subquotients of $E_1^{s,-s-a} = \bigoplus_{x \text{ codim } s} K_a(k(x))^{(j-s)}_{\mathbb{Q}}$. From 3.4(ii), this is nonzero only for $j - s = a$, hence $j \geqslant a$. From 3.4(i), we know that $E_1^{s,-s-a} = 0$ for $a > d + r - s$, because the function field of a subvariety of codimension s on a variety of dimension d over a field of transcendence degree r has transcendence degree $d + r - s$.

This proves $K'_a(X)^{(j)}_{\mathbb{Q}} = 0$ unless $a \leqslant j \leqslant d + r$, and the corollary is a consequence of this and [12, Prop. 5]. □

Remark. For nonproper schemes we cannot expect a bound which is independent of the dimension, because for example $K'_a(A[t, t^{-1}]) = K'_a(A) \oplus K'_{a-1}(A)$.

However, one might expect that $K_a(X)_{\mathbb{Q}} = 0$ for $a >$ tr deg k and X a smooth projective variety over k.

COROLLARY 3.6. *If Parshin's conjecture holds, the rational Gersten–Quillen spectral sequence degenerates at* E_2 *with split filtration for a smooth variety* X *over a field of characteristic* p,

$$K_a(X)^{(j)}_{\mathbb{Q}} = H^{j-a}(X, \mathcal{K}_j)_{\mathbb{Q}}.$$

Proof. One gets the degeneration of the spectral sequence and the splitting of the filtration in a standard way using Adams operators. □

We have the following corollary for the homology of $\mathrm{SL}(k)$:

COROLLARY 3.7. *Assume Parshin's conjecture and let k be a field of transcendence degree* r *over* $\mathbb{F}_q$. *Then*

$$H_n(\mathrm{SL}(k), \mathbb{Q})_{\mathrm{prim}} = 0 \quad \textit{for} \quad n > r.$$

Furthermore, the outer automorphism of $\mathrm{SL}(k)$ *acts like* $(-1)^n$ *on* $H_n(\mathrm{SL}(k), \mathbb{Q})$.

Proof. The first statement follows from Theorem 3.4 and

$$H_n(\mathrm{SL}(k), \mathbb{Q})_{\mathrm{prim}} = H_n(\mathrm{BSL}(k)^+, \mathbb{Q})_{\mathrm{prim}} = \pi_n(\mathrm{BSL}(k)^+, \mathbb{Q}) = K_n(k)_{\mathbb{Q}}$$

for $n \geqslant 2$. The second statement follows because the outer automorphism corresponds to the Adams operator ψ^{-1}, which is 1 on even Adams eigenspaces and -1 on odd Adams eigenspaces. As the generators in degree n of the homology of $\mathrm{SL}(k)$ come from K_n, which is concentrated in degree n, we get the corollary. □

4. Relations to Motivic Cohomology

In this section, we explain how our results fit into the context of mixed motives.

Higher rational K-groups are linked to this theory as they are expected to agree with motivic cohomology groups,

$$K_{2j-i}(X)^{(j)}_{\mathbb{Q}} = H^i_{\mathcal{M}}(X, \mathbb{Q}(j)).$$

All statements in this section are consequences of the work of Beilinson [2] and Jannsen [5]. Beilinson gives the following consequence of 3.4:

COROLLARY 4.1. *Assume Parshin's conjecture, let $X_{\mathbb{Z}}$ be a flat proper model of the smooth projective variety X over $\mathbb{Q}$ and define $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}$ to be the image of $H^i_{\mathcal{M}}(X_{\mathbb{Z}}, \mathbb{Q}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$.*

(a) *One has $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}} = H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$ unless $j \leqslant i \leqslant 2j - 1$ and $j \leqslant \dim X + 1$.*

(b) *If X has potential good reduction at every prime, then the above inequalities may be replaced by $2j - 2 = i - 1 \leqslant 2 \dim X$.*

Proof. By way of the localization sequence, the cokernel is contained in $\bigoplus_p K'_{2j-i-1}(X_p)^{(j-1)}_{\mathbb{Q}}$, where X_p runs through the fibers of $X_{\mathbb{Z}}$ at the primes p.

(a) By Corollary 3.5, the terms $K'_{2j-i-1}(X_p)^{(j-1)}_{\mathbb{Q}}$ are trivial unless $0 \leqslant 2j - i - 1 \leqslant j - 1$ and $j - 1 \leqslant \dim X$.

(b) By hypothesis, we can assume that the fibers X_p are smooth and projective. Then by Theorem 3.3 we must have $2j - i - 1 = 0$ and $j - 1 \leqslant \dim X$. □

COROLLARY 4.2. *Let k be of characteristic p and assume Parshin's conjecture. Then $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = 0$ unless $0 \leqslant j \leqslant \min\{i, \dim X + \operatorname{tr deg} k\}$. In particular, motivic cohomology vanishes in negative degrees.*

Proof. This is a reformulation of Corollary 3.5. □

4.1. FINITE FIELDS

A weight argument as in [10, Theorem 2.49] shows that if the category of mixed motives over $\mathbb{F}_q$ exists, then every mixed motive is a direct sum of pure motives. Thus, the category $\mathcal{MM}_{\mathbb{F}_q}$ agrees with the category $\mathcal{M}_{\mathbb{F}_q}$ and in particular is semisimple.

By the interpretation of higher K-groups as groups of extensions in $\mathcal{MM}_{\mathbb{F}_q}$, Parshin's conjecture holds. Thus we assume Parshin's conjecture when we speculate about properties of $\mathcal{MM}_{\mathbb{F}_q}$, as it is implied by its existence.

Let U be a smooth quasi-projective variety. Then one expects $h^i(U)$ to have weights $i \leqslant w \leqslant 2i$. On the other hand,

$$K_a(U)^{(j)}_{\mathbb{Q}} = H^{2j-a}_{\mathcal{M}}(U, \mathbb{Q}(j)) = \operatorname{Hom}(1, h^{2j-a}(U)(j))$$

and the latter is trivial unless $h^{2j-a}(U)$ has a weight $2j$. From this we conclude $0 \leqslant a \leqslant j$ and get back Corollary 3.5.

If Y is a projective variety, then $h^i(Y)$ has weights $0 \leqslant w \leqslant i$. From this the analogous argument shows that $a \leqslant 0 \leqslant j$, hence $K_a(Y)^{(j)}_{\mathbb{Q}} = 0$ for $a > 0$. Note that this is wrong for $K'_a(Y)$.

4.2. GLOBAL FUNCTION FIELDS

Let k be a global field and C be the corresponding smooth proper curve, respectively number ring. The cohomological dimension of k is expected to be 1, and we get for X smooth and proper over k, $i < 2j$ identities

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = \mathrm{Ext}^1_{\mathcal{MM}}(1, h^{i-1}(X)(j)).$$

Recall that the L-function of a smooth projective variety X over k is given by

$$L(h^i(X), s) = \prod_{\nu \in \mathbb{C}} \det(1 - (N\nu)^{-s} Fr^*_\nu | H^i(\bar{X}, \mathbb{Q}_l))^{(-1)}.$$

There is a (conjectured) analytic continuation and a functional equation

$$L(h^i(X), s) = \epsilon(s) \cdot L(h^i(X), i + 1 - s)$$

with center $(i + 1)/2$. Beilinson's conjecture on special values of L-functions states that, for $i + 1 - n < (i)/2$,

$$\begin{aligned} \mathrm{ord}_{s=i+1-n} L(h^i(X), s) &= \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}(1, h^i(X)(n)) \\ &= \dim H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}. \end{aligned}$$

In the global function field case we have (because there are no infinite places and thus no Γ-factors in the L-series)

$$\mathrm{ord}_{s=i+1-n} L(h^i(X), s) = 0 \quad \text{for} \quad i + 1 - n < \frac{i}{2}.$$

So according to Beilinson's philosophy, one should expect $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} = 0$.

Using Corollary 4.1, we get back Corollary 3.5 plus the stronger claim that if X has potential good reduction at every prime, then

$$K_a(X)^{(j)}_{\mathbb{Q}} = 0 \quad \text{unless} \quad a \leqslant 1.$$

EXAMPLE. Let X be smooth projective curve over a function field, then there are the following exact sequences

$$\begin{aligned} 0 &\longrightarrow K_2(X_{\mathbb{Z}})^{(2)} \longrightarrow K_2(X)^{(2)}_{\mathbb{Q}} \longrightarrow \bigoplus_\nu K'_1(X_\nu)^{(1)}_{\mathbb{Q}} \\ &\longrightarrow K_1(X_{\mathbb{Z}})^{(2)}_{\mathbb{Q}} \longrightarrow K_1(X)^{(2)}_{\mathbb{Q}} \longrightarrow 0 \end{aligned}$$

$$\begin{aligned} 0 &\longrightarrow K_1(X_{\mathbb{Z}})^{(1)} \longrightarrow K_1(X)^{(1)}_{\mathbb{Q}} \longrightarrow \bigoplus_\nu \mathbb{Q} \\ &\longrightarrow K_0(X_{\mathbb{Z}})^{(1)}_{\mathbb{Q}} \longrightarrow K_0(X)^{(1)}_{\mathbb{Q}} \longrightarrow 0. \end{aligned}$$

The analogon to Beilinson's conjectures indicates that the first terms are zero and that $K_2(X)^{(2)}_{\mathbb{Q}}$ is trivial if X has potential good reduction at every prime.

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