Invent. math. 139, 459–493 (2000)
Digital Object Identifier (DOI) 10.1007/s002229900034

Inventiones mathematicae

The K-theory of fields in characteristic p

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Oblatum 21-I-1998 & 26-VII-1999 / Published online: 18 October 1999

Abstract. We show that for a field k of characteristic p, $H^i(k, \mathbb{Z}(n))$ is uniquely p-divisible for $i \neq n$ (we use higher Chow groups as our definition of motivic cohomology). This implies that the natural map $K_n^M(k) \to K_n(k)$ from Milnor K-theory to Quillen K-theory is an isomorphism up to uniquely p-divisible groups, and that $K_n^M(k)$ and $K_n(k)$ are p-torsion free. As a consequence, one can calculate the K-theory mod p of smooth varieties over perfect fields of characteristic p in terms of cohomology of logarithmic de Rham Witt sheaves, for example $K_n(X, \mathbb{Z}/p^r) = 0$ for $n > dim X$. Another consequence is Gersten's conjecture with finite coefficients for smooth varieties over discrete valuation rings with residue characteristic p. As the last consequence, Bloch's cycle complexes localized at p satisfy all Beilinson-Lichtenbaum-Milne axioms for motivic complexes, except possibly the vanishing conjecture.

1. Introduction

The purpose of this paper is to study the p-part of the motivic cohomology groups $H^i(k, \mathbb{Z}(n))$ and the higher algebraic K-groups $K_n(k)$ of a field k of characteristic p. We take Bloch's higher Chow groups as our definition of the motivic cohomology of a smooth quasi-projective variety over a field,

$$H^p(X, \mathbb{Z}(q)) = \mathrm{CH}^q(X, 2q-p).$$

This agrees with the motivic cohomology defined by the second author by [21, II, Theorem 3.6.6], and with the motivic cohomology groups defined

⋆ The first author acknowledges support by the DFG.
⋆⋆ The second author acknowledges support by the DFG and the NSF.

Mathematics Subject Classification (1991): 19D50, 19D45, 19E08, 14C25

by Voevodsky if one assumes resolution of singularities, [30, Prop. 4.2.9, Theorem 4.3.7]. Motivic cohomology and K-theory are related by the spectral sequence of Bloch and Lichtenbaum [6], having motivic cohomology groups as E_2-term and converging to K-theory. It is known that the motivic cohomology groups $H^i(k, \mathbb{Z}(n))$ are trivial for $i > n$, and agree with the Milnor K-group $K_n^M(k)$ for $i = n$. However, the motivic cohomology groups are not known to vanish for $i < 0$.

The question of determining motivic cohomology groups of k can be divided into the calculation of the rational motivic cohomology groups, $H^i(k, \mathbb{Q}(n))$, and the motivic cohomology groups with mod l^r-coefficients, $H^i(k, \mathbb{Z}/l^r(n))$. For l different from the characteristic of k, the Beilinson-Lichtenbaum conjecture states that there should be an isomorphism

$$H^i(k, \mathbb{Z}/l^r(n)) \cong H^i(k_{\text{ét}}, \mu_{l^r}^{\otimes n}), \tag{1}$$

for $i \leq n$. Suslin and Voevodsky [28] show that, assuming resolution of singularities, the Bloch-Kato conjecture

$$K_n^M(k)/l^r \xrightarrow{\sim} H^n(k_{\text{ét}}, \mu_{l^r}^{\otimes n})$$

for all $n \leq m$ implies (1) for all $i \leq n \leq m$. A partial result along these lines was obtained by the second author in [19] and forms the basis of our arguments. In a forthcoming paper [10], we use the methods of this paper to remove the resolution of singularities hypothesis in the theorem of Suslin and Voevodsky. On the other hand, Voevodsky [31] proved the Bloch-Kato conjecture for $l = 2$ and all fields of characteristic different from 2, thereby proving the Beilinson-Lichtenbaum conjecture in this case.

If k has characteristic p, then a conjecture of Beilinson states that Milnor K-theory and Quillen K-theory should agree rationally:

$$K_n^M(k)_{\mathbb{Q}} \xrightarrow{\sim} K_n(k)_{\mathbb{Q}}. \tag{2}$$

By results of Kahn [17], this follows from Bass's conjecture, and by results of the first author this follows from Tate's conjecture on algebraic cycles [8].

The main result of this paper is

Theorem 1.1. *Let k be a field of characteristic p. Then the motivic cohomology groups $H^i(k, \mathbb{Z}/p^r(n))$ vanish for all $i \neq n$.*

By the spectral sequence of Bloch and Lichtenbaum, this implies a mod p version of the conjecture (2), namely, that the natural map

$$K_n^M(k) \longrightarrow K_n(k)$$

is an isomorphism up to uniquely p-divisible groups. Furthermore, $K_n(k)$ is p-torsion free and there is an isomorphism

$$K_n^M(k)/p^r \longrightarrow K_n(k, \mathbb{Z}/p^r).$$

We derive various applications of this result. For example, we show that Gersten's conjecture with mod p-coefficients holds for smooth varieties over discrete valuation rings with residue characteristic p.

As another application, we determine the motivic cohomology and K-theory sheaves for the Zariski topology on a smooth schemes X over a perfect field of characteristic p. Via local to global spectral sequences, this implies that there is a spectral sequence from motivic cohomology to K-theory

$$H^{s-t}(X, \mathbb{Z}/p^r(-t)) \Rightarrow K_{-s-t}(X, \mathbb{Z}/p^r),$$

and $K_n(X, \mathbb{Z}/p^r) = 0$ for $n > \dim X$.

Finally we show that Bloch's cycle complexes $z^n(-, *)[-2n]$, localized at p, satisfy most of the axioms for Beilinson and Lichtenbaum, as extended by Milne [23]. Further applications to topological cyclic homology will appear in [9].

The proof of the theorem is another variation on the theme that the Bloch-Kato conjecture implies the conjecture of Lichtenbaum and Quillen, and is motivated by the ideas in [19]. In the case at hand, the conjecture of Bloch-Kato should be reinterpreted as the theorem of Bloch-Kato [5]

$$K_n^M(k)/p^r \xrightarrow[\sim]{d\log} H^0(k_{\text{ét}}, \nu_r^n),$$

where $\nu_r^n = W_r\Omega^n_{\log}$ is the logarithmic de Rham Witt sheaf of Milne and Illusie [15] (we write ν^n for ν_1^n).

We give an outline of the proof:

In Paragraph 2, we recall the definition of the logarithmic de Rham-Witt groups and define its relative version as the kernel of the restriction map. Relative motivic cohomology groups are defined as the cone of the restriction map on cycle complexes. We prove two results relating motivic cohomology to Milnor-K-theory in Paragraph 3. For example, for a semi-localization R of a regular finitely generated k-algebra, k a field of characteristic p, we have $H^n(R, \mathbb{Z}/p(n)) \cong \nu^n(R)$. In particular, there is a map from relative motivic cohomology to the relative logarithmic de Rham-Witt groups.

Let $\hat{\square}_m$ be the semi-localization of the algebraic m-cube $\operatorname{Spec} k[t_1, \dots, t_m]$ with respect to the 2^m points where all coordinates are either 0 or 1. Let T_m be the set of ideals $(t_i - \epsilon)$, $1 \le i \le m$, $\epsilon \in \{0, 1\}$, and $S_m = T_m - \{(t_m)\}$. In Paragraph 4, we show, assuming the main theorem for $n-1$, that

$$H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) \cong \nu^n(\hat{\square}_m, S_m)$$

$$H^i(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) = 0 \quad \text{for } i \neq n$$

$$H^i(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) = 0 \quad \text{for } i > n.$$

In particular, the long exact relativization sequence for the sets of ideals T_m and S_m gives an exact sequence

$$0 \longrightarrow H^{n-m}(k, \mathbb{Z}/p(n)) \longrightarrow H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \longrightarrow \\ H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) \xrightarrow{t_m=0} H^n(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) \longrightarrow 0.$$

Thus, it suffices to show that the map $H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \longrightarrow H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n))$ is injective. Comparing with the analogous sequence for the relative logarithmic de Rham Witt sheaves, the snake lemma shows that this in turn follows from the surjectivity of

$$H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \xrightarrow{\alpha_m} \nu^n(\hat{\square}_m, T_m)$$

for a different value of m.

To prove this statement, let $\partial\hat{\square}_{m+1}$ be the boundary of $\hat{\square}_{m+1}$, i.e. the closed subscheme defined by the ideal $\prod_i t_i(t_i - 1)$, and view $\hat{\square}_m$ as the face $t_{m+1} = 0$ of $\partial\hat{\square}_{m+1}$. Since Bloch's higher Chow groups give reasonable motivic cohomology groups only for smooth schemes, we define motivic cohomology of $\partial\hat{\square}_{m+1}$ to be the cohomology of the complex formed by cycles on the faces of $\partial\hat{\square}_{m+1}$ which agree on the intersections. We define

$$H^n(X, Y_1, \dots, Y_m, \mathbb{Z}/p(n))^{\mathrm{ker}} = \\ \ker\Big(H^n(X, \mathbb{Z}/p(n)) \longrightarrow \oplus_j H^n(Y_j, \mathbb{Z}/p(n))\Big).$$

Paragraph 5 is devoted to the proof of the existence of the following commutative diagram. Here s is a functorial splitting of the canonical map (we omit the coefficients $\mathbb{Z}/p(n)$).

$$\begin{array}{ccccc}
H^n(\hat{\square}_m, T_m) & \xrightarrow{\alpha_m} & H^n(\hat{\square}_m, T_m)^{\mathrm{ker}} & \xrightarrow{\cong} & \nu^n(\hat{\square}_m, T_m) \\
\| & & \| & & \uparrow{\scriptstyle\gamma} \\
H^n(\partial\hat{\square}_{m+1}, S_{m+1}) & \xrightarrow{\alpha_m} & H^n(\partial\hat{\square}_{m+1}, S_{m+1})^{\mathrm{ker}} & \longleftarrow & \nu^n(\partial\hat{\square}_{m+1}, S_{m+1}) \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} \\
H^n(\partial\hat{\square}_{m+1}) & \longrightarrow & H^n(\partial\hat{\square}_{m+1})^{\mathrm{ker}} & \longleftarrow & \nu^n(\partial\hat{\square}_{m+1})
\end{array}$$

The proof that the map γ is surjective (Paragraph 6) relies on the understanding of the structure of torsion sections $\nu^n(\partial\hat{\square}_m, T_m) \subseteq \nu^n(\partial\hat{\square}_m)$, i.e. the sections in $\nu^n(\partial\hat{\square}_m)$ which vanish on each of the faces $(t_i - \epsilon)$.

We finish the proof of the main theorem in Paragraph 7, where we show that for each section $x \in \nu^n(\partial\hat{\square}_{m+1})$, there is a semi-local smooth scheme U_x over k containing $\partial\hat{\square}_{m+1}$ as a closed subscheme, such that x lifts to $\nu^n(U_x) \cong H^n(U_x)$. But $H^n(U_x)$ maps to all groups in the lower row of the above diagram. Since the right vertical composition is surjective, the same holds for the middle composition, and this implies the surjectivity of α_m.

Finally, in Paragraph 8 we derive consequences of the main theorem.

Acknowledgement. It is a pleasure to thank Bruno Kahn for his helpful comments.

2. Preliminaries

For a commutative ring R, let Ω_R be its absolute Kähler differentials and $\Omega^n_R = \Lambda^n_R \Omega_R$. Then Ω^*_R forms a complex with differentials d, and we denote by $Z\Omega^n_R$ and $d\Omega^{n-1}_R$ the cycles and boundaries, respectively. For two rings R and S, we have

$$\Omega_{R\otimes S} \cong (R \otimes \Omega_S) \oplus (S \otimes \Omega_R).$$

Consequently,

$$\Omega^n_{R\otimes S} \cong \bigoplus_{i+j=n} \Omega^i_R \otimes \Omega^j_S, \tag{3}$$

and $\Omega^*_{R\otimes S}$ is the total complex associated to the double complex $\Omega^*_R \otimes \Omega^*_S$.

For a principal ideal (f) of R, the second exact sequence for differentials gives us an exact sequence

$$f\Omega_R + Rdf \longrightarrow \Omega_R \longrightarrow \Omega_{R/(f)} \longrightarrow 0,$$

and hence we get

$$\Omega^n_{R/(f)} = \Omega^n_R / f\Omega^n_R + df\Omega^{n-1}_R. \tag{4}$$

For an $\mathbb{F}_p$-algebra R, the inverse Cartier operator C^{-1} is the map

$$\begin{aligned} \Omega^n_R &\xrightarrow{C^{-1}} \Omega^n_R / d\Omega^{n-1}_R \\ a_0 da_1 \wedge \ldots \wedge da_n &\mapsto a_0^p a_1^{p-1} \cdots a_n^{p-1} da_1 \wedge \ldots \wedge da_n. \end{aligned}$$

Note that the Cartier operator C is defined only for smooth R, and gives an isomorphism $Z\Omega^n_R / d\Omega^{n-1}_R \xrightarrow{C} \Omega^n_R$ in this case. We define the logarithmic de Rham Witt group $\nu^n(R)$ of R as the kernel of the Artin-Schreier map

$$\nu^n(R) = \ker\left(\Omega^n_R \xrightarrow{1-C^{-1}} \Omega^n_R / d\Omega^{n-1}_R\right).$$

If R is essentially smooth over a perfect field of characteristic p, then by Illusie [15] ν^n is the subsheaf of Ω^n generated locally for the étale topology by $d\log a_1 \wedge \ldots \wedge d\log a_n$, and $\nu^n(R)$ its global sections. Moreover, for R semi-local, there is an exact sequence [14]

$$0 \longrightarrow \nu^n(R) \longrightarrow \bigoplus_{x\in R^{(0)}} \nu^n(k(x)) \longrightarrow \bigoplus_{x\in R^{(1)}} \nu^{n-1}(k(x)) \longrightarrow \ldots . \qquad (5)$$

If $I_1, \ldots, I_m$ are ideals of R, we define the relative logarithmic de Rham-Witt groups as the kernel of the restriction maps

$$\nu^n(R, I_1, \ldots, I_m) = \ker\Big(\nu^n(R) \longrightarrow \bigoplus_{j=1}^{m} \nu^n(R/I_j)\Big).$$

The Milnor K-groups of a field F are defined as the quotient of the tensor algebra on the group of units $F^\times$ by the Steinberg relations

$$K_*^M(F) = T^*F^\times/\langle a\otimes(1-a) \mid a \in F-\{0,1\}\rangle.$$

For a field F with nontrivial discrete valuation, uniformizer π and residue field k, the tame symbol homomorphism

$$K_n^M(F) \xrightarrow{\delta} K_{n-1}^M(k)$$

is defined by $\{\pi, u_2, \ldots, u_n\} \mapsto \{\bar{u}_2, \ldots, \bar{u}_n\}$ and $\{u_1, u_2, \ldots, u_n\} \mapsto 0$ for $u_j \in F$ units in the valuation ring. For R an essentially smooth semi-local ring over a field k, we define

$$K_n^M(R) = \ker\Big(\bigoplus_{x\in R^{(0)}} K_n^M(k(x)) \xrightarrow{\delta_x} \bigoplus_{y\in R^{(1)}} K_{n-1}^M(k(y))\Big).$$

There exists a universally exact Gersten resolution for Milnor K-theory [7, Example 7.3(5)], hence we have

$$K_n^M(R)/p = \ker\Big(\bigoplus_{x\in R^{(0)}} K_n^M(k(x))/p \xrightarrow{\delta_x} \bigoplus_{y\in R^{(1)}} K_{n-1}^M(k(y))/p\Big). \qquad (6)$$

The fundamental theorem of Bloch-Gabber-Kato [5] relates Milnor K-theory and logarithmic de Rham Witt sheaves of fields:

$$K_n^M(F)/p \xrightarrow[\sim]{d\log} \nu^n(F). \qquad (7)$$

The Quillen K-groups of a ring R are the homotopy groups of the classifying space of the Q-construction, similarly for K-groups with coefficients. There is a long exact sequence

$$\ldots \longrightarrow K_n(R) \xrightarrow{\times p^r} K_n(R) \longrightarrow K_n(R, \mathbb{Z}/p^r) \longrightarrow K_{n-1}(R) \xrightarrow{\times p^r} \ldots .$$

The product structure of K-theory induces a canonical map from Milnor K-theory of fields to Quillen K-theory.

For R an essentially smooth semi-local ring over a field, there is a Gersten sequence [25, theorem 5.11]

$$0 \longrightarrow K_n(R,\mathbb{Z}/p) \longrightarrow \bigoplus_{x\in R^{(0)}} K_n(k(x),\mathbb{Z}/p) \longrightarrow \bigoplus_{x\in R^{(1)}} K_{n-1}(k(x),\mathbb{Z}/p) \longrightarrow \dots, \quad (8)$$

and a spectral sequence [25, prop. 5.5, 5.11]

$$E_2^{st} = H^s(X_{\mathrm{Zar}}, (\mathcal{K}/p)_{-t}) \Rightarrow K_{-s-t}(X,\mathbb{Z}/p).$$

Via the Gersten sequences, the map from Milnor to Quillen K-groups of fields induces a map $K_n^M(R) \longrightarrow K_n(R)$ for any regular ring of essentially finite type over a field. This is clear for a smooth ring over a perfect field, and in general one uses a colimit argument as in the proof of Proposition 3.1 below.

We recall the definition of the cubical version of motivic cohomology groups. Let k be a field, and let

$$\square_q = \operatorname{Spec} k[t_1,\dots,t_q].$$

We have the $2q$ faces of codimension 1, $\square_i^\epsilon \subseteq \square_q$ defined by $(t_i-\epsilon)$ for $i=1,\dots,q$ and $\epsilon\in\{0,1\}$. We call an arbitrary intersection of these, including $\square_q$ itself, a *face* of $\square_q$. Let $\partial^+\square_q$ be the divisor consisting of all faces $\square_i^\epsilon$ except $\square_q^0$.

For a regular k-variety X, the cubical version of Bloch's cycle complex is defined as follows [20]: Let $\mathcal{S}$ be a finite set of closed subsets of X with $X\in\mathcal{S}$, and assume for simplicity that the intersection of two sets in $\mathcal{S}$ is again contained in $\mathcal{S}$. If $Y\subseteq X$ is a subscheme of X, then by abuse of notation we denote the set of closed subsets $\{S\in\mathcal{S}|S\subseteq Y\}$ again by $\mathcal{S}$ when we talk about cycles on Y.

The group $z^n(X,q)_{\mathcal{S}}$ is the group of codimension n cycles Z on $X\times\square_q$ such that

- Z intersects $S\times D$ properly on $X\times\square_q$, for $S\in\mathcal{S}$ and D a face of $\square_q$
- $Z\cdot(X\times\partial^+\square_q)=0$

Intersection with the face $\square_q^0$ defines a map $d_q: z^n(X,q)_{\mathcal{S}} \to z^n(X,q-1)_{\mathcal{S}}$. Since

$$d_{q-1}\circ d_q(Z) = \square_{q-1}^0\cdot(\square_q^0\cdot Z) = \square_q^0\cdot(\square_{q-1}^0\cdot Z) = 0,$$

we get a complex $(z^n(X,*)_{\mathcal{S}}, d)$, which we will also denote by $z^n(X)_{\mathcal{S}}$,

$$\dots \xrightarrow{d_{q+1}} z^n(X,q)_{\mathcal{S}} \xrightarrow{d_q} \dots \xrightarrow{d_1} z^n(X,0)_{\mathcal{S}}.$$

By associativity of the intersection product, we have for smooth $S \in \mathcal{S}$

$$(S \times \partial^+\square_q) \cdot_{S\times\square_q} ((S \times \square_q) \cdot Z) = (S \times \partial^+\square_q) \cdot_{X\times\square_q} Z$$
$$= (S \times \partial^+\square_q) \cdot_{X\times\partial^+\square_q} ((X \times \partial^+\square_q) \cdot_{X\times\square_q} Z) = 0,$$

hence intersecting with $S \times \square_q$ determines a map of complexes

$$i_S^* : z^n(X, *)_{\mathcal{S}} \to z^n(S, *)_{\mathcal{S}}. \tag{9}$$

If X is affine, then the inclusion

$$z^n(X, *)_{\mathcal{S}} \hookrightarrow z^n(X, *) \tag{10}$$

is a quasi-isomorphism: By [21, Ch.2, Theorem 3.5.14] this holds for the simplicial version of cycle complexes, and by [20, Theorem 4.7] the simplicial version and the cubical version agree.

Suppose $X = \operatorname{Spec} R$ is affine, with closed subschemes $Y_1, \dots, Y_q$. For each $J \subseteq \{1, \dots, q\}$ including $J = \emptyset$, let $Y_J = \bigcap_{j\in J} Y_j$. We assume that each subscheme Y_J of X is regular, and let $\mathcal{S}$ be the set of all Y_J. Since the complexes we are going to construct are quasi-isomorphic if we increase the number q of closed subschemes by (10), we may increase this number if necessary.

The *relative cycle complex* $z^n(X, Y_1, \dots, Y_m, *)_{\mathcal{S}}$ is defined as the total complex of the double complex, with $J \subseteq \{1, \dots, m\}$,

$$0 \longrightarrow z^n(X, *)_{\mathcal{S}} \xrightarrow{d_0} \bigoplus_{|J|=1} z^n(Y_J, *)_{\mathcal{S}} \xrightarrow{d_1} \bigoplus_{|J|=2} z^n(Y_J, *)_{\mathcal{S}} \xrightarrow{d_2} \dots .$$

The complex $\oplus_{|J|=s} z^n(Y_J, *)_{\mathcal{S}}$ lies in degree s, and each complex is viewed as a cohomological complex concentrated in negative degrees. In other words, $\oplus_{|J|=s} z^n(Y_J, t)$ lies in degree $(s, -t)$. The maps d_j are the alternating sum of the restriction maps (9).

If $m + 1 \le q$, then one easily sees that the map of complexes

$$z^n(X, Y_1, \dots, Y_{m+1}, *)_{\mathcal{S}} \longrightarrow z^n(X, Y_1, \dots, Y_m, *)_{\mathcal{S}},$$

identifies the first term with

$$\operatorname{cone}\Big(z^n(X, Y_1, \dots, Y_m, *)_{\mathcal{S}} \longrightarrow z^n(Y_{m+1}, Y_1 \cap Y_{m+1}, \dots, Y_m \cap Y_{m+1}, *)_{\mathcal{S}}\Big)[-1].$$

We define relative motivic cohomology groups of X to be

$$H^{2n-i}(X, Y_1, \dots, Y_m, \mathbb{Z}(n)) = H_i(z^n(X, Y_1, \dots, Y_m, *)_{\mathcal{S}})$$
$$H^{2n-i}(X, Y_1, \dots, Y_m, \mathbb{Z}/p^r(n)) = H_i(z^n(X, Y_1, \dots, Y_m, *)_{\mathcal{S}} \otimes \mathbb{Z}/p^r).$$

By (10), this is independent of the choice of $\mathcal{S}$. There is a long exact sequence

$$\dots \longrightarrow H^i(X, Y_1, \dots , Y_m, \mathbb{Z}(n)) \xrightarrow{\times p^r} H^i(X, Y_1, \dots , Y_m, \mathbb{Z}(n)) \longrightarrow$$
$$H^i(X, Y_1, \dots , Y_m, \mathbb{Z}/p^r(n)) \longrightarrow H^{i+1}(X, Y_1, \dots , Y_m, \mathbb{Z}(n)) \longrightarrow \dots .$$

Since the cubical and simplicial version of the cycle complexes agree, there is a hypercohomology spectral sequence [3]

$$E_2^{s,t} = H^s(X, \mathcal{H}^t(\mathbb{Z}(n))) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)),$$

and similarly with $\mathbb{Z}/p^r$-coefficients. For R essentially smooth over k and semi-local there is a Gersten resolution [1]

$$0 \longrightarrow H^i(R, \mathbb{Z}/p(n)) \longrightarrow \bigoplus_{x\in R^{(0)}} H^i(k(x), \mathbb{Z}/p(n)) \longrightarrow$$
$$\bigoplus_{x\in R^{(1)}} H^{i-1}(k(x), \mathbb{Z}/p(n-1)) \longrightarrow \dots . \quad (11)$$

3. Motivic cohomology and K-theory

For a field k, $H^i(k, \mathbb{Z}(n)) = 0$ for $i > n$, and there is an isomorphism

$$K_n^M(k) \xrightarrow{\sim} H^n(k, \mathbb{Z}(n)), \quad (12)$$

in particular, $H^n(k, \mathbb{Z}/p^r(n)) = K_n^M(k)/p^r$.

In the cubical version, the map of (12) is given by [29]:

$$\{u_1, \dots , u_n\} \mapsto \left(\frac{u_1}{u_1 - 1}, \dots , \frac{u_n}{u_n - 1}\right) \in (\square_k^n)^{(n)}.$$

In the simplical version, the map is given by Nesterenko-Suslin [24, theorem 4.9]:

$$\{u_1, \dots , u_n\} \mapsto \left(\frac{-u_1}{1 - \sum u_i}, \dots , \frac{-u_n}{1 - \sum u_i}, \frac{1}{1 - \sum u_i}\right) \in (\Delta_k^n)^{(n)}. \quad (13)$$

Proposition 3.1. *Let R be a semi-localization of a regular finite type k-algebra for k a field of characteristic p. Then there are isomorphisms*

$$H^n(R, \mathbb{Z}/p(n)) \xleftarrow{\sim} K_n^M(R)/p \xrightarrow[\sim]{d\log} \nu^n(R).$$

Proof. We use Quillen's method to reduce to the case R essentially smooth over a perfect field: There exists a subfield k' of k which is finitely generated over $\mathbb{F}_p$ and a semi-localization R' of a regular k'-algebra of finite type such that R is a localization of $R' \otimes_{k'} k$. By letting k_i run through the finitely generated subfields of k containing k', we can assume that k is finitely generated over $\mathbb{F}_p$, since all functors in the proposition commute with direct limits. But if R is a localization of a regular k-algebra of finite type, and k is finitely generated over $\mathbb{F}_p$, then R is a localization of a finite type regular $\mathbb{F}_p$-algebra, so we can assume $k = \mathbb{F}_p$.

Consider the following diagram, where the vertical maps are the symbol maps and the horizontal maps are the residue maps:

$$\begin{array}{ccc}
\bigoplus_{x\in R^{(0)}} H^n(k(x), \mathbb{Z}/p(n)) & \longrightarrow & \bigoplus_{x\in R^{(1)}} H^{n-1}(k(x)), \mathbb{Z}/p(n-1)) \\
\uparrow & & \uparrow \\
\bigoplus_{x\in R^{(0)}} K_n^M(k(x))/p & \longrightarrow & \bigoplus_{x\in R^{(1)}} K_{n-1}^M(k(x))/p \\
{\scriptstyle d\log}\downarrow & & {\scriptstyle d\log}\downarrow \\
\bigoplus_{x\in R^{(0)}} \nu^n(k(x)) & \longrightarrow & \bigoplus_{x\in R^{(1)}} \nu^{n-1}(k(x))
\end{array}$$

The upper square is commutative by the following lemma, and the lower diagram is commutative up to sign by [14]. The vertical maps are isomorphisms by (7) and (12). This implies that the induced map on the kernel of the horizontal maps is an isomorphism as well. But the kernel of the upper horizontal map is $H^n(R, \mathbb{Z}/p(n))$ by (11), of the middle horizontal map $K_n^M(R)/p$ by (6), and of the lower horizontal map $\nu^n(R)$ by (5). Q.E.D.

Lemma 3.2. *Let R be a discrete valuation ring over the field k with quotient field F and residue field $\mathfrak{f}$. Then the following diagram commutes*

$$\begin{array}{ccc}
K_n^M(F) & \xrightarrow{\delta} & K_{n-1}^M(\mathfrak{f}) \\
\downarrow & & \downarrow \\
H^n(F, \mathbb{Z}/p(n)) & \xrightarrow{\delta} & H^{n-1}(\mathfrak{f}, \mathbb{Z}/p(n-1)).
\end{array}$$

Proof. By multi-linearity of symbols, we only have to check commutativity of the diagram for symbols of the form $\{u_1, \dots, u_{n-1}, \pi\}$ and $\{u_1, u_2, \dots, u_n\}$ with $u_i \neq 1$ units of R. The first symbol maps to the point

$$\left(\frac{u_1}{u_1 - 1}, \dots, \frac{u_{n-1}}{u_{n-1} - 1}, \frac{\pi}{\pi - 1}\right)$$

in $z^n(F, n)$. This point extends to a curve in $z^n(R, n)$, and the boundary map for motivic cohomology is calculated by intersecting it with faces to get an element of $z^n(R, n-1) = z^{n-1}(\mathfrak{f}, n-1)$. But this curve only meets the face $t_n = 0$ of $\square_{n,R}$, in the point

$$\left(\frac{u_1}{u_1 - 1}, \ldots, \frac{u_{n-1}}{u_{n-1} - 1}\right) \in z^{n-1}(\mathfrak{f}, n-1)$$

(recall that Spec $\mathfrak{f}$ is the subscheme of Spec R given by $\pi = 0$). Similarly, a symbol of the second type is mapped to the point

$$\left(\frac{u_1}{u_1 - 1}, \ldots, \frac{u_n}{u_n - 1}\right),$$

which does not meet any of the faces of $\square_{n,R}$. Hence its image in $z^n(R, n-1)$ is zero. Q.E.D.

If R is the semi-localization of a smooth k-algebra and $I_1, \ldots, I_m$ ideals of R such that R/I_j is essentially smooth, then Proposition 3.1 implies that there is a canonical map

$$H^n(R, I_1, \ldots I_m, \mathbb{Z}/p(n)) \longrightarrow \nu^n(R, I_1, \ldots, I_m). \tag{14}$$

In fact, defining

$$H^n(R, I_1, \ldots, I_m, \mathbb{Z}/p(n))^{\mathrm{ker}} = \ker\Big(H^n(R, \mathbb{Z}/p(n)) \longrightarrow \bigoplus_j H^n(R/I_j, \mathbb{Z}/p(n))\Big), \tag{15}$$

there is a map

$$H^n(R, I_1, \ldots, I_m, \mathbb{Z}/p(n)) \longrightarrow H^n(R, I_1, \ldots, I_m, \mathbb{Z}/p(n))^{\mathrm{ker}} \xrightarrow[\sim]{d\log} \nu^n(R, I_1, \ldots, I_m).$$

Bloch and Lichtenbaum [6] prove that there is a spectral sequence relating motivic cohomology and algebraic K-theory of fields:

$$H^{s-t}(k, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(k).$$

We will use the following proposition in the proof of Theorem 8.1.

Proposition 3.3. *The composition of the isomorphism of Nesterenko-Suslin with the edge homomorphism of the Bloch-Lichtenbaum spectral sequence,* $K_n^M(k) \xrightarrow{\theta_n} H^n(k, \mathbb{Z}(n)) \longrightarrow K_n(k)$, *agrees with the natural map from Milnor K-theory to Quillen K-theory.*

Proof. We use the simplicial definition of motivic cohomology and first recall how the edge morphism $\epsilon_n : H^n(k, \mathbb{Z}(n)) \longrightarrow K_n(k)$ is constructed, see [6] for details:

Let $\partial\Delta^n_k$ denote the set of faces $\{t_i = 0 \mid i = 0, \dots, n\}$ of Δ^n_k and $\partial_0\Delta^n_k$ be the set of faces $\{t_i = 0 \mid i = 0, \dots, n-1\}$. By the homotopy property, $K_p(\Delta^n_k, \partial_0\Delta^n_k) = 0$ for all $p \geq 0$. Identifying Δ^{n-1}_k with the face $t_n = 0$ of Δ^n_k gives the relativization sequences

$$K_{p+1}(\Delta^n_k, \partial_0\Delta^n_k) \xrightarrow{t_n=0} K_{p+1}(\Delta^{n-1}_k, \partial\Delta^{n-1}_k) \longrightarrow K_p(\Delta^n_k, \partial\Delta^n_k) \longrightarrow K_p(\Delta^n_k, \partial_0\Delta^n_k).$$

This in turn gives isomorphisms

$$K_{p+1}(\Delta^{n-1}_k, \partial\Delta^{n-1}_k) \xrightarrow[\sim]{\partial} K_p(\Delta^n_k, \partial\Delta^n_k),$$

and hence the identification

$$K_n(k) \cong K_0(\Delta^n_k, \partial\Delta^n_k). \tag{16}$$

We note that the terms in the relativization sequence have a natural *right*-$K_*(k)$-module structure via the structure morphism, compatible with the maps in the sequence.

Let Z be a closed subscheme of Δ^n_k which is disjoint from $\partial\Delta^n_k$. The canonical identification $K_p(Z) \cong K_p(Z, \emptyset, \dots, \emptyset)$ gives the functorial push-forward homomorphism

$$i^Z_* : K_p(Z) \to K_p(\Delta^n_k, \partial\Delta^n_k).$$

In particular, a zero dimensional subscheme z on Δ^n_k which has support disjoint from $\partial\Delta^n_k$ canonically determines an element $i^z_*(1) \in K_0(\Delta^n_k, \partial\Delta^n_k)$. The map sending z to $i^z_*(1)$ factors through

$$\epsilon_n : H^n(k, \mathbb{Z}(n)) \to K_0(\Delta^n_k, \partial\Delta^n_k),$$

and the composition of ϵ_n with the identification (16) is the edge homomorphism of the spectral sequence. We have to show that the following diagram commutes

$$\begin{array}{ccc} K^M_n(k) & \xrightarrow{\iota_n} & K_n(k) \\ \Big\downarrow{\scriptstyle\theta_n} & & \Big\downarrow{\scriptstyle\partial}\,{\scriptstyle\cong} \\ H^n(k, \mathbb{Z}(n)) & \xrightarrow{\epsilon_n} & K_0(\Delta^n_k, \partial\Delta^n_k) \end{array}$$

We first consider the case $n = 1$.

For a commutative ring R and ideal I, let $\mathcal{H}_{R,I}$ be the category of finitely generated R-modules M of finite projective dimension, with

$$\mathrm{Tor}^R_p(M, R/I) = 0$$

for all $p > 0$. Let $\mathcal{H}_{R/I}$ be the category of finitely generated R/I modules of finite projective dimension. By Quillen's resolution theorem, $\Omega BQ\mathcal{H}_{R,I}$ is a model for the K-theory space $K(R)$ of R, and $\Omega BQ\mathcal{H}_{R/I}$ is a model for $K(R/I)$. The functor

$$M \mapsto M \otimes_R R/I$$

is exact on $\mathcal{H}_{R,I}$, so we may use the homotopy fiber of the induced map

$$\Omega BQ\mathcal{H}_{R,I} \to \Omega BQ\mathcal{H}_{R/I}$$

as a model for the relative K-theory space $K(R, I)$.

Let (M, M', g) be a triple with $M, M' \in \mathcal{H}_{R,I}$, and

$$g : M/IM \to M'/IM'$$

an isomorphism. The triple (M, M', g) gives rise to an element $[M, M', g] \in K_0(R, I)$ as follows: The map g determines a path $[g]$ from M/IM to M'/IM' in $\Omega BQ\mathcal{H}_{R/I}$, and $[M, M', g]$ is the pair consisting of the point $M - M'$ of $\Omega BQ\mathcal{H}_{R,I}$, together with the path from $M/IM - M'/IM'$ to 0 in $\Omega BQ\mathcal{H}_{R/I}$ gotten by translating [g] by $-M'/IM'$. With the obvious notion of exact sequences of triples (M, M', g), it is not hard to see that the functor sending (M, M', g) to $[M, M', g]$ is additive. In addition, if g is an automorphism of $(R/I)^n$, then the image of g under the boundary homomorphism

$$\partial : K_1(R/I) \to K_0(R, I)$$

is the class $[R^n, R^n, g]$.

We apply these considerations to $K_0(\Delta^1_k, \partial\Delta^1_k)$, setting

$$R = k[t_0, t_1]/(t_0 + t_1 - 1); \quad I = t_0 t_1.$$

Let $1 \neq u \in k^\times$ be a unit, giving the point $z = (\frac{-u}{1-u}, \frac{1}{1-u})$ of Δ^1_k. The inclusion of z into Δ^1_k gives the map

$$i^z_* : K_0(k(z)) \to K_0(\Delta^1_k, \partial\Delta^1_k).$$

Explicitly, we have $i^z_*(N) = [N, 0, 0]$, where N is a finite dimensional $k(z)$-vector space viewed as an R-module via the map $R \longrightarrow k(z)$.

We identify $K_1(k)$ with the component $K_1(\{(1, 0)\})$ of $K_1(\partial\Delta^1_k)$, so the image of a unit $u \in K_1(k) = k^\times$ under the boundary isomorphism

$$\partial : K_1(k) \to K_0(\Delta^1_k, \partial\Delta^1_k)$$

is given by the triple $(R, R, (1, u))$,

$$(1, u) \in k((0, 1))^\times \times k((1, 0))^\times = k(\partial\Delta^1)^\times.$$

Let

$$\alpha = \frac{1-u}{u}\left(t_0 + \frac{u}{1-u}\right) \in R.$$

Note that $\alpha(0,1)=1$, $\alpha(1,0)=u^{-1}$, and R/α is the R-module $k(z)$. The short exact sequence of triples

$$0 \longrightarrow (R, R, \mathrm{id}) \xrightarrow{(\mathrm{id}, \times\alpha)} (R, R, (1, u^{-1})) \longrightarrow (0, R/\alpha, 0) \longrightarrow 0$$

and the fact that the first term represents $0 \in K_0(\Delta^1_k, \partial\Delta^1_k)$ then shows that the boundary of $(1, u^{-1})$ is the class $[0, k(z), 0]$.

Since $[0, N, 0] = -[N, 0, 0]$, for each R-module N with $N/IN = 0$, we see that

$$i^z_*(1) = \partial(u),$$

completing the case $n = 1$.

We prove the general case by induction on n. Take $u_1, \dots, u_n$ in $k^\times$ with $\sum_{i=j}^n u_i \neq 1$ for all $j = 1, \dots, n$. Let

$$z = \left(\frac{-u_1}{1-\sum_i u_i}, \dots, \frac{-u_n}{1-\sum_i u_i}, \frac{1}{1-\sum_i u_i}\right) \in \Delta^n_k.$$

The line $L_u \subseteq \Delta^n$ defined by the equations

$$t_{i-1} = \frac{-u_i}{1-\sum_i u_i}; \quad i = 1, \dots, n-1$$

contains the point z, and intersects $\partial\Delta^n$ only in the faces $t_n = 0$ and $t_{n-1} = 0$.

We calculate the class of $i^z_*(1) \in K_0(\Delta^n_k, \partial\Delta^n_k)$ by factoring the embedding through L_u:

$$\begin{array}{ccccc}
 & & & & K_0(k) \\
 & & & & \downarrow{\scriptstyle i^z_*} \\
 & & K_1(k) & \xrightarrow{\sim} & K_0(L_u, \{p, q\}) \\
 & & \downarrow{\scriptstyle i^q_*} & & \downarrow{\scriptstyle i^{L_u}_*} \\
K_n(k) & \xrightarrow{\sim} & K_1(\Delta^{n-1}_k, \partial\Delta^{n-1}_k) & \xrightarrow{\sim} & K_0(\Delta^n_k, \partial\Delta^n_k).
\end{array}$$

Here p and q are the points of intersection of L_u with the face $t_{n-1} = 0$ and $t_n = 0$. One easily checks that

$$q = \left(\frac{-u_1}{1-\sum u_i}, \dots, \frac{-u_{n-1}}{1-\sum u_i}, \frac{1-u_n}{1-\sum u_i}\right).$$

The map

$$(t_0, \dots, t_n) \mapsto \left(\frac{t_{n-1}}{t_{n-1}+t_n}, \frac{t_n}{t_{n-1}+t_n}\right)$$

defines an affine-linear isomorphism of $(\mathcal{L}_u, t_{n-1} = 0, t_n = 0)$ with $(\Delta^1_k, (0,1), (1,0))$, and sends the point z to

$$\left(\frac{-u_n/1-\sum_i u_i}{1-u_n/1-\sum_i u_i}, \frac{1/1-\sum_i u_i}{1-u_n/1-\sum_i u_i}\right) = \left(\frac{-u_n}{1-u_n}, \frac{1}{1-u_n}\right).$$

Using the case $n = 1$, we see that the image $i^z_*(1)$ in $K_1(k)$ in the above diagram is u_n. We have to calculate $i^q_*(u_n) \in K_1(\Delta^{n-1}_k, \partial\Delta^{n-1}_k) \cong K_n(k)$. Since this map is a map of right $K_*(k)$-modules, we have $i^q_*(u_n) = i^q_*(1) \cup u_n$, where $i^q_*(1)$ is the image of 1 under $i^q_* : K_0(k) \longrightarrow K_0(\Delta^{n-1}_k, \partial\Delta^{n-1}_k)$. One checks that the image of the symbol $\{\frac{u_1}{1-u_n}, \dots, \frac{u_{n-1}}{1-u_n}\}$ under the map (13) is the point q, hence by induction hypothesis the image of $i^q_*(1)$ in $K_{n-1}(k)$ is $\{\frac{u_1}{1-u_n}, \dots, \frac{u_{n-1}}{1-u_n}\}$. Consequently, we have

$$i^z_*(1) = i^q_*(q_n) = i^q_*(1) \cup u_n = \left\{\frac{u_1}{1-u_n}, \dots, \frac{u_{n-1}}{1-u_n}, u_n\right\}$$

in $K_n(k)$. By the Steinberg relation, the last term agrees with $\{u_1, \dots, u_{n-1}, u_n\}$. Q.E.D.

4. The semi-local m-cube

Let $\hat{\square}_m$ be the semi-localization of $\square_m$ with respect to the 2^m points where all coordinates are 0 or 1. We define the set of ideals

$$T_m = \{(t_i - \epsilon) \mid i = 1, \dots, m;\ \epsilon \in \{0, 1\}\}.$$

We order the ideals I_s of T_m by

$$I_s = \begin{cases} (t_s - 1) & \text{for } s \le m \\ (t_{s-m}) & \text{for } s > m, \end{cases}$$

and let T^s_m be the subset of T_m consisting of the first s ideals. Let

$$S_m = T^{2m-1}_m = T_m - \{(t_m)\}.$$

For an ideal I_s, let $\hat{\square}^s_m$ be the closed subscheme of $\hat{\square}_m$ defined by I_s, and $\hat{\square}^{s,t}_m$ the closed subscheme defined by $I_s + I_t$. For $r > s$, let T^s_m/I_r be the set of ideals in the ring of functions of $\hat{\square}^r_m \cong \hat{\square}_{m-1}$ given by the image of the ideals in T^s_m, after deleting those ideals which become the unit ideal, i.e.

$$T^s_m/I_r = \begin{cases} T^s_{m-1} & \text{for } r \le m \text{ or } s < r - m \\ T^{s-1}_{m-1} & \text{for } r > m \text{ and } s \ge r - m. \end{cases}$$

Throughout this paragraph, let $\mathcal{S}$ be the set of all faces of $\hat{\square}_m$, i.e. the set of closed subsets of $\hat{\square}_m$ given by the ideals of T_m and all their intersections.

For $1 \le s \le m$ let

$$i_s : (t_1, \dots, t_{m-1}) \mapsto (t_1, \dots, t_{s-1}, 0, t_s, \dots, t_{m-1})$$
$$j_s : (t_1, \dots, t_{m-1}) \mapsto (t_1, \dots, t_{s-1}, 1, t_s, \dots, t_{m-1})$$

be the inclusion of the faces $t_s = 0$ and $t_s = 1$ into $\hat{\square}_m$. We have the identities

$$i_t i_s = \begin{cases} i_{s+1} i_t & s \geq t \\ i_s i_{t-1} & s < t \end{cases}; \qquad j_t i_s = \begin{cases} i_{s+1} j_t & s \geq t \\ i_s j_{t-1} & s < t \end{cases}; \tag{17}$$
$$j_t j_s = \begin{cases} j_{s+1} j_t & s \geq t \\ j_s j_{t-1} & s < t \end{cases}.$$

Similarly, we define projection maps for $1 \leq s \leq m$ and $1 \leq s < m$, respectively,

$$p_s : (t_1, \dots, t_m) \mapsto (t_1, \dots, t_{s-1}, t_{s+1}, \dots, t_m)$$
$$q_s : (t_1, \dots, t_m) \mapsto (t_1, \dots, t_{s-1}, 1 - (t_s - 1)(t_{s+1} - 1), t_{s+2}, \dots, t_m).$$

The following identities hold

$$p_s j_t = \begin{cases} j_{t-1} p_s & t > s \\ \mathrm{id} & t = s \\ j_t p_{s-1} & t < s \end{cases}; \qquad p_s i_t = \begin{cases} i_{t-1} p_s & t > s \\ \mathrm{id} & t = s \\ i_t p_{s-1} & t < s \end{cases} \tag{18}$$

$$q_s j_t = \begin{cases} j_{t-1} q_s & t > s+1 \\ j_s p_s & t = s, s+1 \\ j_t q_{s-1} & t < s \end{cases}; \qquad q_s i_t = \begin{cases} i_{t-1} q_s & t > s+1 \\ \mathrm{id} & t = s, s+1 \\ i_t q_{s-1} & t < s \end{cases}. \tag{19}$$

We have the subcomplex $z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}^{\mathrm{ker}}$ of $z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}$ defined by

$$z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}^{\mathrm{ker}} = \ker\Big(z^n(\hat{\square}_m, *)_{\mathcal{S}} \longrightarrow \bigoplus_{t \leq s} z^n(\hat{\square}_m^t, *)_{\mathcal{S}} \Big).$$

Proposition 4.1. *The canonical map*

$$z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}^{\mathrm{ker}} \to z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}$$

is a quasi-isomorphism for $s \leq 2m$ and $n > 0$, and the inclusion

$$z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}^{\mathrm{ker}} \subseteq z^n(\hat{\square}_m, T_m^{s-1}, *)_{\mathcal{S}}^{\mathrm{ker}}$$

is functorially split for $s<2m$.

Proof. By contravariant functoriality, there are maps

$$i_s^*, j_s^* : z^n(\hat{\square}_m, *)_{\mathcal{S}} \longrightarrow z^n(\hat{\square}_{m-1}, *)_{\mathcal{S}}$$
$$p_s^*, q_s^* : z^n(\hat{\square}_{m-1}, *)_{\mathcal{S}} \longrightarrow z^n(\hat{\square}_m, *)_{\mathcal{S}}.$$

Let

$$\iota_s = \begin{cases} j_s^* & \text{for } s \leq m \\ i_{s-m}^* & \text{for } s > m \end{cases}; \qquad \rho_s = \begin{cases} p_s^* & \text{for } s \leq m \\ q_{s-m}^* & \text{for } s > m \end{cases}.$$

We can assume that the proposition holds for $\hat{\square}_{m-1}$ and proceed by induction on s. For $s \leq 2m$, consider the following commutative diagram of complexes:

$$\begin{array}{ccccc}
z^n(\hat{\square}_m, T^s_m, *)^{\mathrm{ker}}_{\mathcal{S}} & \xrightarrow{\mathrm{incl}} & z^n(\hat{\square}_m, T^{s-1}_m, *)^{\mathrm{ker}}_{\mathcal{S}} & \xrightarrow{\iota_s} & z^n(\hat{\square}^s_m, T^{s-1}_m/I_s, *)^{\mathrm{ker}}_{\mathcal{S}} \\
\downarrow & & \downarrow & & \downarrow \\
z^n(\hat{\square}_m, T^s_m, *)_{\mathcal{S}} & \longrightarrow & z^n(\hat{\square}_m, T^{s-1}_m, *)_{\mathcal{S}} & \longrightarrow & z^n(\hat{\square}^s_m, T^{s-1}_m/I_s, *)_{\mathcal{S}} \; .
\end{array}$$

The vertical maps are the natural maps, and by induction on m and s we can assume that the middle and right vertical map is a quasi-isomorphism. If we can show that the map ι_s is split surjective for $s < 2m$ and surjective for $s = 2m$ and $n > 0$, then the second statement of the proposition follows, and both rows of the diagram define distinguished triangles in the derived category of complexes. Since by induction the two right vertical maps are quasi-isomorphisms, the left vertical map will be a quasi-isomorphism as well.

Consider the following diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
z^n(\hat{\square}_m, T^s_m, *)^{\mathrm{ker}}_{\mathcal{S}} & \longrightarrow & z^n(\hat{\square}_m, T^{s-1}_m, *)^{\mathrm{ker}}_{\mathcal{S}} & \xrightarrow{\iota_s} & z^n(\hat{\square}^s_m, T^{s-1}_m/I_s, *)^{\mathrm{ker}}_{\mathcal{S}} \\
\downarrow & & \downarrow & & \downarrow \\
z^n(\hat{\square}_m, I_s, *)_{\mathcal{S}} & \longrightarrow & z^n(\hat{\square}_m, *)_{\mathcal{S}} & \xrightarrow{\iota_s} & z^n(\hat{\square}^s_m, *)_{\mathcal{S}} \\
 & & \downarrow{\scriptstyle \prod_{t<s}\iota_t} & & \downarrow{\scriptstyle \prod_{t<s}\iota_t} \\
 & & \bigoplus_{t<s} z^n(\hat{\square}^t_m, *)_{\mathcal{S}} & & \bigoplus_{t<s} z^n(\hat{\square}^{t,s}_m, *)_{\mathcal{S}}
\end{array} \tag{20}$$

Let α be an element of $z^n(\hat{\square}^s_m, T^{s-1}_m/I_s, *)^{\mathrm{ker}}_{\mathcal{S}}$, i.e. an element of $z^n(\hat{\square}^s_m, *)_{\mathcal{S}}$ mapping to zero under ι_t for $t < s$. If $s < 2m$, let $\bar{\alpha} = \rho_s(\alpha) \in z^n(\hat{\square}_m, *)_{\mathcal{S}}$. We have to show that $\bar{\alpha}$ lies in the kernel of ι_t for $t < s$. This follows from (18) for $t < s \leq m$:

$$\iota_t \bar{\alpha} = j^*_t p^*_s \alpha = p^*_{s-1} j^*_t \alpha = 0.$$

For $s > t > m$ this follows from (19):

$$\iota_t \bar{\alpha} = i^*_{t-m} q^*_{s-m} \alpha = q^*_{s-m-1} i^*_{t-m} \alpha = 0$$

and for $t \leq m < s$,

$$\iota_t \bar{\alpha} = j^*_t q^*_{s-m} \alpha = \begin{cases} q^*_s j^*_{t-1} \alpha = 0 & t \geq s-m+1 \\ p^*_{s-m} j^*_t \alpha = 0 & t = s-m, s-m+1 \\ q^*_{s-m-1} j^*_t \alpha = 0 & t < s-m. \end{cases} \tag{21}$$

It remains to show for $n > 0$ the surjectivity of the restriction map

$$z^n(\hat{\square}_m, S_m, q)^{\mathrm{ker}}_{\mathcal{S}} \xrightarrow{i_m^*} z^n(\hat{\square}_m^{2m}, T_{m-1}, q)^{\mathrm{ker}}_{\mathcal{S}}. \tag{22}$$

The group $z^n(\hat{\square}_{m-1}, 0)_{\mathcal{S}}$ is zero, as a cycle on $\hat{\square}_{m-1}$ which intersects a vertex properly must necessarily avoid the vertex, and $\hat{\square}_{m-1}$ is the semi-local scheme of the set of vertices in $\square_{m-1}$. Thus

$$z^n(\hat{\square}_m^{2m}, T_{m-1}, 0)^{\mathrm{ker}}_{\mathcal{S}} \subseteq z^n(\hat{\square}_{m-1}, 0)_{\mathcal{S}} = 0,$$

proving surjectivity in this case. Now suppose $q > 0$. Identify $\square_q \times \square_m$ with $\square_{q+m}$ by

$$((x_1, \dots, x_q), (t_1, \dots, t_m)) \to (x_1, \dots, x_q, t_m, \dots, t_1)$$

and make a similar identification of $\square_q \times \square_{m-1}$ with $\square_{q+m-1}$. This identifies $\square_q \times \hat{\square}_m$ and $\square_q \times \hat{\square}_{m-1}$ with subschemes of $\square_{q+m}$ and $\square_{q+m-1}$, and identifies the map i_m^* of (22) with the map i_{q+1}^* for $\square_{q+m}$. Let $\bar{i}_{q+1}$ and $\bar{q}_q$ be the restriction of i_{q+1} and q_q, respectively. Then the pull-back along $\bar{q}_q$ gives a splitting to (22) because $i_{q+1}^* q_q^* = \mathrm{id}$. It is a map from $z^n(\hat{\square}_m^{2m}, q)_{\mathcal{S}}$ to $z^n(\hat{\square}_m, q)_{\mathcal{S}}$, because the restriction of $\bar{q}_q^*(x)$ to all but the last face of $\square_q$ is trivial if the same holds for x. Similarly, it is a map from $z^n(\hat{\square}_m^{2m}, T_{m-1}, q)^{\mathrm{ker}}_{\mathcal{S}}$ to $z^n(\hat{\square}_m, S_m, q)^{\mathrm{ker}}_{\mathcal{S}}$, because the restriction of $\bar{q}_q^*(x)$ to all but the last face of $\hat{\square}_m$ is trivial if this holds for x. Note however that the map i_m^* of (22) is not split surjective as a map of complexes, because $\bar{q}_q^*$ does not commute with the differential $\bar{i}_q^*$ of the complexes. Q.E.D.

We are now ready to calculate some relative motivic cohomology groups:

Proposition 4.2. *a) There is an isomorphism*

$$H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) \xrightarrow{\sim} \nu^n(\hat{\square}_m, S_m).$$

b) Let $i \neq n$ and assume that $H^{i-1}(k, \mathbb{Z}/p(n-1)) = 0$ for all fields of characteristic p. Then

$$H^i(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) = 0.$$

Proof. We first show by induction on m that for $s < 2m$, there is an exact sequence

$$0 \longrightarrow H^i(\hat{\square}_m, T_m^s, \mathbb{Z}/p(n)) \xrightarrow{\gamma} H^i(\hat{\square}_m, \mathbb{Z}/p(n)) \xrightarrow{(\iota_t)} \bigoplus_{t \le s} H^i(\hat{\square}_m^t, \mathbb{Z}/p(n)).$$

Indeed, by Proposition 4.1, there is an exact sequence

$$0 \longrightarrow H^i(\hat{\square}_m, T^s_m, \mathbb{Z}/p(n)) \longrightarrow H^i(\hat{\square}_m, T^{s-1}_m, \mathbb{Z}/p(n)) \longrightarrow H^i(\hat{\square}^s_m, T^{s-1}_m/I_s, \mathbb{Z}/p(n)) \longrightarrow 0,$$

and by induction on m, the last term injects into $H^i(\hat{\square}^s_m, \mathbb{Z}/p(n))$.

Taking $i = n$ and $s = 2m - 1$, this proves (a) in view of Proposition 3.1. Note that γ factors through $H^i(\hat{\square}_m, (t_1 - 1, \ldots, t_m - 1), \mathbb{Z}/p(n))$, so for (b) it suffices to show that this latter group vanishes. The long exact relativization sequence

$$\longrightarrow H^i(\hat{\square}_m, (t_1 - 1, \ldots, t_m - 1), \mathbb{Z}/p(n)) \longrightarrow H^i(\hat{\square}_m, \mathbb{Z}/p(n)) \longrightarrow H^i(k, \mathbb{Z}/p(n)) \longrightarrow$$

is split by the structure map and gives

$$H^i(\hat{\square}_m, \mathbb{Z}/p(n)) \cong H^i(\hat{\square}_m, (t_1-1, \ldots, t_m-1), \mathbb{Z}/p(n)) \oplus H^i(k, \mathbb{Z}/p(n)).$$

Thus we have to show that the map $H^i(k, \mathbb{Z}/p(n)) \to H^i(\hat{\square}_m, \mathbb{Z}/p(n))$ is an isomorphism, or by homotopy invariance, that the restriction map

$$H^i(\mathbb{A}^m, \mathbb{Z}/p(n)) \to H^i(\hat{\square}_m, \mathbb{Z}/p(n))$$

is an isomorphism. For this, we know by the same proof as in [27, theorem 2.4], that the hypercohomology spectral sequence

$$H^s(\mathbb{A}^m, \mathcal{H}^t(\mathbb{Z}/p(n))) \Rightarrow H^{s+t}(\mathbb{A}^m, \mathbb{Z}/p(n))$$

is concentrated on the line $s = 0$: $H^i(\mathbb{A}^m, \mathbb{Z}/p(n)) = H^0(\mathbb{A}^m, \mathcal{H}^i(\mathbb{Z}/p(n)))$. Similarly, the Gersten conjecture for semi-local regular rings gives

$$H^i(\hat{\square}_m, \mathbb{Z}/p(n)) = H^0(\hat{\square}_m, \mathcal{H}^i(\mathbb{Z}/p(n))).$$

Comparing the Gersten resolution for the sheaf $\mathcal{H}^i(\mathbb{Z}/p(n))$ on $\mathbb{A}^m$ and $\hat{\square}_m$ we get the exact sequence

$$0 \longrightarrow H^0(\mathbb{A}^m, \mathcal{H}^i(\mathbb{Z}/p(n))) \longrightarrow H^0(\hat{\square}_m, \mathcal{H}^i(\mathbb{Z}/p(n))) \longrightarrow \bigoplus_{x \in (\mathbb{A}^m - \hat{\square}_m)^{(1)}} H^{i-1}(k(x), \mathbb{Z}/p(n-1)).$$

By assumption, the last term vanishes, giving the isomorphism we need. Q.E.D.

Corollary 4.3. *We have*

$$H^{n-1}(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) \cong H^{n-m}(k, \mathbb{Z}/p(n))$$
$$H^i(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) = 0 \qquad \text{for} \quad i > n.$$

In particular, there is an exact sequence

$$0 \longrightarrow H^{n-m}(k, \mathbb{Z}/p(n)) \longrightarrow H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \xrightarrow{\beta_m} H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) \longrightarrow H^n(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) \longrightarrow 0.$$

Proof. By definition of T_m and S_m, there is a long exact relativization sequence

$$\ldots \longrightarrow H^i(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \longrightarrow H^i(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) \xrightarrow{t_m=0} H^i(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) \longrightarrow H^{i+1}(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \longrightarrow \ldots . \tag{23}$$

We see from this sequence for varying m, together with Proposition 4.2(b), that

$$H^{n-1}(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) \cong H^{n-2}(\hat{\square}_{m-2}, T_{m-2}, \mathbb{Z}/p(n)) \cong \ldots \cong H^{n-m}(k, \mathbb{Z}/p(n)),$$

and that, for $i > n$,

$$H^i(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \cong H^{i+1}(\hat{\square}_{m+1}, T_{m+1}, \mathbb{Z}/p(n)) \cong \ldots \cong H^{2n+1}(\hat{\square}_{m+2n+1-i}, T_{m+2n+1-i}, \mathbb{Z}/p(n)).$$

But by Proposition 4.1, the latter group is the cohomology in degree zero of the complex $z^n(\hat{\square}_{m+2n+1-i}, T_{m+2n+1-i}, *)^{\ker}_{\mathcal{S}}$, which is zero. Q.E.D.

Corollary 4.4.

$$H^i(\hat{\square}_m, S_m, \mathbb{Z}(n)) = 0 \quad \text{for} \;\; i > n$$

$$H^i(\hat{\square}_m, T_m, \mathbb{Z}(n)) = 0 \quad \text{for} \;\; i > n.$$

In particular, the map $H^n(\hat{\square}_m, S_m, \mathbb{Z}/l(n)) \longrightarrow H^n(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/l(n))$ *is surjective.*

Proof. The only place where we used $\mathbb{Z}/p$-coefficients in this section is the hypothesis in Proposition 4.2(b). Hence the same proof as in Proposition 4.2 together with the (trivial) fact that $H^i(F, \mathbb{Z}(n)) = 0$ for $i > n$ and any field F proves the first statement. The second statement follows as in the proof of Corollary 4.3. Similarly, the same holds with mod l coefficients, and we get the last statement with the long relativization sequence. Q.E.D.

5. The boundary of the semi-local m-cube

Let $\partial\hat{\square}_m$ be the boundary of $\hat{\square}_m$, i.e. the closed subscheme defined by $\prod_i t_i(t_i-1)$. Let

$$b : \partial\hat{\square}_m \longrightarrow \hat{\square}_m$$

be the inclusion. Again, we let $\mathcal{S}$ be the set of all faces of $\hat{\square}_m$, order the ideals of T_m as in the previous paragraph, and denote the subset of T_m consisting of the first s ideals by T_m^s. We remind the reader that for ideals $I_s, I_t \in T_m$, $\partial\hat{\square}_m^s \cong \hat{\square}_{m-1}$ denotes the closed subscheme defined by I_s and $\partial\hat{\square}_m^{s,t} \cong \hat{\square}_{m-2}$ denotes the closed subscheme defined by $I_s + I_t$.

Bloch's higher Chow groups form a Borel-Moore homology theory, hence our definition of motivic cohomology via higher Chow groups is reasonable only for smooth schemes. In order to deal with $\partial\hat{\square}_m$, we give the following ad hoc construction of a cycle complex:

$$z^n(\partial\hat{\square}_m,*)_{\mathcal{S}} = \ker\Big(\bigoplus_{1\le s\le 2m} z^n(\partial\hat{\square}_m^s,*)_{\mathcal{S}} \longrightarrow \bigoplus_{1\le u<v\le 2m} z^n(\partial\hat{\square}_m^{u,v},*)_{\mathcal{S}}\Big).$$

If i is the inclusion of $\partial\hat{\square}_m^{u,v}$ into $\partial\hat{\square}_m^s$, then the map $z^n(\partial\hat{\square}_m^s,*)_{\mathcal{S}} \longrightarrow z^n(\partial\hat{\square}_m^{u,v},*)_{\mathcal{S}}$ is i^* for $s=u<v$, $-i^*$ for $u<v=s$, and zero otherwise. In other words, we consider cycles on the faces which agree on their intersections. This definition is motivated by the blow-up long exact sequence for motivic cohomology, see [4].

More generally, we define the following relative complex, where ι_t is the projection to the corresponding summand:

$$\begin{aligned} z^n(\partial\hat{\square}_m, T_m^s,*)_{\mathcal{S}} &= \ker\Big(z^n(\partial\hat{\square}_m,*)_{\mathcal{S}} \xrightarrow{(\iota_t)} \bigoplus_{t\le s} z^n(\partial\hat{\square}_m^t,*)_{\mathcal{S}}\Big) \\ &= \ker\Big(\bigoplus_{t>s} z^n(\partial\hat{\square}_m^t,*)_{\mathcal{S}} \longrightarrow \bigoplus_{1\le u<v\le 2m} z^n(\partial\hat{\square}_m^{u,v},*)_{\mathcal{S}}\Big). \end{aligned} \tag{24}$$

In particular, the second description shows that we have

$$z^n(\partial\hat{\square}_m, S_m,*)_{\mathcal{S}} = z^n(\hat{\square}_{m-1}, T_{m-1})^{\ker}_{\mathcal{S}}.$$

We define the relative motivic cohomology groups

$$H^{2n-i}(\partial\hat{\square}_m, T_m^s, \mathbb{Z}/p(n)) = H_i(z^n(\partial\hat{\square}_m, T_m^s,*)_{\mathcal{S}} \otimes \mathbb{Z}/p).$$

To simplify notation, we drop the coefficients $\mathbb{Z}/p(n)$ for the rest of this section and define the groups

$$H^n(\partial\hat{\square}_m)^{\ker} = \ker\Big(\bigoplus_{1\le s\le 2m} H^n(\partial\hat{\square}_m^s) \longrightarrow \bigoplus_{1\le u<v\le 2m} H^n(\partial\hat{\square}_m^{u,v})\Big),$$

with the same sign convention as above, and more generally

$$H^n(\partial\hat{\square}_m, T_m^s)^{\ker} = \ker\Big(H^n(\partial\hat{\square}_m)^{\ker} \xrightarrow{(\iota_t)} \bigoplus_{t\le s} H^n(\partial\hat{\square}_m^t)\Big)$$
$$= \ker\Big(\bigoplus_{t>s} H^n(\partial\hat{\square}_m^t) \longrightarrow \bigoplus_{1\le u<v\le 2m} H^n(\partial\hat{\square}_m^{u,v})\Big). \quad (25)$$

Again, these are the cohomology classes on faces which agree on the intersections of two faces. Note that there are restriction maps

$$b^* : z^n(\hat{\square}_m, T_m^s, *)_{\mathcal{S}}^{\ker} \longrightarrow z^n(\partial\hat{\square}_m, T_m^s, *)_{\mathcal{S}},$$

because cycles coming from $\hat{\square}_m$ agree on the intersection of the faces of $\partial\hat{\square}_m$. On the other hand, the cohomology of the kernel of a map between two complexes maps to the kernel of the induced map on cohomology, hence by Proposition 4.1 the map b^* induces restriction maps

$$b^* : H^n(\hat{\square}_m, T_m^s) \longrightarrow H^n(\partial\hat{\square}_m, T_m^s) \longrightarrow H^n(\partial\hat{\square}_m, T_m^s)^{\ker}.$$

If we let ι_t be the map induced by the inclusion $\partial\hat{\square}_m^t \to \partial\hat{\square}_m$, then by Proposition 3.1 and (25) we have

$$\nu^n(\partial\hat{\square}_m, T_m^s) = \ker\Big(\nu^n(\partial\hat{\square}_m) \xrightarrow{(\iota_t)} \bigoplus_{t\le s} \nu^n(\partial\hat{\square}_m^t)\Big) \xrightarrow{(\iota_t)}$$
$$\ker\Big(\bigoplus_{t>s} \nu^n(\partial\hat{\square}_m^t) \to \bigoplus_{u<v} \nu^n(\partial\hat{\square}_m^{u,v})\Big) \cong H^n(\partial\hat{\square}_m, T_m^s)^{\ker}. \quad (26)$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
H^n(\partial\hat{\square}_m, T_m^s) & \longrightarrow & H^n(\partial\hat{\square}_m, T_m^s)^{\ker} & \longleftarrow & \nu^n(\partial\hat{\square}_m, T_m^s) \\
\downarrow & & \downarrow & & \downarrow \\
H^n(\partial\hat{\square}_m, T_m^{s-1}) & \longrightarrow & H^n(\partial\hat{\square}_m, T_m^{s-1})^{\ker} & \longleftarrow & \nu^n(\partial\hat{\square}_m, T_m^{s-1})\ .
\end{array}$$

Proposition 5.1. *The vertical maps in the above diagram are compatibly split for $s < 2m$. In particular, there is a commutative diagram:*

$$\begin{array}{ccccc}
H^n(\partial\hat{\square}_m, S_m) & \longrightarrow & H^n(\partial\hat{\square}_m, S_m)^{\ker} & \longleftarrow & \nu^n(\partial\hat{\square}_m, S_m) \\
{\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow \\
H^n(\partial\hat{\square}_m) & \longrightarrow & H^n(\partial\hat{\square}_m)^{\ker} & \longleftarrow & \nu^n(\partial\hat{\square}_m)\ .
\end{array}$$

Proof. We will show in each case that there are maps i_s^*, j_s^*, p_s^* and q_s^* satisfying the equations (18) and (19); the proof of Proposition 4.1 gives the desired splitting. For ν^n, we have the four maps by contravariant functoriality, and there is a short exact sequence

$$0 \longrightarrow \nu^n(\partial\hat{\square}_m, T_m^s) \longrightarrow \nu^n(\partial\hat{\square}_m, T_m^{s-1}) \xrightarrow{\iota_s} \nu(\partial\hat{\square}_m^s, T_m^{s-1}/I_s).$$

Here ι_s is induced by j_s^* for $s \leq m$ and by i_{s-m}^* for $s > m$. As in diagram (20), one checks that the map ι_s is split by ρ_s, where $\rho_s = p_s^*$ for $s \leq m$ and $\rho_s = q_s^*$ for $s > m$.

For z^n and $H^n(-)^{\mathrm{ker}}$ we have exact sequences

$$\begin{aligned} 0 \longrightarrow z^n(\partial\hat{\square}_m, T_m^s, *)_{\mathcal{S}} \longrightarrow& \\ & z^n(\partial\hat{\square}_m, T_m^{s-1}, *)_{\mathcal{S}} \xrightarrow{\iota_s} z^n(\partial\hat{\square}_m^s, T_m^{s-1}/I_s, *)_{\mathcal{S}}, \\ 0 \longrightarrow H^n(\partial\hat{\square}_m, T_m^s)^{\mathrm{ker}} \longrightarrow& \\ & H^n(\partial\hat{\square}_m, T_m^{s-1})^{\mathrm{ker}} \xrightarrow{\iota_s} H^n(\partial\hat{\square}_m^s, T_m^{s-1}/I_s)^{\mathrm{ker}}. \end{aligned}$$

Here ι_s is the projection to the corresponding summand in (24), and (25), and one sees from the second description in (24) and (25) that the image of ι_s is contained in the relative cohomology group indicated. It suffices to show that the map ι_s is split surjective. We define maps $\bar{p}_s^*$ and $\bar{q}_s^*$ by factoring through $\hat{\square}_m$:

$$z^n(\hat{\square}_m, *)_{\mathcal{S}} \xrightarrow{p_s^*, q_s^*} z^n(\hat{\square}_m, *)_{\mathcal{S}} \xrightarrow{b^*} z^n(\partial\hat{\square}_m, *)_{\mathcal{S}}.$$

$$H^n(\hat{\square}_m) \xrightarrow{p_s^*, q_s^*} H^n(\hat{\square}_m) \xrightarrow{b^*} H^n(\partial\hat{\square}_m)^{\mathrm{ker}},$$

Note that the maps $\bar{p}_s^*$ and $\bar{q}_s^*$ have image in the kernel defining the relative cycle complex and cohomology groups, respectively. Furthermore, together with the maps ι_s they satisfy the equations (18) and (19), hence the proof in diagram (20) extends to this situation.

Compatibility for the map between cohomology groups is obvious, because the map $\mathrm{id} - \rho_s\iota_s$ on complexes induces the map $\mathrm{id} - \rho_s\iota_s$ on cohomology.

For the compatibility for the map between ν^n and $H^n(-)^{\mathrm{ker}}$ with the splitting, we have to check that the splitting is compatible with the map of (26). Since the splitting is given by $x \mapsto x - \rho_s\iota_s x$, it suffices to show that

$$(\iota_t\rho_s\iota_s x)_t = \rho_s\iota_s(\iota_t x)_t \in \bigoplus_{t>s} \nu^n(\partial\hat{\square}_m^t).$$

But this follows from the following commutative diagram. The upper composition is $\rho_s \iota_s x$, since the map ρ_s for ν^n factors through $\nu^n(\hat{\square}_m)$:

$$\begin{array}{ccccccc}
\nu^n(\partial\hat{\square}_m) & \xrightarrow{\iota_s} & \nu^n(\hat{\square}_{m-1}) & \xrightarrow{\rho_s} & \nu^n(\hat{\square}_{m-1}) & \xrightarrow{b^*} & \nu^n(\partial\hat{\square}_m) \\
{\scriptstyle(\iota_t)}\downarrow & & \| & & \| & & {\scriptstyle(\iota_t)}\downarrow \\
\bigoplus_t \nu^n(\hat{\square}_{m-1}) & \xrightarrow{\mathrm{proj}_s} & \nu^n(\hat{\square}_{m-1}) & \xrightarrow{\rho_s} & \nu^n(\hat{\square}_{m-1}) & \xrightarrow{(\iota_t)} & \bigoplus_t \nu^n(\hat{\square}_{m-1})
\end{array}$$

Q.E.D.

6. Torsion in ν^n

In this section we are going to prove the following theorem

Theorem 6.1. *Let τ be a section of $\nu^n(\partial\hat{\square}_m, T_m)$. Then we can write τ as a sum of elements of the form $df_1 \wedge \ldots \wedge df_n$, with $\prod_i f_i = 0$ on $\partial\hat{\square}_m$.*

Consider $\hat{\square}_m$ as the face $t_{m+1} = 0$ of $\partial\hat{\square}_{m+1}$, and let

$$p_{m+1} : \partial\hat{\square}_{m+1} \longrightarrow \hat{\square}_m$$

be the projection. For the proof of Theorem 1.1, we need the following:

Corollary 6.2. *The restriction map*

$$\nu^n(\partial\hat{\square}_{m+1}, S_{m+1}) \longrightarrow \nu^n(\hat{\square}_m, T_m)$$

is surjective.

Proof. Let $\omega \in \nu^n(\hat{\square}_m, T_m)$ and consider the following short exact sequence:

$$0 \longrightarrow \nu^n(\hat{\square}_m, \partial\hat{\square}_m) \longrightarrow \nu^n(\hat{\square}_m, T_m) \longrightarrow \nu^n(\partial\hat{\square}_m, T_m).$$

By Theorem 6.1, each element of $\nu^n(\partial\hat{\square}_m, T_m)$ is a sum of elements of the form $df_1 \wedge \ldots \wedge df_n$ with $\prod_i f_i = 0$ on $\partial\hat{\square}_m$. Let $\hat{f}_i$ be lifts of the functions f_i to $\hat{\square}_m$. We pull these functions back to $\partial\hat{\square}_{m+1}$ via p_{m+1}, and let f_i' be the function $p_{m+1}^*(\hat{f}_i) \cdot (1 - t_{m+1})$ on $\partial\hat{\square}_{m+1}$. Let $g = \frac{1}{1 - f_1' \cdots f_n'}$, then

$$\frac{dg}{g} \wedge \frac{df_2'}{f_2'} \wedge \ldots \wedge \frac{df_n'}{f_n'} = \frac{1}{1 - f_1' \cdots f_n'} df_1' \wedge \ldots \wedge df_n'$$

is a section of $\nu^n(\partial\hat{\square}_{m+1}, S_{m+1})$ with restriction $df_1 \wedge \ldots \wedge df_n$ to $\partial\hat{\square}_m$ and with trivial restriction to the other faces. Thus we can assume that ω is contained in $\nu^n(\hat{\square}_m, \partial\hat{\square}_m)$.

Cover $\partial\hat{\square}_{m+1}$ by the two open sets $U = \partial\hat{\square}_{m+1} - \{t_{m+1} = 0\}$ and $V = \partial\hat{\square}_{m+1} - \{t_{m+1} = 1\}$. We lift ω via p_{m+1} to $\nu^n(V)$. Since $\omega|_{\partial\hat{\square}_m} = 0$, this lift agrees with the zero section of $\nu^n(U)$ on $U \cap V$. Thus we can glue these sections, to get a section of $\nu^n(\partial\hat{\square}_{m+1})$ which vanishes on $\partial\hat{\square}_{m+1} - \{t_{m+1} = 0\}$. Q.E.D.

We proceed with the proof of Theorem 6.1. We call the differentials on $\partial\hat{\square}_m$ which vanish on each face *torsion differentials*:

$$\Omega^n_{\partial\hat{\square}_m,tor} = \ker\big(\Omega^n_{\partial\hat{\square}_m} \longrightarrow \bigoplus_{1\le s\le 2m} \Omega^n_{\partial\hat{\square}^s_m}\big).$$

Lemma 6.3. *Each section of $\Omega^n_{\partial\hat{\square}_m,tor}$ can be written as a sum of terms of the form $f_0 df_1 \dots df_n$, with $\prod_i f_i = 0$.*

Proof. For each vertex v of $\partial\hat{\square}_m$, let T_v be the product of the functions t_i, $t_i - 1$, $1 \le i \le m$, which are non-zero at v, T'_v the product of the functions which vanish at v. Note that $T_v T'_v = 0$ on $\partial\hat{\square}_m$. It suffices to show that, for each v, each section ω of $\Omega^n_{\partial\hat{\square}_m,tor}$ can be written as a sum of sections of the form $f_0 df_1 \wedge \dots \wedge df_n$, with $T'_v | f_0 \prod_i f_i$. Indeed, since $T_v(v) = \pm 1$ and $T_{v'}(v) = 0$ for $v \neq v'$, the function $u = \sum_v T_v$ is a unit on $\partial\hat{\square}_m$, and

$$\omega = \sum_v \frac{1}{u} T_v \omega$$

is in the desired form.

To prove the local statement, we may assume $v = (0, \dots, 0)$. We lift ω to a section $\tilde{\omega}$ of $\Omega^n_{\hat{\square}_m}$ such that $\tilde{\omega}$ maps to zero in $\Omega^n_{\hat{\square}^s_m}$ for each $1 \le s \le m$. Now $\Omega^n_{\hat{\square}_m}$ is a free $\mathcal{O}_{\hat{\square}_m}$-module with basis $dt_{i_1} \wedge \dots \wedge dt_{i_n}$, $1 \le i_1 < \dots < i_n \le m$, and the kernel of the restriction map $\Omega^n_{\hat{\square}_m} \to \Omega^n_{\hat{\square}^s_m}$ is $t_s \Omega^n_{\hat{\square}_m} + dt_s \Omega^{n-1}_{\hat{\square}_m}$. Thus, if we write

$$\tilde{\omega} = \sum_{I=(i_1<\dots<i_n)} a_I dt_{i_1} \wedge \dots \wedge dt_{i_n},$$

either $t_s | a_I$, or s is in I, for each index I. Since this holds for each s, we have $T'_v | a_I t_{i_1} \cdots t_{i_n}$ as desired. Q.E.D.

Proposition 6.4. *We have*

$$\nu^n(\partial\hat{\square}_m, T_m) = d\Omega^{n-1}_{\partial\hat{\square}_m} \cap \Omega^n_{\partial\hat{\square}_m,tor}.$$

Proof. We first recall that

$$C^{-1}(f_0 df_1 \wedge \ldots \wedge df_n) = f_0^p (f_1 \cdots f_n)^{p-1} df_1 \wedge \ldots \wedge df_n,$$

By Lemma 6.3, this implies that C^{-1} is zero on $\Omega^n_{\partial\hat{\square}_m,tor}$. The proposition now follows in view of the following commutative diagram, where $1 \leq s \leq 2m$, and the fact that the upper right horizontal map is the canonical surjection.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \nu^n(\partial\hat{\square}_m, T_m) & \longrightarrow & \Omega^n_{\partial\hat{\square}_m,tor} & \xrightarrow{\mathrm{id}-C^{-1}} & \Omega^n_{\partial\hat{\square}_m,tor}/d\Omega^{n-1}_{\partial\hat{\square}_m} \cap \Omega^n_{\partial\hat{\square}_m,tor} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \nu^n(\partial\hat{\square}_m) & \longrightarrow & \Omega^n_{\partial\hat{\square}_m} & \xrightarrow{\mathrm{id}-C^{-1}} & \Omega^n_{\partial\hat{\square}_m}/d\Omega^{n-1}_{\partial\hat{\square}_m} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \oplus_s \nu^n(\partial\hat{\square}^s_m) & \longrightarrow & \oplus_s \Omega^n_{\partial\hat{\square}^s_m} & \xrightarrow{\mathrm{id}-C^{-1}} & \oplus_s \Omega^n_{\partial\hat{\square}^s_m}/d\Omega^{n-1}_{\partial\hat{\square}^s_m}
\end{array}$$

Q.E.D.

By Lemma 6.3 and Proposition 6.4, Theorem 6.1 is equivalent to the statement:

$$d\Omega^{n-1}_{\partial\hat{\square}_m,tor} = d\Omega^{n-1}_{\partial\hat{\square}_m} \cap \Omega^n_{\partial\hat{\square}_m,tor}.$$

Obviously,

$$d\Omega^{n-1}_{\partial\hat{\square}_m,tor} \subseteq d\Omega^{n-1}_{\partial\hat{\square}_m} \cap \Omega^n_{\partial\hat{\square}_m,tor} \subseteq Z\Omega^n_{\partial\hat{\square}_m,tor},$$

hence it suffices to show that the complex

$$\ldots \longrightarrow \Omega^n_{\partial\hat{\square}_m,tor} \longrightarrow \Omega^{n+1}_{\partial\hat{\square}_m,tor} \longrightarrow \ldots$$

is exact. We begin with the following special case:

Proposition 6.5. *Let $R_m = \mathbb{F}_p[t_1, \ldots, t_m]/(t_1 \cdots t_m)$. Then for any $\mathbb{F}_p$-algebra A, the complex*

$$\Omega^*_{R_m\otimes A,tor} := \ker\left(\Omega^*_{R_m\otimes A} \longrightarrow \bigoplus_i \Omega^*_{R_m/t_i\otimes A}\right)$$

is exact.

Proof. First note that $\Omega^*_{R_m\otimes A}$ is isomorphic to the total complex of the double complex $\Omega^*_{R_m}\otimes\Omega^*_A$ by (3). Since Ω^j_A is an $\mathbb{F}_p$-module, tensoring with Ω^j_A over $\mathbb{F}_p$ is exact. Hence $\Omega^*_{R_m\otimes A,tor}$ is the total complex of the double complex $\Omega^*_{R_m,tor}\otimes\Omega^*_A$. On the other hand, the total complex is exact if the rows of the double complex are exact, so it suffices to prove the proposition for R_m.

We proceed by induction on m and assume the result for $m-1$ and all $\mathbb{F}_p$-algebras A. We have

$$\Omega^n_{R_m}=\bigoplus_{i_1<\ldots<i_n}R_m dt_{i_1}\wedge\ldots\wedge dt_{i_n}/\Big(\sum_i\prod_{l\neq i}t_l dt_i\Big)\Omega^{n-1}_{R_m}.$$

Assigning degree one to t_m and dt_m and degree zero to t_i and dt_i for $i<m$, we can give each element of $\Omega^n_{R_m}$ a degree, since $\sum_i\prod_{l\neq i}t_l dt_i$ is homogeneous. This is compatible with differentials, hence decomposes $\Omega^n_{R_m}$ into a direct sum of subcomplexes.

Suppose ω is in $Z\Omega^n_{R_m,tor}$. Then $d\omega=0$ and we can assume that ω has nonzero degree i in t_m. Write

$$\omega=\omega_1 t_m^{i-1}dt_m+\omega_2 t_m^i.$$

The map $\Omega^n_{R_m}\longrightarrow\Omega^n_{R_{m-1}}$ induced by $t_m\mapsto 1$ sends ω to ω_2, hence ω_2 is in $Z\Omega^n_{R_{m-1},tor}$. By induction, $\omega_2=d\tau$ for some $\tau\in\Omega^{n-1}_{R_{m-1},tor}$. Then τt_m^i is in $\Omega^{n-1}_{R_m,tor}$,

$$\omega-d(\tau t_m^i)=\omega-\omega_2 t_m^i-i\tau t_m^{i-1}dt_m=(\omega_1-i\tau)t_m^{i-1}dt_m,$$

and we can assume $\omega_2=0$.

Let $S=\mathbb{F}_p[t_m,t_m^{-1}]$ and consider the localization map $R_m\longrightarrow R_{m-1}\otimes S$. From $\Omega^*_S=S\oplus S dt_m$ we get

$$\Omega^n_{R_{m-1}\otimes S,tor}=S\otimes\Omega^n_{R_{m-1},tor}\oplus S\otimes\Omega^{n-1}_{R_{m-1},tor}dt_m.$$

Thus

$$0=d\omega=d(\omega_1 t_m^{i-1}dt_m)=t_m^{i-1}d\omega_1 dt_m$$

in $\Omega^n_{R_{m-1}\otimes S,tor}$ implies that ω_1 is in $\Omega^{n-1}_{R_{m-1},tor}$ and $d\omega_1=0$. By induction, $\omega_1=d\tau_1$ for some τ_1 in $\Omega^{n-2}_{R_{m-1},tor}$. Then $\tau_1 t_m^{i-1}dt_m$ is in $\Omega^{n-1}_{R_m,tor}$, and $d(\tau_1 t_m^{i-1}dt_m)=\omega_1 t_m^{i-1}dt_m=\omega$. Q.E.D.

Lemma 6.6. *The complex* $\Omega^*_{(\partial\hat{\square}_m)_v,tor}$ *is exact for the localization* $(\partial\hat{\square}_m)_v$ *of* $\partial\hat{\square}_m$ *at the point* $v = (0, \ldots, 0)$.

Proof. Applying the proposition to $A = k$, we see that the complex $\Omega^*_{S_m,tor}$ is exact for $S_m = k[t_1, \ldots, t_m]/(t_1 \cdots t_m)$. Since $(\partial\hat{\square}_m)_v \cong \mathrm{Spec}(S_m)_v$, it suffices to show that exactness is preserved by localization. Recall that $d(x^p\omega) = x^p d\omega$ for all differentials, because we are in characteristic p. Let $\frac{\omega}{s} \in \Omega^n_{(S_m)_v,tor}$ with $d\frac{\omega}{s} = 0$.Then

$$0 = d\frac{\omega s^{p-1}}{s^p} = \frac{1}{s^p} d(\omega s^{p-1}) \in \Omega^n_{(S_m)_v,tor}.$$

This means that there is a $t \in (S_m)_v^\times$, such that $td(\omega s^{p-1}) = 0$ and hence $d\omega t^p s^{p-1} = t^p d\omega s^{p-1}$ is trivial in $\Omega^n_{S_m,tor}$. By the proposition, $\omega t^p s^{p-1} = d\tau$ in $\Omega^n_{S_m,tor}$ for some τ, and

$$d\frac{\tau}{t^p s^p} = \frac{1}{t^p s^p} d\tau = \frac{\omega}{s}.$$

Q.E.D.

To finish the proof of Theorem 6.1, we show that the complex $\Omega^*_{\partial\hat{\square}_m,tor}$ is exact if it is exact for the localization of $\partial\hat{\square}_m$ at each of the vertices v. Let $\omega \in \Omega^n_{\partial\hat{\square}_m,tor}$ with $d\omega = 0$. By the lemma, for each v the image of ω in $\Omega^n_{(\partial\hat{\square}_m)_v,tor}$ can be written as $d\tau_v$ for $\tau_v = \frac{\sigma_v}{s_v} \in \Omega^{n-1}_{(\partial\hat{\square}_m)_v,tor}$, $\sigma_v \in \Omega^{n-1}_{\partial\hat{\square}_m}$ and s_v a function on $\partial\hat{\square}_m$ which does not vanish at v.

We first show that we can choose $\sigma_v \in \Omega^{n-1}_{\partial\hat{\square}_m,tor}$. We can assume that $v = (0, \ldots, 0)$. Since τ_v vanishes in $\Omega^{n-1}_{(\partial\hat{\square}^i_m)_v}$ for each $i > m$, there are functions $u_{v,i}$ on $\partial\hat{\square}_m$ which do not vanish at v, such that $u_{v,i}\tau_v$ is zero in $\Omega^{n-1}_{\partial\hat{\square}^i_m}$. We now replace σ_v and s_v by their product with $\prod_i u_{v,i}(t_i - 1)$, a unit at v. Then τ_v does not change, but σ_v is contained in $\Omega^{n-1}_{\partial\hat{\square}_m,tor}$.

Finally, choose functions c_v on $\partial\hat{\square}_m$ such that $\sum c_v s_v = 1$. Then $\sum c_v^p s_v^p = \left(\sum c_v s_v\right)^p = 1$, and

$$d\sum_v c_v^p s_v^{p-1}\sigma_v = \sum_v c_v^p s_v^p d\frac{\sigma_v}{s_v} = \sum_v c_v^p s_v^p \omega = \omega.$$

Q.E.D.

7. The theorem

In this section we finish the proof of Theorem 1.1. Using coefficient sequences, one easily sees that it suffices to consider the case $r = 1$. By Corollary 4.3, the group $H^{n-m}(k, \mathbb{Z}/p(n))$ is the kernel of β_m in the following commutative diagram:

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
H^{n-m}(k, \mathbb{Z}/p(n)) & & 0 \\
\downarrow & & \downarrow \\
H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) & \xrightarrow{\alpha_m} & \nu^n(\hat{\square}_m, T_m) \\
{\scriptstyle \beta_m}\downarrow & & \downarrow \\
H^n(\hat{\square}_m, S_m, \mathbb{Z}/p(n)) & \xrightarrow{\cong} & \nu^n(\hat{\square}_m, S_m) \\
\downarrow & & \downarrow \\
H^n(\hat{\square}_{m-1}, T_{m-1}, \mathbb{Z}/p(n)) & \xrightarrow{\alpha_{m-1}} & \nu^n(\hat{\square}_{m-1}, T_{m-1}) \\
\downarrow & & \\
0 & &
\end{array}$$

The injectivity of β_m is equivalent to the injectivity of α_m, and the snake lemma in a diagram for a different m shows that α_m is injective if and only if α_{m+1} is surjective. Thus Theorem 1.1 follows from

Theorem 7.1. *For any m, the map*

$$\alpha_m : H^n(\hat{\square}_m, T_m, \mathbb{Z}/p(n)) \longrightarrow \nu^n(\hat{\square}_m, T_m)$$

is a surjection.

We first show that we can lift sections of $\nu^n(\partial\hat{\square}_{m+1}, S_{m+1})$ to sections of smooth schemes:

Proposition 7.2. *Let $x \in \nu^n(\partial\hat{\square}_{m+1})$. Then there exists a smooth semi-local scheme U_x containing $\partial\hat{\square}_{m+1}$ as a closed subscheme, such that x lies in the image of the restriction map $\nu^n(U_x) \longrightarrow \nu^n(\partial\hat{\square}_{m+1})$.*

Proof. We prove more generally the following statement: Let k be a field, $X \subseteq \mathbb{A}^N_k$ be an open subset of an affine space, and Y the subscheme of X defined by some polynomial f. Then for every section $\omega_Y \in \nu^n(Y)$, there is an étale neighborhood U of Y in X such that ω_Y is the restriction of a section of $\nu^n(U)$.

Let $x_1, \ldots, x_N$ be the standard coordinates of $\mathbb{A}^N_k$. For indices $I = (1 \le i_1 < \ldots < i_n \le N)$, let $dx^I = dx_{i_1} \wedge \ldots \wedge dx_{i_n}$ and $x^I = x_{i_1} \cdots x_{i_n}$. Lift ω_Y to a section $\omega \in \Omega^n_X$, and write $\omega = \sum_I a_I dx^I$. The condition for $\omega|_Y$ to be a section of $\nu^n(Y)$ is $(1 - C^{-1})\omega|_Y \in d\Omega^{n-1}_Y$, i.e.

$$\sum_I (a_I - a_I^p (x^I)^{p-1}) dx^I = d\chi + f\eta + df \cdot \tau$$

for some $n-1$ forms χ, τ and an n-form η on X. As $d(f\tau) = df \cdot \tau + fd\tau$, we may absorb the term $df \cdot \tau$ into $d\chi$ and $f\eta$, and assume $\tau = 0$. Write $d\chi = \sum_I b_I dx^I$, giving the equation

$$\sum_I (a_I - a_I^p (x^I)^{p-1} - b_I) dx^I = f\eta.$$

In $\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, the right hand side vanishes. Since $\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is free over $\mathcal{O}_Y$ with generators dx^I, we get

$$a_I - a_I^p (x^I)^{p-1} - b_I = 0 \tag{27}$$

on Y for each I. Now define the closed subscheme of $\mathbb{A}^1_X$

$$\mathbb{A}_I = \mathrm{Spec}_{\mathcal{O}_X} \mathcal{O}_X[t_I]/(t_I - (x^I)^{p-1} t_I^p - b_I).$$

Since the coefficient of t_I is 1, $\mathbb{A}_I$ is flat over X. Since

$$d(t_I - (x^I)^{p-1} t_I^p - b_I) = dt_I \in \Omega_{\mathbb{A}_I/X},$$

we have $\Omega_{\mathbb{A}_I/X} = 0$, so $\mathbb{A}_I$ is unramified, hence étale over X.

Let $q : U \to X$ be the fiber product of the $\mathbb{A}_I$ for all I over X. The relation (27) determines a section $a_Y : Y \longrightarrow U$ by sending t_I to a_I, identifying Y with the closed subscheme of U defined by $t_I - a_I = 0$ for all I, hence U is an étale neighborhood of Y in X.

Since $q : U \to X$ is étale, the pull-backs of the dx^I give a basis of Ω^n_U over $\mathcal{O}_U$. Let $y_i = q^*(x_i)$, then $q^*(dx^I) = dy^I$ and $q^*(d\chi) = \sum_I q^*(b_I) dy^I$. Let $\omega_U = \sum_I t_I dy^I \in \Omega^n_U$, then

$$(1 - C^{-1})(\omega_U) = \sum_I (t_I - t_I^p (y^I)^{p-1}) dy^I.$$

Since we are on U, we have $t_I - t_I^p (y^I)^{p-1} = q^* b_I$, hence $(1 - C^{-1})(\omega_U) = q^* db = d(q^* b)$. Thus ω_U defines a section of ν^n over U. As Y is given by $t_I = a_I$, ω_U extends the given form $\sum_I a_I dx^I$. Q.E.D.

To finish the proof of Theorem 7.1, consider now the following diagram, where we omit the coefficients $\mathbb{Z}/p(n)$:

$$\begin{array}{ccccc}
H^n(\hat{\square}_m, T_m) & \xrightarrow{\alpha_m} & H^n(\hat{\square}_m, T_m)^{\mathrm{ker}} & \xrightarrow{\cong} & \nu^n(\hat{\square}_m, T_m) \\
\| & & \| & & \uparrow \\
H^n(\partial\hat{\square}_{m+1}, S_{m+1}) & \xrightarrow{\alpha_m} & H^n(\partial\hat{\square}_{m+1}, S_{m+1})^{\mathrm{ker}} & \longleftarrow & \nu^n(\partial\hat{\square}_{m+1}, S_{m+1}) \\
{\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow & & {\scriptstyle s}\uparrow \\
H^n(\partial\hat{\square}_{m+1}) & \longrightarrow & H^n(\partial\hat{\square}_{m+1})^{\mathrm{ker}} & \longleftarrow & \nu^n(\partial\hat{\square}_{m+1}) \\
\uparrow & & \uparrow & & \uparrow \\
\oplus_x H^n(U_x) & = & \oplus_x H^n(U_x) & \xleftarrow{\cong} & \oplus_x \nu^n(U_x) .
\end{array}$$

The upper and lower squares are commutative by definition of the cohomology groups and construction of the maps, and the middle squares are commutative by Proposition 5.1. The composition in the right vertical column is surjective by Corollary 6.2 and Proposition 7.2, hence the composition in the middle column is surjective. But then the map α_m is surjective, concluding the proof of the theorem.

8. Consequences

Theorem 8.1. *For any field k of characteristic p, the groups $K_n^M(k)$ and $K_n(k)$ are p-torsion free. The natural map*

$$K_n^M(k)/p^r \longrightarrow K_n(k, \mathbb{Z}/p^r)$$

is an isomorphism, and the natural map

$$K_n^M(k) \longrightarrow K_n(k)$$

is an isomorphism up to uniquely p-divisible groups.

Proof. Since $H^{n-1}(k, \mathbb{Z}/p(n)) = 0$, the long exact coefficient sequence shows that $K_n^M(k)$ is p-torsion free, reproving Izhboldin's theorem [16]. Appyling the Bloch-Lichtenbaum spectral sequence [6] with coefficients [26, Appendix B],

$$H^{s-t}(k, \mathbb{Z}/p(-t)) \Rightarrow K_{-s-t}(k, \mathbb{Z}/p),$$

we get $K_n^M(k)/p \cong H^n(k, \mathbb{Z}/p(n)) \cong K_n(k, \mathbb{Z}/p)$, induced by the natural map from Milnor K-theory to Quillen K-theory by Proposition 3.3. In particular, the coefficient sequence for K-theory shows that $K_{n-1}(k)$ is p-torsion free and that $K_n^M(k)/p \cong K_n(k)/p$.

Since the composition of the natural map with the Chern class map from Quillen K-theory to Milnor K-theory is multiplication by $(n-1)!$ on $K_n^M(k)$, and since $K_n^M(k)$ is p-torsion free, we get, working modulo prime to p-torsion, a short exact sequence

$$0 \longrightarrow K_n^M(k) \longrightarrow K_n(k) \longrightarrow Q \longrightarrow 0.$$

This gives us

$$0 \longrightarrow {}_pQ \longrightarrow K_n^M(k)/p \longrightarrow K_n(k)/p \longrightarrow Q/p \longrightarrow 0.$$

Since the middle map is an isomorphism, Q is uniquely p-divisible. Q.E.D.

The following consequence of our result has been pointed out to us by B. Kahn:

Theorem 8.2. *Let R be a semi-local ring, essentially smooth over a discrete valuation ring. Then Gersten's conjecture for R holds with finite coefficients, i.e. for any m there is an exact sequence*

$$0 \longrightarrow K_n(R, \mathbb{Z}/m) \longrightarrow \bigoplus_{x \in R^{(0)}} K_n(k(x), \mathbb{Z}/m) \longrightarrow \bigoplus_{x \in R^{(1)}} K_{n-1}(k(x), \mathbb{Z}/m) \longrightarrow \dots .$$

Proof. The case m prime to the residue characteristic p of the base DVR was handled by Gillet [12], so we can restrict ourselves to the case $m = p^r$. By [2], or [11, Corollary 6], it suffices to show this in the case when R is a discrete valuation ring. The statement is a special case of the mod p Gersten conjecture for regular local rings containing a field [25] and [13], if R has equal characteristic p. In general, let F be the quotient field, k the residue field of R and t a uniformizer. We have a localization sequence

$$\dots \longrightarrow K_{n+1}(R, \mathbb{Z}/p^r) \longrightarrow K_{n+1}(F, \mathbb{Z}/p^r) \xrightarrow{\partial} K_n(k, \mathbb{Z}/p^r) \longrightarrow K_n(R, \mathbb{Z}/p^r) \longrightarrow \dots .$$

Let $\{x_1, \dots, x_n\} \in K_n^M(k)$ be a lifting of some element in $K_n(k, \mathbb{Z}/p^r) = K_n^M(k)/p^r$. Then one easily sees that $\{t, x_1, \dots, x_n\} \mod p^r$ lifts this element to $K_{n+1}^M(F)/p^r$, hence to $K_{n+1}(F, \mathbb{Z}/p^r)$. The resulting short exact sequence proves Gersten's conjecture. Q.E.D.

Note by the same argument, there is a Gersten resolution with rational coefficients, hence with integral coefficients, if Beilinson's conjecture (2) holds.

To get consequences for the K-theory of smooth varieties over perfect fields of characteristic p, we have to sheafify our result.

For a scheme X, we let $(\mathcal{K}/p^r)_n$ denote the Zariski sheaf associated to the presheaf $U \mapsto K_n(U, \mathbb{Z}/p^r)$, and $\mathcal{H}^i(\mathbb{Z}/p^r(n))$ the ith cohomology sheaf of the motivic complex.

Theorem 8.3. *Let X be a smooth variety over a perfect field of characteristic p. Then*

$$\mathcal{H}^i(\mathbb{Z}/p^r(n)) = 0 \quad \text{for } i \neq n$$
$$\mathcal{H}^n(\mathbb{Z}/p^r(n)) \cong \nu_r^n$$
$$(\mathcal{K}/p^r)_n \cong \nu_r^n.$$

Proof. The sheaves ν_r^n, $\mathcal{H}^i(\mathbb{Z}/p^r(n))$ and $(\mathcal{K}/p^r)_n$ admit Gersten resolutions by (5), (11) and (8). Hence $\mathcal{H}^i(\mathbb{Z}/p^r(n)) = 0$ for $i \neq n$ follows from the corresponding result for fields, and Proposition 3.1 implies that $\mathcal{H}^n(\mathbb{Z}/p^r(n)) = \nu_r^n$. The result for K-theory follows from Proposition 3.1 and Theorem 8.1. Q.E.D.

Theorem 8.4. *Let X be a smooth variety of dimension d over a perfect field of characteristic p. Then*

$$H^s(X_{\mathrm{Zar}}, \nu_t^r) \cong H^{s+t}(X, \mathbb{Z}/p^r(t)),$$

and there is a spectral sequence from motivic cohomology to K-theory

$$H^{s-t}(X, \mathbb{Z}/p^r(-t)) \Rightarrow K_{-s-t}(X, \mathbb{Z}/p^r).$$

In particular, we have $K_n(X, \mathbb{Z}/p^r) = 0$ for $n > \dim X$.

Proof. The hypercohomology spectral sequence for motivic cohomology collapses to a line, proving the first equality. The second statement is the first equality plugged into the Brown-Gersten spectral sequence for K-theory, and the third statement follows because $\nu_r^n = 0$ for $n > \dim X$. Q.E.D.

Our result also implies that Bloch's cycle complexes satisfy most of the Beilinson-Lichtenbaum [22] axioms for motivic complexes, as extended by Milne [23]. Let X be a smooth variety over a perfect field of characteristic p, and consider the complex of presheaves $\mathbb{Z}(n) = z^n(-, *)[-2n]$ on the small étale site of X. It is in fact a complex of sheaves for the étale topology, in particular for the Zariski topology. Let $\mathbb{Z}_{(p)}(n) = \mathbb{Z}(n) \otimes \mathbb{Z}_{(p)}$, and $\epsilon : X_{\text{ét}} \longrightarrow X_{\mathrm{Zar}}$ the change of topology map. We claim that $\mathbb{Z}_{(p)}(n)$ satisfies all the axioms of Beilinson-Lichtenbaum, except possibly acyclicity below degree 0 (i.e. the Soulé-Beilinson vanishing conjecture). This is clear for all axioms, except the following two:

Theorem 8.5. *There are exact triangles in the derived category of sheaves on X_{Zar} and $X_{\text{ét}}$, respectively:*

$$\mathbb{Z}(n)_{\mathrm{Zar}} \xrightarrow{\times p} \mathbb{Z}(n)_{\mathrm{Zar}} \xrightarrow{d\log} \tau_{\leq n} R\epsilon_* \nu^n[-n]$$
$$\mathbb{Z}(n)_{\text{ét}} \xrightarrow{\times p} \mathbb{Z}(n)_{\text{ét}} \xrightarrow{d\log} \nu^n[-n].$$

Proof. The cohomology sheaves $\mathcal{H}^i(\mathbb{Z}(n)_{\text{Zar}})$ are trivial for $i > n$ by Gersten resolution and the (trivial) fact that this holds for fields. Consequently, the map of complexes $\tau_{\leq n}\mathbb{Z}(n)_{\text{Zar}} \longrightarrow \mathbb{Z}(n)_{\text{Zar}}$ is a quasi-isomorphism. On the other hand, $\nu^n[-n]$ is (trivially) isomorphic to $\tau_{\leq n}R\epsilon_*\nu^n[-n]$. We define the map $d\log$ to be the following composition of maps of Zariski sheaves

$$\mathbb{Z}(n) \xleftarrow{\text{qis}} \tau_{\leq n}\mathbb{Z}(n) \longrightarrow \mathcal{H}^n(\tau_{\leq n}\mathbb{Z}(n)) \cong \mathcal{H}^n(\mathbb{Z}(n)) \xrightarrow{d\log} \nu^n \xrightarrow{\sim} \tau_{\leq n}R\epsilon_*\nu^n[-n].$$

By Theorem 8.3, the sheaves $\mathcal{H}^i(\mathbb{Z}(n))$ are uniquely p-divisible for $i \neq n$, and there is an exact sequence

$$0 \longrightarrow \mathcal{H}^n(\mathbb{Z}(n)_{\text{Zar}}) \xrightarrow{\times p} \mathcal{H}^n(\mathbb{Z}(n)_{\text{Zar}}) \longrightarrow \nu^n \longrightarrow 0.$$

This proves the theorem for the Zariski topology. The same argument works for any étale covering of X, hence the statement for the étale topology. Q.E.D.

Theorem 8.6. *(Hilbert's theorem 90) Let X be a smooth variety over a perfect field of characteristic p, then*

$$R^{n+1}\epsilon_*\mathbb{Z}_{(p)}(n)_{\text{ét}} = 0.$$

Proof. Since $\mathcal{H}^i(\mathbb{Z}(n)_{\text{ét}}) = 0$ for $i > n$, and since motivic cohomology with $\mathbb{Q}$-coefficients is the same as the étale version, we know that $R^{n+1}\epsilon_*\mathbb{Z}_{(p)}(n)_{\text{ét}}$ is p-power torsion. Consider the following map induced by $\mathbb{Z}(n)_{\text{Zar}} \longrightarrow R\epsilon_*\mathbb{Z}(n)_{\text{ét}}$,

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \mathcal{H}^n(\mathbb{Z}(n)_{\text{Zar}} \otimes \mathbb{Z}/p) & \longrightarrow & {}_p\mathcal{H}^{n+1}(\mathbb{Z}_{(p)}(n)_{\text{Zar}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & R^n\epsilon_*(\mathbb{Z}(n)_{\text{ét}} \otimes \mathbb{Z}/p) & \longrightarrow & {}_pR^{n+1}\epsilon_*\mathbb{Z}_{(p)}(n)_{\text{ét}} & \longrightarrow & 0\,.
\end{array}$$

The upper right hand group is trivial, so it suffices to show that the left vertical map is an isomorphism. But by the previous theorem, both terms agree with ν^n. Q.E.D.

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