P-ADIC K-THEORY OF HECKE CHARACTERS OF IMAGINARY QUADRATIC FIELDS AND AN ANALOGUE OF BEILINSON'S CONJECTURES

THOMAS GEISSER

0. Introduction

In this paper we combine ideas of Soulé [23] and Deninger [5, 6] to prove a p-adic analogue of Beilinson's conjectures for motives associated to Hecke characters of imaginary quadratic fields.

Let E be an elliptic curve defined over an imaginary quadratic field K with complex multiplication by the ring of integers of K. In [23], Soulé proved the following theorem:

Let p be a prime which splits in K and $l \geq 0$ such that $p-1$ divides neither l, $l+1$ nor $l+2$. Then there exists a $\mathbb{Z}_p$-submodule $\mathcal{V}_l \subseteq K_{2l+2}(E, \mathbb{Z}_p)$ and a regulator map $r_l : K_{2l+2}(E, \mathbb{Z}_p) \to \mathbb{Z}_p^2$, such that the index of $r_l(\mathcal{V}_l)$ in $\mathbb{Z}_p^2$ equals n_l, where n_l is the p-adic valuation of the value at $s = -l$ of a p-adic L-series analog to $L(E, s)$.

On the other hand let φ be a Hecke character of an imaginary quadratic field K of positive weight w. Then Deninger constructed a motive M in $\mathcal{M}_{\mathbb{Q}}(K)$, the category of Chow motives over K with coefficients in $\mathbb{Q}$, such that the L-series of M coincides with the L-series of φ. The motive M arises naturally as a factor of the Grothendieck restriction $\mathcal{R}_{F/K}(h_1(E))^{\otimes w}$ for a CM-elliptic curve E of Shimura type over a finite extension F of K. Then he proved parts of the Beilinson conjectures for M, i.e. he related the leading coefficient of $L(M, -l)$ to a map from $H_{\mathcal{A}}^{w+1}(M, \mathbb{Q}(l+w+1)) = K_{2l+w+1}(M)_{\mathbb{Q}}^{(l+w+1)}$ to Deligne cohomology.

Here we combine the ideas of both papers to prove a generalization of Soulé's theorem for motives M_Ω attached to Hecke characters of infinity type (a, b) and weight $w = a + b > 0$ in the category $\mathcal{M}_{\mathbb{Z}_p}(K)$ of Chow motives over K with coefficients in $\mathbb{Z}_p$. More precisely, we first prove a Grothendieck-Riemann-Roch theorem for K-groups with coefficients $K_a(X, \mathbb{Z}/p^n)$ and p big enough relative to $\dim X$. Then we show that the functors $K_a(-, \mathbb{Z}/p^n)^{(i)}$ factor through $\mathcal{M}_{\mathbb{Z}_p}$, and finally we prove the following theorem:

Theorem 0.1. *Let $l \geq 0$ and $p > (3[F : K] + 1)w + 2l + 1$ be a prime split in the imaginary quadratic field K, $a + l > 0$ and $b + l > 0$. Then there exists a submodule $\mathcal{V} \subseteq K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)}$ such that the length as an $\mathcal{O}_\Omega$-module of the cokernel of the regulator $R|_{\mathcal{V}}$ restricted to $\mathcal{V}$ equals the valuation of the p-adic L-function*

$$G(\varphi_\Omega \kappa^l, u_1^{-a-l} - 1, u_2^{-b-l} - 1).$$

The p-adic L-series here is a p-adic analogue of $L(\varphi, -l)$.

As a corollary we reprove Soulé's theorem, generalized to elliptic curves of Shimura type over any field F/K and with the precise description of the Adams eigenspaces which are involved.

Theorem 0.2. *Let E be an elliptic curve over F with complex multiplication by K and CM-character φ of Shimura type. If $l > 0$ and $p > 3[F : K] + 2l + 2$, then there exists a submodule $\mathcal{V} \subseteq K_{2l+2}(h_1(E), \mathbb{Z}_p)^{(l+2)} \subseteq K_{2l+2}(E, \mathbb{Z}_p)$ such that the index of the regulator map $R|_{\mathcal{V}}$ restricted to $\mathcal{V}$ equals the the p-adic valuation of*

$$\prod_{\lambda} G(\varphi_\lambda \kappa^l, u_1^{-1-l} - 1, u_2^{-l} - 1) G(\bar{\varphi}_\lambda \kappa^l, u_1^{-l} - 1, u_2^{-1-l} - 1).$$

The product runs over all $\mathbb{C}_p$-valued characters arising from φ inducing $\mathfrak{p}$ on K.

The product of the p-adic L-series is a p-adic analogue of

$$L(E, -l) = \prod_{\epsilon} L(\varphi_\epsilon, -l) \cdot L(\bar{\varphi}_\epsilon, -l)$$

where ϵ runs over the grossencharacters arising from the elliptic curve.

We now give a more detailed description of the content of this paper.

First we recall some facts about algebraic and étale K-theory and prove some theorems of K-theory which are well known after tensoring with $\mathbb{Q}$, by just inverting a bounded number of primes (for example the existence of an Adams eigenspace decomposition or the Grothendieck-Riemann-Roch theorem).

This suffices to show that the functors $K_a(X, \mathbb{Z}/p^n)^{(i)}$ factor through the category of Chow motives $\mathcal{M}_{\mathbb{Z}_p}(K)$, at least if p is big enough. This implies that the functors $K_a^{\text{ét}}(X, \mathbb{Z}/p^n)^{(i)}$ and $H^a_{\text{ét}}(X, \mathbb{Z}/p^n(i))$ also factor through $\mathcal{M}_{\mathbb{Z}_p}(K)$.

Next we recall some properties of Hecke characters and of the extension $F(E_{p^\infty})/K$, where $F(E_{p^\infty})$ is the field generated by the p-power torsion of a CM-elliptic curve.

We construct motives M in $\mathcal{M}_{\mathbb{Z}_p}(K)$ for Hecke characters of the imaginary quadratic field K of positive weight. This is done following Deninger [6]: take an elliptic curve over a field F/K with complex multiplication by K, consider the w-fold tensor product of the Grothendieck restriction $\mathcal{R}_{F/K} h_1(E)$ and decompose it by idempotents of its endomorphism algebra. Then every Hecke character of positive weight w arises in this fashion.

The quotient $\mathcal{U}/\mathcal{C}$ of local units modulo elliptic units plays a vital role in the proof of the main theorem, and we have to modify some results of de Shalit [19] to the two variable situation.

We follow ideas of Soulé [23] in constructing special elements in the p-adic K-groups $K_{2l+w+1}(M, \mathbb{Z}/p^n)^{(w+l+1)}$.

We then define the p-adic regulator map for p-adic K-theory, again following Soulé. We make the crucial observation that in the local situation (i.e. after tensoring our motive with $K_{\mathfrak{p}}$ for a prime $\mathfrak{p}$ dividing the split prime p) the map constructing elements composed with the regulator map is an isomorphism (up to some special cases).

Finally the main theorem is proven and specialized to elliptic curves (i.e. $w = 1$).

0.1. **Acknowledgments.** This paper is based on the Ph.D. thesis of the author under the supervision of C.Deninger. For a more detailed version see [9].

I would like to thank the following people for remarks and useful discussions: C.Deninger, U.Jannsen, B.Köck, K.Künnemann, M.Levine, G.McConnell, C.Soulé and J.Wildeshaus. Furthermore I would like to thank H.J.Nastold for his support, and B.Mazur and the Harvard Department of Mathematics for their hospitality during my stay at Harvard, where parts of this paper were written.

0.2. **Conventions.** All schemes are supposed to be of finite type over a field k, in particular we mean "smooth over k" whenever we talk about smooth schemes.

For a scheme X over k we denote by $\bar{X} = X \times_k \bar{k}$ the base extension to the algebraic closure $\bar{k}$ of k.

For an abelian group A we denote by ${}_nA$ the n-torsion elements of A.

If the scheme X is regular and separated, we identify the K and K'-groups of X.

The groups $H^a(X, \mathcal{F})$ always denote étale cohomology groups.

1. K-groups

1.1. **Higher K-groups.** For a scheme X let $BQ\mathcal{P}$ be the classifying space to the Quillen category $Q\mathcal{P}$ of the category $\mathcal{P}$ of locally free $\mathcal{O}_X$-modules. Similarly, let $BQ\mathcal{M}$ be the same construction applied to the category $\mathcal{M}$ of coherent $\mathcal{O}_X$-modules. Then one defines the algebraic K-groups to be

$$K_a(X) = \pi_{a+1}(BQ\mathcal{P}(X)), \qquad K_a(X, \mathbb{Z}/n) = \pi_{a+1}(BQ\mathcal{P}(X), \mathbb{Z}/n);$$
$$K'_a(X) = \pi_{a+1}(BQ\mathcal{M}(X)), \qquad K'_a(X, \mathbb{Z}/n) = \pi_{a+1}(BQ\mathcal{M}(X), \mathbb{Z}/n);$$

Here the homotopy groups with coefficients are given by maps $M_n^a \to BQ\mathcal{P}(X)$ up to homotopy, where M_n^a is the Moore space mod n [4]. For $a \geq 2$ the cofibration $M_n^a \to S^a \xrightarrow{\cdot n} S^a$ gives rise to the universal coefficient sequence

$$0 \to K_{a-1}(X)/n \to K_{a-1}(X, \mathbb{Z}/n) \to {}_nK_{a-2}(X) \to 0.$$

We define K-groups with $\mathbb{Z}_p$-coefficients to be

$$K_a(X, \mathbb{Z}_p) := \varprojlim K_a(X, \mathbb{Z}/p^n).$$

This does not agree with the usual definition $K_a(X)^\wedge_p$ as the homotopy groups of the p-completed K-spectrum [3, I 4], but there is an exact sequence [3]

$$0 \to \varprojlim{}^1{}_{p^n}K_a(X) \to K_a(X)^\wedge_p \to \varprojlim K_a(X, \mathbb{Z}/p^n) \to 0.$$

Many properties of K-groups are given by properties of the spaces BQP and fibration sequences of these as given in [17]. In particular K-groups are contravariantly functorial and via the identification $K_a(X) = K'_a(X)$ covariantly functorial for proper maps of regular quasiprojective schemes.

A different construction of K-groups is Quillen's $+$-construction,

$$K_a(A) = \pi_a(BGL(A)^+), \qquad K_a(A, \mathbb{Z}/n) = \pi_a(BGL(A)^+, \mathbb{Z}/n).$$

On these groups one can define a λ-algebra structure (see [14]). The universal coefficient sequence is (by naturality of λ^k) a sequence of λ-morphisms. There are various ways to extend the $+$-construction from rings to schemes, see for example [22] where the above construction is sheafified for regular, noetherian schemes of finite dimension. This gives us the structure of a K_0-λ-algebra on $K_a(X)$ and $K_a(X, \mathbb{Z}/n)$ with locally nilpotent γ-filtration. Since Moore spaces exist only for $a \geq 2$, one cannot define a λ-ring structure on $K_1(X, \mathbb{Z}/n)$ via the $+$-construction. However, by a construction of Grayson [12], we know that there exist Adams operators on the Q-construction which agree with the ones defined via the λ-ring structure above.

In [15] Loday constructs a cup product

$$\cup : K_a(A) \times K_b(A) \to K_{a+b}(A).$$

For K-groups with coefficients we use the pairing $BQ \vee BQ \to BQ^2$ of Waldhausen [26, par. 9] and the map $M_n^{a+b} \to M_n^a \wedge M_n^b$ of Browder [4] to construct a cup product. For $a \geq 2$ this agrees with Loday's product induced by the pairing of $BGL^+(A)$, see the discussion in [20, II 2.1].

The following lemma from linear algebra allows us to prove that K-theory with coefficients is the direct sum of its Adams eigenspaces under certain hypothesis:

Lemma 1.1. *Let M be an abelian group with descending filtration $F^\cdot$ such that $F^aM = M$ and $F^{b+1}M = 0$ for some natural numbers a and b. Let $k \in \mathbb{Z}$ such that k and $1-k^i$ act invertibly on M for $i = 1,\dots,b-a$ and let ψ^k be an operator on M commuting with the filtration such that ψ^k acts like k^q on the q-th graded piece. Define*

$$M^{(q)} = \{x \in M | \psi^k(x) = k^q x\}.$$

Then the $M^{(q)}$ split the filtration, i.e.

$$F^qM = \bigoplus_{q \leq i \leq b} M^{(i)}.$$

Furthermore if ψ^j is an operator commuting with ψ^k and acting like j^q on the q-th graded piece, then ψ^j acts like j^q on $M^{(q)}$. If j and $1-j^i$ act also invertibly on M, then $M^{(q)} = \{x \in M | \psi^j(x) = j^q x\}$.

Proof: Clearly $M^{(j)} \cap M^{(l)} = 0$ for $j \neq l$, and one sees with induction on the number of summands that the sums occurring are direct.

Let $x \in M^{(q)}$ and suppose $x \in F^jM$ for some $a \leq j < q$. Then $k^q x = \psi^k(x) = k^j x \mod F^{j+1}M$, so $(k^q - k^j)x \in F^{j+1}M$ and $x \in F^{j+1}M$. By induction we get $M^{(q)} \subseteq F^qM$ and thus $\oplus_{q \leq i \leq b} M^{(i)} \subseteq F^qM$.

To show the opposite inclusion we use descending induction on q:

Assume $F^{q+1}M = \oplus_{q+1 \leq i \leq b} M^{(i)}$ (this is true for $q = b$) and let $x \in F^qM$. Then $\psi^k(x) = k^q x + x'$ with $x' \in F^{q+1}M$, $x' = x_{q+1} + \dots + x_b$ where $x_j \in M^{(j)}$. Define $\lambda_j = \frac{1}{k^j - k^q}$, then $-k^q\lambda_j = 1 - k^j\lambda_j$ and we obtain for $\hat{x} = x - \sum_{j=q+1}^b \lambda_j x_j$:

$$\psi^k(\hat{x}) = \psi^k(x - \sum_{j=q+1}^b \lambda_j x_j) = k^q x + \sum_{j=q+1}^b (1 - k^j\lambda_j) x_j = k^q x - \sum_{j=q+1}^b k^q \lambda_j x_j = k^q \hat{x}.$$

Thus $x = \hat{x} + \sum_{j=q+1}^b \lambda_j x_j \in \oplus_{q \leq j \leq b} M^{(j)}$.

Because $\psi^k \circ \psi^j = \psi^j \circ \psi^k$ we have $\psi^j(M^{(q)}) \subseteq M^{(q)}$. For $x \in M^{(q)}$ we have by assumption $\psi^j(x) = j^q x + x' \in M^{(q)}$ where $x' \in F^{q+1}M$. So $\psi^j(x) - j^q x = x' \in M^{(q)} \cap F^{q+1}M = 0$ and we get $\psi^j(x) = j^q x$, thus proving $M^{(q)} \subseteq \{x \in M | \psi^j(x) = j^q x\}$. If j satisfies the condition on invertibility, then by symmetry the sets $M^{(q)}$ defined with k and j are equal. □

Remark: 1) If M is a $\mathbb{Z}/p$-vector space, we have $M^{(q)} = M^{(q+p-1)}$, so we clearly need some condition on k which implies $p - 1 > b - a$. On the other hand, if M is a $\mathbb{Z}_{(p)}$-module and $p - 1 > b - a$, we can take k to be a primitive root of unity mod p.

2) In the special case of Adams operators on a nilpotent γ-algebra M, we say that M has an Adams eigenspace decomposition, if $M = \oplus M^{(q)}$, where $M^{(q)}$ is the k^q-eigenspace of ψ^k for some k as in the lemma. For $j \neq k$, the operator ψ^j acts like j^q on $M^{(q)}$, but the j^q-eigenspace of ψ^j may be bigger than $M^{(q)}$.

3) As a corollary of the proof we see that if $\psi^k(x) - k^n x \in F^{n+1}$, then $x \in F^n$. This follows because we can modify x by an element of F^{n+1} to $\hat{x}$ such that $\psi^k(\hat{x}) = k^n \hat{x}$, thus $\hat{x} \in F^n$ and then $x \in F^n$.

1.2. **Etale K-theory.** We list some properties of étale K-groups $K_a^{\text{ét}}(X, \mathbb{Z}/p^n)$ and its connections to K-theory with coefficients, for more details see Dwyer and Friedlander [7].

For a noetherian scheme X of finite cohomological dimension, there exists a canonical map
$$\rho : K_a(X, \mathbb{Z}/p^n) \to K_a^{\text{ét}}(X, \mathbb{Z}/p^n).$$
According to Soulé [21, 2.3] the étale K-groups form a λ-ring and ρ is a morphism of λ-rings.

There is a fourth quadrant spectral sequence
$$E_r^{st}(X) \Rightarrow K_{-s-t}^{\text{ét}}(X, \mathbb{Z}/p^n)$$
where
$$E_2^{st}(X) = \begin{cases} H^s(X, \mathbb{Z}/p^n(-t/2)) & t \text{ even;} \\ 0 & t \text{ odd.} \end{cases}$$
The cup product structure on $\oplus K_\cdot(X, \mathbb{Z}/p^n)$ is compatible with ρ, where the product structure on $\oplus K_\cdot^{\text{ét}}(X, \mathbb{Z}/p^n)$ is the abutment of the cup product in étale cohomology.

Proposition 1.2. *(Soulé* [21, theorem 1]*) Let p be odd, X a noetherian scheme whose cohomological étale p-dimension d is finite. Assume $p \geq d/2 + 1$. Then the spectral sequence of Dwyer and Friedlander degenerates, $E_2 = E_\infty$, and the induced filtration on $K_a^{\acute{e}t}(X, \mathbb{Z}/p^n)$ admits a natural splitting.*

Proposition 1.3. *We have for $p \geq \frac{d+3}{2}$:*
$$K_a^{\acute{e}t}(X, \mathbb{Z}/p^n) = \bigoplus_{\frac{a}{2} \leq i \leq \frac{a+d}{2}} K_a^{\acute{e}t}(X, \mathbb{Z}/p^n)^{(i)}$$
and there is an isomorphism
$$K_a^{\acute{e}t}(X, \mathbb{Z}/p^n)^{(i)} \cong E_\infty^{2i-a,-2i} \cong H^{2i-a}(X, \mathbb{Z}/p^n(i)).$$

Proof: Let $F_{\text{ét}}^k K_a^{\text{ét}}(X, \mathbb{Z}/p^n)$ be the filtration by étale dimension attached to the spectral sequence of Dwyer and Friedlander.

By Soulé [21, prop.2] we know that ψ^k acts like k^i on $E_r^{s,-2i}$ and thus also on the s-th graded piece $F_{\text{ét}}^s / F_{\text{ét}}^{s+1} K_{2i-s}^{\text{ét}}(X, \mathbb{Z}/p^n)$.

Define a new filtration $F^i := F_{\text{ét}}^{2i-a} K_a^{\text{ét}}(X, \mathbb{Z}/p^n)$. Then ψ^k acts like k^i on F^i/F^{i+1}, we have $F^{\frac{a}{2}} = K_a^{\text{ét}}(X, \mathbb{Z}/p^n)$ and $F^i = 0$ for $2i > d + a$. Applying lemma 1.1, we see that if we chose a primitive root of unity mod p for k, the first equation follows.

The second equation follows from
$$K_a^{\text{ét}}(X, \mathbb{Z}/p^n)^{(i)} \cong F^i/F^{i+1} = F_{\text{ét}}^{2i-a}/F_{\text{ét}}^{2(i+1)-a} K_a^{\text{ét}}(X, \mathbb{Z}/p^n) \cong H^{2i-a}(X, \mathbb{Z}/p^n(i)).$$
□

According to Soulé [21, theorem 5] this isomorphism composed with ρ is, up to an automorphism, the same as the Chern class map for $p \geq i$.

We will also need the following

Proposition 1.4. *(Thomason* [25, theorem 4.11]*) Let X be a regular scheme over a number field or a local field. Then the map ρ becomes an isomorphism if we localize K-theory at the inverse of the Bott element* [25, A.7]*,*

$$\rho : K_a(X, \mathbb{Z}/p^n)[\beta^{-1}] \cong K_a^{ét}(X, \mathbb{Z}/p^n).$$

This allows us to transport structure from K-theory to étale K-theory, for example we get covariance for proper maps of quasiprojective, regular varieties. On the other hand, the induced contravariance and product structure agree with the contravariance and product structure defined by [7], because ρ respects these structures.

1.3. **The length of the γ-filtration.** We want to show that K-groups with coefficients can be decomposed into the direct sum of Adams eigenspaces. For K-groups this is well known after tensoring with $\mathbb{Q}$, and Soulé has given a bound for the denominators needed. The same proof, which is a version of 1.1 that uses all Adams operators ψ^k at the same time, works for any λ-ring with nilpotent γ-filtration, in particular for K_a with coefficients if $a \geq 2$.

Theorem 1.5. *a) Let X be a regular noetherian scheme of dimension d. Then we have the following decompositions:*

$$K_a(X) \otimes \mathbb{Z}[\tfrac{1}{(a+d-1)!}] = \bigoplus_{q=2}^{a+d} K_a(X) \otimes \mathbb{Z}[\tfrac{1}{(a+d-1)!}]^{(q)} \qquad \textit{for} \quad a \geq 2;$$

$$K_1(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}] = \bigoplus_{q=1}^{d+2} K_1(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}]^{(q)};$$

$$K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] = \bigoplus_{q=0}^{d} K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}]^{(q)}.$$

b) For $a \geq 2$ and $p > a + d$ we have:

$$K_a(X, \mathbb{Z}/p^n) = \bigoplus_{q=1}^{a+d} K_a(X, \mathbb{Z}/p^n)^{(q)}$$

Proof: Part (a) is [22, prop 5], using [22, theorem 4 iv]. The proof depends on the following result:

For $k \geq a + d + 1$ we have $\gamma^k = 0$ on $K_a(X)$ and for $k \geq d + 3$ we have $\gamma^k = 0$ on $K_1(X)$.

The same statement remains true for K-groups with coefficients (with the same proof, see [22, théorème 1]) and we use 1.1 to prove (b), taking any primitive root of unity mod p for k. Note that in this case we only know $K_a(X, \mathbb{Z}/p^n) = F_\gamma^1 K_a(X, \mathbb{Z}/p^n)$, and thus we have to invert $a + d$ as well. □

Corollary 1.6. *For $a \geq 2$ and $p > a + d$, the universal coefficient sequence decomposes into sequences of Adams eigenspaces*

$$0 \to K_a(X)^{(i)}/p^n \to K_a(X, \mathbb{Z}/p^n)^{(i)} \to {}_{p^n}K_{a-1}(X)^{(i)} \to 0.$$

Proof: Since the sequence is a sequence of λ-rings, the Adams operators ψ^i commute with the maps of the sequence. Thus the i-eigenspace is mapped to the

i-eigenspace. Now since all groups in the exact sequence decompose into a direct sum of eigenspaces, the eigenspace functor is exact. □

2. The Grothendieck-Riemann-Roch theorem

2.1. **Chern and Todd characters.** In this section we define the Chern character and the Todd character with bounded denominators and prove some properties of them.

Let X have dimension d, and let N_m be m-th Newton polynomial. It has coefficients in $\mathbb{Z}$, so we can define the following morphism of rings [8, I.4.1]

$$\begin{aligned} ch : K_0(X) &\to \prod_{m=0}^{d} gr_\gamma^m K_0(X) \otimes \mathbb{Z}[\tfrac{1}{d!}] \\ x &\mapsto \epsilon(x) + \sum_{m=1}^{d} \frac{1}{m!} N_m(c_1(x), \dots, c_m(x)). \end{aligned}$$

Here ϵ is the augmentation $K_0(X) \to \mathbb{Z}$ and

$$c_m(x) = \gamma^m(x - \epsilon(x)) \mod F_\gamma^{m+1} K_0(X).$$

Similarly we get for $a \geq 2$ and $p > d + a$ the map

$$\begin{aligned} ch : K_a(X, \mathbb{Z}/p^n) &\to \prod_{m=1}^{a+d} gr_\gamma^m K_a(X, \mathbb{Z}/p^n) \\ x &\mapsto \sum_{m=1}^{a+d} \frac{1}{m!} N_m(c_1(x), \dots, c_m(x)) = \sum_{m=1}^{a+d} \frac{(-1)^{m-1}}{(m-1)!} c_m(x). \end{aligned}$$

The last equality holds true because $K_a(X, \mathbb{Z}/p^n)$ has trivial multiplication.

Let H_m be the m-th Hirzebruch polynomial. It is determined by the property that for $k \geq m$ and the elementary symmetric functions $\sigma_i(x_1, \dots, x_k)$ we have

$$H_m\big(\sigma_1(x_1, \dots, x_k), \dots, \sigma_m(x_1, \dots, x_k)\big) = \text{coefficient of } t^m \text{ in } \prod_{i=1}^{k} e^{x_i t} \frac{x_i t}{e^{x_i t} - 1}$$

If a prime p divides the denominator of H_m, then p is less than or equal to $m + 1$ [13, Lemma 1.7.3], so we get a homomorphism of groups [8, par. I.4]

$$\begin{aligned} td : K_0(X) &\to 1 + \prod_{m=1}^{d} gr_\gamma^m K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}] \\ x &\mapsto 1 + \sum_{m=1}^{d} H_m(c_1(x), \dots, c_m(x)). \end{aligned}$$

Proposition 2.1. *a) The map*

$$ch : K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}] = \bigoplus_{j=0}^{d} K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}]^{(j)} \to \bigoplus_{j=0}^{d} gr_\gamma^j K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}]$$

is an isomorphism of graded rings and coincides on the j-th Adams eigenspace with the natural map $K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]^{(j)} \xrightarrow{\sim} gr_\gamma^j K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$.

b) For $a \geq 2$ and $p > a+d$ the map

$$ch : K_a(X, \mathbb{Z}/p^n) = \bigoplus_{j=1}^{a+d} K_a(X, \mathbb{Z}/p^n)^{(j)} \to \bigoplus_{j=1}^{a+d} gr_\gamma^j K_a(X, \mathbb{Z}/p^n)$$

is an isomorphism of graded rings and coincides on the j-th Adams eigenspace with the natural map $K_a(X, \mathbb{Z}/p^n)^{(j)} \xrightarrow{\sim} gr_\gamma^j K_a(X, \mathbb{Z}/p^n)$.

Proof: a) Since ch is a homomorphism of λ-rings, we get for all m, j the following diagram

$$\begin{array}{ccc} K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]^{(m)} & \xrightarrow{ch} & gr_\gamma^j K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}] \\ \big\downarrow \psi^k = k^m & & \big\downarrow \psi^k = k^j \\ K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]^{(m)} & \xrightarrow{ch} & gr_\gamma^j K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}] \end{array}$$

Thus $(k^j - k^m)ch(x) = 0$ for all k, so the greatest common divisor of the numbers $(k^j - k^m)$ annihilates $ch(x)$. By Soulé [21, 3.3.1] we know that this greatest common divisor is only divisible by primes less than or equal to $d+1$, which we have inverted. So $ch(x) = 0$ for $m \neq j$, and we see that ch respects the grading.

Now for $x \in F_\gamma^m K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$ we have $c_m(x) = \gamma^m(x) = (-1)^{m-1}(m-1)!\, x$ mod $F_\gamma^{m+1} K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$. Observing that $N_m(0, \ldots, 0, c_m) = (-1)^{m-1} m c_m$, this proves the last claim and then the first claim follows.

b) Using 1.5, the same proof works for $K_a(X, \mathbb{Z}/p^n)$, if we use ψ^k for k a primitive root of unity mod p. □

2.2. **The Adams-Riemann-Roch theorem.** Let $i : X \to Z$ be a closed embedding of smooth schemes of constant codimension c defined by the coherent ideal $\mathcal{I}$ of $\mathcal{O}_Z$. We define $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ to be the conormal sheaf of i. It is a locally free $\mathcal{O}_X$-module of rank c. Furthermore let $\theta^j(\mathcal{N})$ be the Bott cannibalistic class of $\mathcal{N}$.

Proposition 2.2. *(Adams-Riemann-Roch)*

For a closed immersion $i : X \to Z$ of smooth schemes with conormal sheaf $\mathcal{N}$ the following diagram commutes for $a \geq 2$:

$$\begin{array}{ccc} K_a(X, \mathbb{Z}/p^n) & \xrightarrow{\theta^k(\mathcal{N})\psi^k} & K_a(X, \mathbb{Z}/p^n) \\ i_* \big\downarrow & & \big\downarrow i_* \\ K_a(Z, \mathbb{Z}/p^n) & \xrightarrow{\psi^k} & K_a(Z, \mathbb{Z}/p^n) \end{array}$$

In other words, we have for every $x \in K_a(X, \mathbb{Z}/p^n)$:

$$\psi^k(i_*(x)) = i_*(\theta^k(\mathcal{N})\psi^k(x)).$$

Proof: The proposition follows exactly as [24, cor. 1.3]. The proof there works for any K_0-algebra with ring morphisms ψ^k which are compatible with the operations of K_0 and pull backs, which is the case in our situation. For more details see [9]. □

Define

$$u = ch^{-1}(td(\mathcal{N}^\vee)) \in K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d+1)!}].$$

By the properties of the maps ch and td it is a unit with augmentation 1. We will use properties of u to prove the Grothendieck-Riemann-Roch theorem. Let us examine the behavior of u with respect to Adams operators.

Lemma 2.3. *The following equation holds in* $K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$*:*

$$\theta^k(\mathcal{N})\psi^k(u) = k^c u.$$

Proof: If we apply ch to the equation, we see that it suffices to prove

$$ch(\theta^k(\mathcal{N})) \cdot \phi^k(td(\mathcal{N}^\vee)) = k^c td(\mathcal{N}^\vee)$$

where ϕ^k is the map which is multiplication by k^m on $gr_\gamma^m K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$. Let $p : \mathbb{D}(\mathcal{N}) \to X$ be the flag scheme of $\mathcal{N}$ over X. The induced map $gr_\gamma^\cdot K_0(X) \to gr_\gamma^\cdot K_0(\mathbb{D}(\mathcal{N}))$ is injective and the class of $p^*\mathcal{N}$ decomposes into a sum of classes of invertible modules. Observe also that since p^* is a morphism of graded algebras, its image lies in $\oplus_{i=0}^d gr_\gamma^i K_0(\mathbb{D}(\mathcal{N}))$, so the following diagram commutes:

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{p^*} & K_0(\mathbb{D}(\mathcal{N})) \\
\Big\downarrow{\scriptstyle ch} & & \Big\downarrow{\scriptstyle ch} \\
\bigoplus_{i=0}^d gr_\gamma^i K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}] & \xrightarrow{p^*} & \bigoplus_{i=0}^d gr_\gamma^i K_0(\mathbb{D}(\mathcal{N})) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]
\end{array}$$

As the same diagram for td commutes as well, it is enough to prove the lemma for an invertible sheaf $\mathcal{N}$. Let $a = c_1(\mathcal{N})$ be the first Chern class of $\mathcal{N}$, then we have the following formal identities:

$$\begin{aligned}
ch(\theta^k(\mathcal{N})) &= ch(1 + [\mathcal{N}] + \cdots + [\mathcal{N}]^{k-1}) = 1 + e^a + \cdots + e^{(k-1)a} \\
td(\mathcal{N}^\vee) &= \frac{-a}{e^{-a} - 1} e^{-a} = \frac{-a}{1 - e^a} \\
\phi^k(td(\mathcal{N}^\vee)) &= \frac{-ka}{1 - e^{ka}}
\end{aligned}$$

and hence we get formally:

$$ch(\theta^k(\mathcal{N}))\phi^k(td(\mathcal{N}^\vee)) = (1 + e^a + \cdots + e^{(k-1)a})\frac{-ka}{1 - e^{ka}} = -ka\frac{1}{1 - e^a} = k \cdot td(\mathcal{N}^\vee)$$

□

Corollary 2.4. *The map* $x \mapsto i_*(ux)$ *maps* $K_a(X, \mathbb{Z}/p^n)^{(j)}$ *to* $K_a(Z, \mathbb{Z}/p^n)^{(j+c)}$.

Proof: By the Adams-Riemann-Roch theorem we have

$$\psi^k(i_*(ux)) = i_*(\theta^k(\mathcal{N})\psi^k(u)\psi^k(x)) = i_*(k^c u \cdot k^j x) = k^{j+c} i_*(ux).$$

□

2.3. **The Grothendieck-Riemann-Roch theorem.** We prove the Grothendieck-Riemann-Roch theorem for K-groups with coefficients. We follow [24], but keep track of denominators. It will turn out that it suffices to invert enough primes so that the Chern character and the Todd character are defined on K_0. The key observation is that the proof in [24] makes calculations in K_0 and then only uses the fact that the higher K-groups are modules under K_0.

For a smooth scheme X the tangent sheaf $\mathcal{T} = (\Omega^1_{X/k})^\vee$ is a locally free $\mathcal{O}_X$-module of finite rank. One defines the Todd class of X to be

$$Td(X) = td([\mathcal{T}_X]) \in \bigoplus_{j=0}^{d_X} gr^j_\gamma K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d_X+1)!}].$$

Theorem 2.5. *(Grothendieck-Riemann-Roch)*

a) Let X, Y be smooth projective schemes over k of dimension d_X and d_Y, respectively, and let $f : X \to Y$ be a morphism of pure codimension c. Let $a \geq 2$, $d = \max\{d_X, d_Y\}$ and $p > d + a$. Then the homomorphism $f_* : K_a(X, \mathbb{Z}/p^n) \to K_a(Y, \mathbb{Z}/p^n)$ has degree c, i.e.

$$f_* : F^m_\gamma K_a(X, \mathbb{Z}/p^n) \subseteq F^{m+c}_\gamma K_a(Y, \mathbb{Z}/p^n)$$

and hence f induces a map

$$f_* : gr^m_\gamma K_a(X, \mathbb{Z}/p^n) \to gr^{m+c}_\gamma K_a(Y, \mathbb{Z}/p^n).$$

b) The following diagram commutes for large p:

$$\begin{array}{ccc}
K_a(X, \mathbb{Z}/p^n) & \xrightarrow{Td(X)\cdot ch} & gr^{\cdot}_\gamma K_a(X, \mathbb{Z}/p^n) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\
K_a(Y, \mathbb{Z}/p^n) & \xrightarrow{Td(Y)\cdot ch} & gr^{\cdot}_\gamma K_a(Y, \mathbb{Z}/p^n).
\end{array}$$

Remark: 1) The proof of (a) works for $K_a(X) \otimes \mathbb{Z}[\frac{1}{(a+d-1)!}]$ respectively $K_1(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$ respectively $K_0(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]$, because we can prove the theorem for $K_a(X) \otimes \mathbb{Z}_{(p)}$, all $p \geq a + d$.

2) In (b) it suffices to take $p > a + d'$, where d' is the dimension of a space $\mathbb{P}^r_Y$ such that f factors through $\mathbb{P}^r_Y$

3) A gap in the proof of the theorem in [9] was pointed out by J.Nekovar.

Proof: Note that the Adams eigenspace decomposition of $K_a(X, \mathbb{Z}/p^n)$ and of $K_a(Y, \mathbb{Z}/p^n)$ are defined because $p > d + a$.

As in [24] we factor f into $X \xrightarrow{i} \mathbb{P}^r_Y \xrightarrow{p} Y$ such that i is a closed embedding.

a) Via the five lemma it is easy to conclude from the corresponding statement for K-theory that we have a canonical isomorphism

$$\begin{aligned}
K_a(Y, \mathbb{Z}/p^n) \otimes K_0(\mathbb{P}^r) &\xrightarrow{\sim} K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n) \\
(y, \xi) &\mapsto p^*(y) \cdot q^*(\xi)
\end{aligned}$$

where p and q are the projections of $\mathbb{P}^r_Y$ to Y and $\mathbb{P}^r$ respectively.

If we let $x = 1 - [\mathcal{O}_{\mathbb{P}^r}(-1)]$, then the elements $x^j \in F^j_\gamma K_0(\mathbb{P}^r)$ form a basis of $K_0(\mathbb{P}^r)$ for $j = 0, \dots, r$, and we get

$$\bigoplus_{j=0}^{r} K_a(Y, \mathbb{Z}/p^n) \cdot x^j \xrightarrow{\sim} K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n).$$

If we denote the structure morphism $\mathbb{P}^r \to \operatorname{Spec} k$ by s, then the projection formula and base change show that

$$p_*(p^* y \cdot q^* \xi) = p_*(q^* \xi) \cdot y = s_* \xi \cdot y,$$

where $s_* \xi \in \mathbb{Z} = K_0(k)$ acts on $K_a(Y, \mathbb{Z}/p^n)$ via the structure morphism.

Consider the embedding $i : X \to \mathbb{P}^r_Y$ and let $x \in F^m_\gamma K_a(X, \mathbb{Z}/p^n)$. Write $x = ux'$ with $x' = u^{-1}x \in F^m_\gamma K_a(X, \mathbb{Z}/p^n)$. Decompose $x' = \sum_{j\geq m} x'_j$ with $x'_j \in K_a(X, \mathbb{Z}/p^n)^{(j)}$. By 2.4 we get $i_*(x) = i_*(ux') = \sum_{j\geq m} i_*(ux'_j)$ where $i_*(ux'_j) \in K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n)^{(j+c+r)}$.

To prove the theorem it will suffice to show that for $z = i_*(ux'_j) \in K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n)^{(j+c+r)}$ we have $p_*(z) \in F^{j+c}_\gamma K_a(Y, \mathbb{Z}/p^n)$.

We write $z = \sum_{i=0}^r x^i y_i$ with $y_i \in K_a(Y, \mathbb{Z}/p^n)$, then by the discussion above $p_*(z) = \sum \alpha_i y_i$ with $\alpha_i = s_*(x^i) \in \mathbb{Z}$.

We will show by induction on i that $y_i \in F^{j+c+r-i}_\gamma K_a(Y, \mathbb{Z}/p^n)$. As $x^i \in F^i_\gamma K_0(\mathbb{P}^r)$, we have

$$\psi^k(x^i) = k^i x^i + \sum_{i<l} \beta_{il} x^l,$$

and from

$$k^{j+c+r} z = \psi^k(z) = \sum \psi^k(x^i)\psi^k(y_i)$$

we get by comparing the coefficient of x^i:

$$k^{j+r+c} y_i = k^i \psi^k(y_i) + \sum_{l<i} \beta_{li} \psi^k(y_l).$$

By induction hypothesis we know that y_l and hence $\psi^k(y_l)$ is contained in $F^{j+c+r-l}_\gamma K_a(Y, \mathbb{Z}/p^n)$ for $l < i$. Thus

$$k^i(\psi^k(y_i) - k^{j+r+c-i} y_i) \in F^{j+c+r-i+1}_\gamma K_a(Y, \mathbb{Z}/p^n)$$

and the same is true without the factor k^i, because we can assume k invertible. The remark after lemma 1.1 tells us that we can conclude from $\psi^k(y_i) - k^{j+c+r-i} y_i \in F^{j+c+r-i+1}_\gamma K_a(Y, \mathbb{Z}/p^n)$ that $y_i \in F^{m+c+r-i}_\gamma K_a(Y, \mathbb{Z}/p^n)$. Noting that $i \leq r$, the theorem follows.

b) We prove b) for i and p separately in two lemmas:

Lemma 2.6. *The following diagram commutes for $p > a + \dim \mathbb{P}^r_Y$:*

$$\begin{array}{ccc} K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n) & \xrightarrow{Td(\mathbb{P}^r_Y)\cdot ch} & gr^{\cdot}_\gamma K_a(\mathbb{P}^r_Y, \mathbb{Z}/p^n) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ K_a(Y, \mathbb{Z}/p^n) & \xrightarrow{Td(Y)\cdot ch} & gr^{\cdot}_\gamma K_a(Y, \mathbb{Z}/p^n). \end{array}$$

Proof: The classical Hirzebruch-Riemann-Roch theorem states that the following diagram is commutative

$$\begin{array}{ccc} K_0(\mathbb{P}^r) & \xrightarrow{Td(\mathbb{P}^r)\cdot ch} & gr^{\cdot}_\gamma K_0(\mathbb{P}^r) \otimes \mathbb{Z}[\frac{1}{(r+1)!}] \\ \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s_*} \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}[\frac{1}{(r+1)!}]. \end{array}$$

This can be proved as in [2, prop.10] with formal calculations, so it suffices to tensor with $\mathbb{Z}[\frac{1}{(r+1)!}]$ in order for the Todd map to be defined.

Now using $Td(\mathbb{P}^r_Y) = p^*(Td(Y)) \cdot q^*(Td(\mathbb{P}^r))$, the projection formula and the base change property, we see that for $y \in K_a(Y, \mathbb{Z}/p^n)$ and $x \in K_0(\mathbb{P}^r)$ we have:

$$\begin{aligned} p_*\big(Td(\mathbb{P}^r_Y) \cdot ch(p^*(y)q^*(x))\big) &= p_*\big(p^*(Td(Y)ch(y)) \cdot q^*(Td(\mathbb{P}^r)ch(x))\big) \\ &= Td(Y)ch(y)s_*\big(Td(\mathbb{P}^r)ch(x)\big) \\ &= Td(Y)ch(y)s_*(x) \\ &= Td(Y)ch\big(p_*(p^*(y)q^*(x))\big). \end{aligned}$$

□

Lemma 2.7. *For a closed embedding $i : X \to Z$ of smooth schemes over k and p sufficiently large, the following diagram commutes:*

$$\begin{array}{ccc} K_a(X, \mathbb{Z}/p^n) & \xrightarrow{Td(X)\cdot ch} & gr^{\cdot}_\gamma K_a(X, \mathbb{Z}/p^n) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K_a(Z, \mathbb{Z}/p^n) & \xrightarrow{Td(Z)\cdot ch} & gr^{\cdot}_\gamma K_a(Z, \mathbb{Z}/p^n). \end{array}$$

Proof: Let us first prove the commutativity of the diagram

$$\begin{array}{ccc} K_a(X, \mathbb{Z}/p^n) & \xrightarrow{td(\mathcal{N}^\vee)^{-1}\cdot ch} & gr^{\cdot}_\gamma K_a(X, \mathbb{Z}/p^n) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K_a(Z, \mathbb{Z}/p^n) & \xrightarrow{ch} & gr^{\cdot}_\gamma K_a(Z, \mathbb{Z}/p^n). \end{array}$$

For this write any given $x \in K_a(X, \mathbb{Z}/p^n)$ as above as $x = ux'$. We have to show $ch(i_*(ux')) = i_*(td(\mathcal{N}^\vee)^{-1}ch(ux'))$. But

$$td(\mathcal{N}^\vee)^{-1}ch(ux') = td(\mathcal{N}^\vee)^{-1}ch(u)ch(x') = td(\mathcal{N}^\vee)^{-1}td(\mathcal{N}^\vee)ch(x') = ch(x')$$

so it suffices to show $ch(i_*(ux')) = i_*(ch(x'))$.

Assume without loss of generality that $x' \in K_a(X, \mathbb{Z}/p^n)^{(m)}$ for some m. Then the assertion is equivalent to the following congruence:

$$ch(i_*(ux')) \equiv i_*(ux') \equiv i_*(x') \equiv i_*(ch(x')) \mod F^{m+d+1}_\gamma K_a(Z, \mathbb{Z}/p^n).$$

The first congruence follows because $i_*(ux') \in K_a(Z, \mathbb{Z}/p^n)^{(d+m)}$ and 2.1. The second congruence follows because u is a unit with augmentation 1 and the third congruence is i_* applied to 2.1 again.

Now the claim of the proposition falls out: For the closed immersion $i : X \to Z$ we have the exact sequence

$$0 \to \mathcal{N} \to i^*\Omega^1_{Z/k} \to \Omega^1_{X/k} \to 0$$

and therefore $i^*(Td(Z)) = Td(X)td(\mathcal{N}^\vee)$. Then for $x \in K_a(X, \mathbb{Z}/p^n)$ we get

$$\begin{aligned} ch(i_*(x)) &= i_*\big(td(\mathcal{N}^\vee)^{-1}ch(x)\big) \\ &= i_*\big(i^*(Td(Z)^{-1})Td(X)ch(x)\big) = Td(Z)^{-1}i_*\big(Td(X)ch(x)\big) \end{aligned}$$

and hence

$$Td(Z)ch(i_*(x)) = i_*(Td(X)ch(x)).$$

This concludes the proof of the Grothendieck-Riemann-Roch theorem. □

2.4. K_0 **and Chow groups.** In this section we are going to prove the following theorem

Theorem 2.8. *Let X be a smooth quasiprojective variety of dimension d. Then we have a cycle map which is covariant, contravariant and compatible with products*

$$CH^p(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] \xrightarrow{\sim} K_0(X)^{(p)} \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}].$$

Proof: Consider the Brown-Gersten-Quillen spectral sequence. According to Soulé [22, th. 4 iv], it degenerates with split filtration at the E_2-level after inverting $(d-1)!$. We thus get

$$K_0(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] = \bigoplus_{p=0}^{d} E_2^{p,-p}(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}] = \bigoplus_{p=0}^{d} CH^p(X) \otimes \mathbb{Z}[\tfrac{1}{(d-1)!}].$$

This isomorphism is contravariant and compatible with products (because the spectral sequence is), and covariant by [10, theorem 7.22].

Observe now that ψ^k acts on $E_2^{p,-p}$ like k^p [22, p.524], so the theorem follows from 1.4. $\square$

Remark: 1) The Brown-Gersten-Quillen spectral sequence comes from and induces the filtration by support on K-groups,

$$F^i_{top} K_0(X) = \text{im } K_0(\mathcal{M}^i) \to K_0(\mathcal{M}),$$

where $\mathcal{M}^i$ is the full subcategory of $\mathcal{M}$ with objects $\mathcal{F}$ such that $\text{codim}_X(\text{supp}\mathcal{F}) \geq i$. On the other hand there is a cycle map

$$CH^p(X) \xrightarrow{cl} gr^p_{top} K_0(X) \quad , \qquad Z \mapsto i_*(\mathcal{O}_Z)$$

for $i : Z \to X$ is the natural inclusion. Soulé in essentially proves that ψ^k induces an action k^p on $gr^p_{top} K_0(X) = E_\infty^{p,-p}$, showing that $gr^p_{top} K_0(X) = K_0(X)^{(p)}$ if the Adams-eigenspace decomposition of $K_0(X)$ exists.

2) Note that for a cycle Z in $CH^p(X)$ the corresponding element in $K_0(X)$ is given by $i_*(\mathcal{O}_Z)$. This is clear for smooth cycles by the preceding discussion, and can be reduced to this case by the localization sequence.

3. Chow motives and K-theory with coefficients

Let Λ be a unitary commutative ring. The category $\mathcal{M}_\Lambda(k)$ of Chow motives over the field k with coefficients in Λ is obtained from the category of smooth projective varieties over k as follows, see [16]:

(1) We first define the intermediate category $\mathcal{C}_\Lambda(k)$: The objects of this category are $h(X)$, for X smooth projective over k. We define

$$\text{Hom}_{\mathcal{C}_\Lambda(k)}(h(X), h(Y)) = CH^{\dim X}(X \times Y) \otimes \Lambda$$

if X is equidimensional.

Composition is defined by intersection: For $a \in \text{Hom}(h(X), h(Y))$ and $b \in \text{Hom}(h(Y), h(Z))$ we have

$$b \circ a = p_{13*}(p_{12}^* a \cdot p_{23}^* b)$$

where p_{ij} are the three projections of $X \times Y \times Z$ to two of the factors.

We have a contravariant functor from the category $\mathcal{V}(k)$ of smooth projective varieties over k to $\mathcal{C}_\Lambda(k)$, by sending X to $h(X)$ and a map $f: X \to Y$ to its graph in $CH^{\dim Y}(Y \times X)$.

(2) The next step is to add images of projectors in $\mathcal{C}_\Lambda(k)$, i.e. we take the Karoubien hull of this category. The objects of the new category are pairs (X,p) with X an object in $\mathcal{C}_\Lambda(k)$ and p an idempotent in $\mathrm{Hom}_{\mathcal{C}_\Lambda(k)}(h(X),h(X))$. The morphisms are given by

$$\mathrm{Hom}\big((X,p),(Y,q)\big) = \{q \circ f \circ p | f \in \mathrm{Hom}_{\mathcal{C}_\Lambda(k)}(h(X),h(Y))\}.$$

(3) The idempotents $e_0 = \{pt\} \times \mathbb{P}^1$ and $e_2 = \mathbb{P}^1 \times \{pt\}$ of $\mathrm{Hom}(h(\mathbb{P}^1),h(\mathbb{P}^1))$ are orthogonal with $e_0 + e_2 = id$. It is easy to see [16, par.6] that $(h(\mathbb{P}^1),e_0) \cong h(\mathrm{Spec}\, k)$. If we let $\mathcal{L} = (h(\mathbb{P}^1),e_2)$, then we have $h(\mathbb{P}^1) = h(\mathrm{Spec}\, k) \oplus \mathcal{L}$. Our last step is formal inversion of $\mathcal{L}$. The objects of the new category are $M(r) = M \otimes \mathcal{L}^{\otimes r}$ with M an object in the category above and $r \in \mathbb{Z}$. The morphisms are given by

$$\mathrm{Hom}\big(M(r),N(s)\big) = \varinjlim \mathrm{Hom}\big(M(r+n),N(s+n)\big).$$

Observe that the terms in this limit are defined and stable for n big enough [16, par.8]. We call the resulting category $\mathcal{M}_\Lambda(k)$.

Remark: Let X be a variety of dimension d with a k-rational point. Via the idempotents $X \times \{pt\}$ and $\{pt\} \times X$ in $CH^{\dim X}(X \times X)$ it is possible to split off the parts $h_0(X)$ and $h_{2d}(X)$ from $h(X)$. Then $h_0(X) = h(\mathrm{Spec}\, k)$ and $h_{2d}(X) = \mathcal{L}^{\otimes d}$. For a curve C this gives us the splitting

$$h(C) = h(\mathrm{Spec}\, k) \oplus h_1(C) \oplus \mathcal{L}.$$

Let $\mathcal{M}^d_\Lambda(k)$ be the full subcategory of $\mathcal{M}_\Lambda(k)$ generated by smooth projective varieties of dimension less than or equal to d by the steps above, i.e. we define the category $\mathcal{C}^d_\Lambda(k)$ to be the full subcategory of $\mathcal{C}_\Lambda(k)$ with objects $h(X)$ for X smooth projective of dimension at most d over k, take the Karoubien hull and formally invert $\mathcal{L}$.

Furthermore let $\mathcal{M}_\Lambda(k,\mathcal{O})$ be the category of Chow motives over k with coefficients in Λ and with multiplication by a ring $\mathcal{O}$, i.e. the objects are pairs consisting of an object of $\mathcal{M}_\Lambda(k)$ and an embedding of $\mathcal{O}$ into $\mathrm{Hom}_{\mathcal{M}_\Lambda(k)}(X,X)$, and the morphisms are compatible with this action.

Theorem 3.1. *The functor $K_a(-,\mathbb{Z}/p^n)^{(i)}$ factors through $\mathcal{M}^d_{\mathbb{Z}_p}(k)$ for $a \geq 2$ and $p > 3d + a$.*

Proof: According to 1.5 there is an Adams eigenspace decomposition of $K_a(X,\mathbb{Z}/p^n)$, $K_a(X \times Y,\mathbb{Z}/p^n)$ and $K_a(X \times Y \times Z,\mathbb{Z}/p^n)$ for $p > 3d + a \geq d_X + d_Y + d_Z + a$.

a) $K_a(-,\mathbb{Z}/p^n)^{(i)}$ factors through $\mathcal{C}^d_\Lambda(k)$:

We need to show that every element $\alpha \in CH^{\dim X}(X \times Y) \otimes \mathbb{Z}_p$ induces a map

$$\alpha_* : K_a(X,\mathbb{Z}/p^n)^{(i)} \to K_a(Y,\mathbb{Z}/p^n)^{(i)}$$

in a functorial way, and for maps $f: Y \to X$ between varieties we have $f^* = [\Gamma_f]_*$.

This follows if we have the following ingredients:

(1) A functor which is contravariant and admits a product.

(2) A map
$$CH^i(X \times Y) \otimes \mathbb{Z}_p \xrightarrow{cl} K_0(X \times Y)^{(i)} \otimes \mathbb{Z}_p$$
which is covariant and contravariant functorial and compatible with products (this is the case by 2.8).

(3) A push-forward map for smooth projective varieties shifting degree and satisfying the projection and base change formula (K-groups have push-forward with projection formula and base change, and the shift of degrees follows from 2.5).

For a given $\alpha \in CH^{\dim X}(X \times Y) \otimes \mathbb{Z}_p$ we define α_* to be the composition
$$K_a(X, \mathbb{Z}/p^n)^{(i)} \xrightarrow{p_X^*} K_a(X \times Y, \mathbb{Z}/p^n)^{(i)} \xrightarrow{\cup cl(\alpha)} K_a(X \times Y, \mathbb{Z}/p^n)^{(i+d_X)} \xrightarrow{p_{Y*}} K_a(Y, \mathbb{Z}/p^n)^{(i)}.$$

The property $[\Gamma_f]_* = f^*$ for maps $f : Y \to X$ between varieties is checked as follows: let γ be the graph map $Y \to X \times Y$, $y \mapsto (y, f(y))$. For the closed embedding γ we have by the projection formula $\gamma_*\gamma^*(z) = z \cup \gamma_*(1) = z \cup cl(\Gamma_f)$. But then
$$[\Gamma_f]_*(x) = p_{Y*}(p_X^* x \cup cl(\Gamma_f)) = p_{Y*}\gamma_*\gamma^*(p_X^* x) = \gamma^* p_X^*(x) = f^*(x).$$

For the compatibility with composition, we have to check that for $\alpha \in CH^{\dim X}(X \times Y) \otimes \mathbb{Z}_p$ and $\beta \in CH^{\dim Y}(Y \times Z) \otimes \mathbb{Z}_p$ the maps $\beta_* \circ \alpha_*$ and $(\beta \circ \alpha)_*$ agree. Consider the following diagram of schemes and maps:

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl] \arrow[dd] \arrow[dr] & \\
X \times Y \arrow[ddd] \arrow[dddr] & & Y \times Z \arrow[dddl] \arrow[ddd] \\
 & X \times Z \arrow[ddl] \arrow[ddr] & \\
 & & \\
X & Y & Z
\end{tikzcd}$$

Denote the projection from A to B by p_B^A, then for $x \in K_a(X, \mathbb{Z}/p^n)^{(i)}$ we have $\alpha_*(x) = {p_Y^{XY}}_*({p_X^{XY}}^* x \cup cl(\alpha))$ and

$$\begin{aligned}
\beta_*(\alpha_*(x)) &= {p_Z^{YZ}}_*({p_Y^{YZ}}^*(\alpha_*(x)) \cup cl(\beta)) \\
&= {p_Z^{YZ}}_*\big({p_Y^{YZ}}^*\big[{p_Y^{XY}}_*({p_X^{XY}}^* x \cup cl(\alpha))\big] \cup cl(\beta)\big) \\
&= {p_Z^{YZ}}_*\big({p_{YZ}^{XYZ}}_*\big[{p_{XY}^{XYZ}}^*({p_X^{XY}}^* x \cup cl(\alpha))\big] \cup cl(\beta)\big) \qquad \text{(base change)} \\
&= {p_Z^{YZ}}_*\big({p_{YZ}^{XYZ}}_*\big[({p_{XZ}^{XYZ}}^* {p_X^{XZ}}^* x) \cup {p_{XY}^{XYZ}}^* cl(\alpha)\big] \cup cl(\beta)\big) \\
&= {p_Z^{YZ}}_*\big({p_{YZ}^{XYZ}}_*\big[({p_{XZ}^{XYZ}}^* {p_X^{XZ}}^* x) \cup {p_{XY}^{XYZ}}^* cl(\alpha) \cup {p_{YZ}^{XYZ}}^* cl(\beta)\big]\big) \quad \text{(proj. formula)} \\
&= {p_Z^{XZ}}_*\big({p_{XZ}^{XYZ}}_*\big[({p_{XZ}^{XYZ}}^* {p_X^{XZ}}^* x) \cup {p_{XY}^{XYZ}}^* cl(\alpha) \cup {p_{YZ}^{XYZ}}^* cl(\beta)\big]\big) \\
&= {p_Z^{XZ}}_*\big({p_X^{XZ}}^* x \cup {p_{XZ}^{XYZ}}_*\big[{p_{XY}^{XYZ}}^* cl(\alpha) \cup {p_{YZ}^{XYZ}}^* cl(\beta)\big]\big) \qquad \text{(projection formula)} \\
&= {p_Z^{XZ}}_*({p_X^{XZ}}^* x \cup cl(\beta \circ \alpha)) \qquad \text{(compatibility of } cl \text{ with } f_*, f^*, \cup) \\
&= (\beta \circ \alpha)_*(x)
\end{aligned}$$

This shows that $K_a(-,\mathbb{Z}/p^n)^{(i)}$ factors through $\mathcal{C}^d_\Lambda(k)$.

b) $K_a(-,\mathbb{Z}/p^n)^{(i)}$ factors through $\mathcal{M}^d_\Lambda(k)$:

We have to check that we have $K_a(X,\mathbb{Z}/p^n)^{(j-1)} = K_a(X\otimes\mathcal{L},\mathbb{Z}/p^n)^{(j)}$ for the Lefschetz motive $\mathcal{L}$.

Now $K_0(\mathbb{P}^1) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot x$, where 1 lies in degree 0 and x in degree 1, and we have an isomorphism

$$\begin{aligned} K_a(X,\mathbb{Z}/p^n)\otimes K_0(\mathbb{P}^1) &\xrightarrow{\sim} K_a(X\times\mathbb{P}^1,\mathbb{Z}/p^n)\\ (x,y) &\mapsto p^*x\cup q^*y \end{aligned}$$

This implies that $K_a(X\times\mathbb{P}^1,\mathbb{Z}/p^n)$ has an Adams eigenspace decomposition with the same denominators as $K_a(X,\mathbb{Z}/p^n)$ and we have

$$\begin{aligned} K_a(X,\mathbb{Z}/p^n)^{(i)}\oplus xK_a(X,\mathbb{Z}/p^n)^{(i-1)} &= K_a(X\otimes\mathbb{P}^1,\mathbb{Z}/p^n)^{(i)}\\ &= K_a(X,\mathbb{Z}/p^n)^{(i)}\oplus K_a(X\otimes\mathcal{L},\mathbb{Z}/p^n)^{(i)} \end{aligned}$$

Observing that the isomorphism leaves $K_a(X,\mathbb{Z}/p^n)^{(i)}$ fixed, we get the desired equality. $\square$

Proposition 3.2. *a) The functor $K^{ét}_a(-,\mathbb{Z}/p^n)^{(i)}$ factors through $\mathcal{M}^d_{\mathbb{Z}_p}(k)$ for $p > 3d+a$, and the morphism of functors*

$$\rho : K_a(-,\mathbb{Z}/p^n)^{(i)} \to K^{ét}_a(-,\mathbb{Z}/p^n)^{(i)}$$

induces a morphism of functors on $\mathcal{M}^d_{\mathbb{Z}_p}(k)$.

b) The functors $H^{2i-a}(-,\mathbb{Z}/p^n(i))$ and $H^{2i-a}(-\times_k\bar{k},\mathbb{Z}/p^n(i))$ factor through $\mathcal{M}_{\mathbb{Z}_p}(k)$, and the morphism of functors

$$\tau : K^{ét}_a(-,\mathbb{Z}/p^n)^{(i)} \xrightarrow{\sim} H^{2i-a}(-,\mathbb{Z}/p^n(i))$$

induces a morphism of functors on $\mathcal{M}^d_{\mathbb{Z}_p}(k)$ for $p > 3d+a$.

Proof: a) By the isomorphism 1.4

$$\rho : K_a(-,\mathbb{Z}/p^n)[\beta^{-1}]^{(i)} \xrightarrow{\sim} K^{\text{ét}}_a(-,\mathbb{Z}/p^n)^{(i)}$$

we define cycle class, push-forwards, pull-backs and products on étale K-theory, and then compatibility is obvious.

b) We use proposition 1.3 and 1.4 to transport the structures from K-theory to étale cohomology. In fact, both the map ρ and the Dwyer-Friedlander spectral sequence are compatible with pull-backs and products. And for proper maps $X \to Y$ of codimension d, the push-forward in K-theory induces by 2.5 a map

$$\begin{aligned} H^{2i-a}(X,\mathbb{Z}/p^n(i)) = K_a(X,\mathbb{Z}/p^n)[\beta^{-1}]^{(i)} &\to\\ K_a(Y,\mathbb{Z}/p^n)[\beta^{-1}]^{(i+c)} &= H^{2(i+c)-a}(Y,\mathbb{Z}/p^n(i+c)). \end{aligned}$$

Again compatibility is obvious by construction. $\square$

Remark: For étale cohomology one could check the properties needed to prove the factorization directly (Bloch-Ogus axioms [1]), but this way we avoid checking compatibilities. Of course, pull-backs and products agree with the normal pull-backs and products for étale cohomology (as ρ and the spectral sequence are contravariant functorial and respect the product), and the push-forward should agree with the usual Gysin morphism in étale cohomology.

Since the morphism of functors "reduction of coefficients" commutes with the cycle map, cup products and push-forwards, it induces a morphism of functors on $\mathcal{M}^d_{\mathbb{Z}_p}(k)$, and we can define

$$\begin{array}{ccc} K_a(M,\mathbb{Z}_p)^{(i)} & & := \varprojlim K_a(M,\mathbb{Z}/p^n)^{(i)} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ K_a^{\text{ét}}(M,\mathbb{Z}_p)^{(i)} & & := \varprojlim K_a^{\text{ét}}(M,\mathbb{Z}/p^n)^{(i)} \\ \downarrow & & \downarrow \\ H^{2i-a}(M,\mathbb{Z}_p(i)) & & := H^{2i-a}(M,\mathbb{Z}/p^n(i)) \end{array}$$

4. Hecke characters of imaginary quadratic fields

4.1. **Hecke characters.** Let K be a number field, $\mathfrak{f}$ be an ideal of K and $X = \sum n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K,\bar{\mathbb{Q}})]$ be a linear combination of embeddings.

A Hecke character of K with values in a number field T of infinity type X and conductor dividing $\mathfrak{f}$ is a group homomorphism $\varphi : \mathcal{I}_\mathfrak{f} \to T^*$ from the ideal classes of K prime to $\mathfrak{f}$ to T^* such that for any principal ideal (α) with $\alpha \equiv 1 \mod \mathfrak{f}$, α totally positive, one has (see [18])

$$\varphi((\alpha)) = \alpha^X = \prod_{\sigma: K \to \bar{\mathbb{Q}}} \sigma(\alpha)^{n_\sigma}.$$

Let $W_\mathfrak{f} \subseteq I_K$ be the standard open subgroup of the idèles of K

$$\prod_{\mathfrak{p} \nmid \mathfrak{f}} \mathcal{O}_\mathfrak{p}^* \times \prod_{\mathfrak{p} | \mathfrak{f}} W_\mathfrak{f}(\mathfrak{p}) \times \prod_{\lambda \text{ real}} \mathbb{R}_+^* \times \prod_{\lambda \text{ complex}} \mathbb{C}^*,$$

where $\mathcal{O}_\mathfrak{p}^*$ are the units of $K_\mathfrak{p}$ and $W_\mathfrak{f}(\mathfrak{p})$ are the units congruent to 1 mod $\mathfrak{f}$.

In the adelic description a Hecke character (CM-character) is a continuous group homomorphism $\chi : I_K \to T^*$ of the idèles of K to T^* such that

$$\chi(W_\mathfrak{f}) = 1 \quad \text{for some } W_\mathfrak{f}$$
$$\chi\,|_{K^*} = X : K^* \to T^*$$

By extending X to a character $X_\mathbb{A} : I_K \to I_T$ one defines the group homomorphism $\chi_\mathbb{A} := \chi \cdot X_\mathbb{A}^{-1} : I_K/K^* \to I_T$ of the idèles of K to the idèles of T. For an infinite place τ of T we get a grossencharacter $\tau \circ \chi_\mathbb{A}$ of type A0 in the sense of Weil.

On the other hand we get for the finite places:

Lemma 4.1. *Let $\chi_\mathfrak{P} : I_K \xrightarrow{\chi_\mathbb{A}} I_T \to T_\mathfrak{P}^*$ be the $\mathfrak{P}$-component of a Hecke character for a finite place $\mathfrak{P}$ of T. Then $\chi_\mathfrak{P}$ factors through the Galois group $Gal(K^{ab}/K)$ and has image in $\mathcal{O}_{T_\mathfrak{P}}^*$.*

Proof: Since $T_\mathfrak{P}$ is totally disconnected, the connected component of the idèle classes C_K^0 is mapped to 1 and by class field theory $C_K/C_K^0 \cong Gal(K^{ab}/K)$. On the other hand $Gal(K^{ab}/K)$ is compact, so it has image in the maximal compact subgroup $\mathcal{O}_{T_\mathfrak{P}}^*$ of $T_\mathfrak{P}^*$. $\square$

Remark: 1) Consider the characters χ_λ given by

$$\chi_\lambda : I_K \xrightarrow{\chi_{\mathbb{A}}} I_T \to T_{\mathfrak{P}}^* \xrightarrow{\lambda} \mathbb{C}_p^*$$

for a place $\mathfrak{P}$ dividing p of T and an embedding λ of $T_{\mathfrak{P}}$ into $\mathbb{C}_p^*$. There are $\sum_{\mathfrak{P}|p}[T_{\mathfrak{P}} : \mathbb{Q}_p] = [T : \mathbb{Q}]$ characters of this type. We call them p-adic grossencharacters associated to χ, because they are the p-adic analogue of the $[T : \mathbb{Q}]$ grossencharacters arising from χ.

2) In case χ is a Hecke character of conductor $\mathfrak{f}$ of an imaginary quadratic field K and p a prime split in K, one sees as in [19, II 1.1] that the character $\chi_{\mathfrak{P}} : Gal(K^{ab}/K) \to \mathcal{O}_{T_{\mathfrak{P}}}$ factors through the Galois group of the ray class field $K(\mathfrak{f}p^\infty)$.

4.2. **Complex multiplication.** Let E be an elliptic curve over a number field F with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, see [19, II 1.3]. Suppose that E is of Shimura type, i.e. the extension $F(E_{tors})/K$ is abelian, where E_{tors} denotes the group of all torsion points of E.

Let $\psi : I_F \to K^*$ be the CM-character of E, i.e.

$$\psi_{\mathbb{A}} : I_F/F^* \to Gal(F^{ab}/F) \to I_K \to \mathcal{O}_K \otimes \mathbb{Z}_p$$

gives the action of G_F on the Tate module T_pE for all p. The character ψ can be extended to a character $\varphi' : I_K \to T'^*$ with values in a number field T' such that $\psi = \varphi' \circ N_K^F$, see [19, II 1.4].

Let $A = R_{F/K}E$ be the Weil restriction of E. It has complex multiplication by a finite $\mathcal{O}_K$-algebra $\mathcal{O}'_T$, [11]. The K-algebra $T = \mathcal{O}'_T \otimes \mathbb{Q}$ is then a commutative semisimple K-algebra of degree $d = [F : K]$ and thus a product of fields. If e runs through the idempotents of T, we have

$$T = \prod_e T_e \quad \text{and} \quad A \sim \prod_e A_e,$$

where $T_e = e \cdot T$ is a CM field containing K, $\sim$ means isogenous, and A_e is an abelian variety with complex multiplication by T_e. Let φ be the CM-character of A, it is T-valued and the components φ_e are the CM-characters of the abelian varieties A_e.

It is easy to see that the characters φ_e are the characters satisfying the condition $\psi = \varphi' \circ N_K^F$. If $\bar{\varphi}$ is the complex conjugate character and κ is the cyclotomic character, then $\bar{\varphi} \cdot \varphi = \kappa$.

Note that $A = R_{F/K}E = R_{F_K}E^\vee = A^\vee$ and that $A^\vee$ also has complex multiplication by φ.

4.3. **The extension** $F(E_{p^\infty})/K$**.** Choose for the rest of this paper a prime number p having the following properties:

- p splits in K into $\mathfrak{p}\bar{\mathfrak{p}}$
- p does not divide $d = [F : K]$
- p is prime to the conductors of F/K and of E
- $p \geq 5$

Let $\mathfrak{f}$ be the lowest common multiple of the conductors of F/K and of φ. We have [19, prop.1.6, 1.7]:

Proposition 4.2. *1) Let $\mathfrak{g}$ be any ideal divisible by $\mathfrak{f}$, then the ray class field $K(\mathfrak{g})$ equals $F(E_{\mathfrak{g}})$.*

2) For any ideal $\mathfrak{m}$ *prime to* $\mathfrak{f}$, $Gal(F(E_{\mathfrak{m}})/F) \cong (\mathcal{O}_K/\mathfrak{m})^*$.

Let $F_n = F(E_{p^n})$, $F_\infty = \bigcup F_n$ and $\mathcal{G} = Gal(F_\infty/K)$. We want to describe the structure of $\mathcal{G}$. By the proposition we see that

$$Gal(F_\infty/F) = \mathcal{O}_{K,p}^* = \mathcal{O}_{\mathfrak{p}}^* \oplus \mathcal{O}_{\bar{\mathfrak{p}}}^* = \Delta_1 \times \Gamma_1 \times \Delta_2 \times \Gamma_2,$$

where $\mathcal{O}_{K,p}$ is the completion of $\mathcal{O}_K$ at p,

$$\Delta_1 = Gal(F(E_{\mathfrak{p}})/F) \cong \mathbb{Z}/(p-1)$$
$$\Gamma_1 = Gal(F(E_{\mathfrak{p}^\infty})/F(E_{\mathfrak{p}})) \cong \mathbb{Z}_p$$

and similarly for Δ_2 and Γ_2.

For $\mathcal{G}$ we get the decomposition

$$\mathcal{G} = H \times \Gamma_1 \times \Gamma_2$$

where $H = Gal(F(E_p)/K) \supseteq \Delta_1 \times \Delta_2$.

The decomposition patterns of primes of F are given by the following [19, prop.1.9]:

Proposition 4.3. *1) All the primes above* $\mathfrak{p}$ *are totally ramified in* $F(E_{\mathfrak{p}^\infty})/F$.

2) All primes not above $\mathfrak{p}$ *and not dividing* $\mathfrak{f}$ *are unramified and finitely decomposed in* $F(E_{\mathfrak{p}^\infty})/F$.

Corollary 4.4. *The decomposition group of* $\mathfrak{p}$ *in* $\mathcal{G}$ *is of finite index and contains the group* $\Gamma_1 \times \Gamma_2'$ *where* Γ_2' *is of finite index in* Γ_2.

Proof: By the proposition the inertia group of $\mathfrak{p}$ in $Gal(F(E_{\mathfrak{p}^\infty})/F)$ is isomorphic to Γ_1 and $\mathfrak{p}$ is unramified and finitely decomposed in $Gal(F(E_{\bar{\mathfrak{p}}^\infty})/F)$. So in $Gal(F(E_{p^\infty})/F)$ the prime $\mathfrak{p}$ has inertia group isomorphic to $\mathbb{Z}_p$ and residue class extension isomorphic to $\mathbb{Z}_p$. □

Let us now prove some properties of the Hecke characters ψ and φ.

Lemma 4.5. *The character* $\psi_{\mathfrak{p}} : I_F \to K_{\mathfrak{p}}^*$ *factors through* $Gal(F_\infty/F)$ *and the character* $\varphi_{\mathfrak{P}} : I_K \to T_{\mathfrak{P}}^*$ *factors through* $\mathcal{G}$.

Proof: The character $\psi_{\mathfrak{p}} : I_F \to I_K \to \mathcal{O}_{K_{\mathfrak{p}}}^* \cong {\mathbb{Z}_p}^*$ gives the action of $Gal(F^{ab}/F)$ on the $\mathfrak{p}$-power torsion points of E and thus factors through the extension $Gal(F(E_{\mathfrak{p}^\infty})/F)$. On the other hand, for any place $\mathfrak{P}$ of T above $\mathfrak{p}$, the character $\varphi_{\mathfrak{P}} : I_K \to I_T \to T_{\mathfrak{P}}^*$ gives the action of $Gal(K^{ab}/K)$ on the $\mathfrak{P}$-power torsion points of one of the abelian varieties A_e and consequently factors through $Gal(K(A_{\mathfrak{P}^\infty})/K)$. By the universal property of the Weil restriction the $\mathfrak{P}$-torsion points of A_e are defined over $F(E_{\mathfrak{p}^\infty})$, so $\varphi_{\mathfrak{P}}$ factors through $Gal(F(E_{\mathfrak{p}^\infty})/K)$. □

By the defining property of ψ, the $\mathfrak{p}$-component of ψ,

$$\kappa_1 = \psi_{\mathfrak{p}} = \varphi_{\mathfrak{P}}|_{Gal(F_\infty/F)} : Gal(F_\infty/F) \to \mathcal{O}_{K_{\mathfrak{p}}}^* \cong {\mathbb{Z}_p}^*$$

gives the action on $T_{\mathfrak{p}}E$. Similarly the character

$$\kappa_2 = \psi_{\bar{\mathfrak{p}}} = \varphi_{\bar{\mathfrak{P}}}|_{Gal(F_\infty/F)} : Gal(F_\infty/F) \to \mathcal{O}_{K_{\bar{\mathfrak{p}}}}^* \cong {\mathbb{Z}_p}^*$$

gives the action on $T_{\bar{\mathfrak{p}}}E$ for all places $\bar{\mathfrak{P}}$ over $\bar{\mathfrak{p}}$.

Observe that κ_1 and κ_2 are trivial on $Gal(F(E_{\bar{\mathfrak{p}}^\infty})/F)$ and $Gal(F(E_{\mathfrak{p}^\infty})/F)$, respectively.

Consider the cyclotomic character $\kappa : Gal(F(\zeta_{p^\infty})/F) \to \mathbb{Z}_p{}^*$ giving the action on the p-power roots of unity. Via the projection $Gal(F(E_{p^\infty})/F) \to Gal(F(\zeta_{p^\infty})/F)$ we can consider κ as a character of $Gal(F_\infty/F)$.

Lemma 4.6. *The characters κ and κ_1 agree as characters on $Gal(F(E_{\mathfrak{p}^\infty})/F)$. Similarly the characters κ and κ_2 agree as characters on $Gal(F(E_{\bar{\mathfrak{p}}^\infty})/F)$.*

Proof: Because of the Galois invariance of the Weil pairing,

$$\kappa = det(\kappa_1 \times \kappa_2) : Gal(F_\infty/F) \longrightarrow \mathbb{Z}_p{}^* \times \mathbb{Z}_p{}^* \xrightarrow{\times} \mathbb{Z}_p{}^*.$$

But we just observed that κ_2 is trivial on $Gal(F(E_{\mathfrak{p}^\infty})/F)$, so the lemma follows. $\square$

5. Motives for Hecke characters of imaginary quadratic fields

In this section we construct for any Hecke character of an imaginary quadratic field K a motive in $\mathcal{M}_{\mathbb{Z}_p}(K)$ such that the Galois group of K acts like the Hecke character on a certain cohomology group of the motive. We follow the exposition in [6, section 1], with some additional considerations because we are in the p-adic situation.

Recall that the Weil restriction A of E has complex multiplication by an $\mathcal{O}_K$-algebra $\mathcal{O}'_T$, and that $T = \mathcal{O}'_T \otimes \mathbb{Q}$ decomposes into a direct sum of fields $T = \oplus_e T_e$. To obtain a similar decomposition of $\mathcal{O}'_T \otimes \mathbb{Z}_p$ we need the following lemma of [5, lemma 4.6.1]:

Lemma 5.1. *The morphism $\pi : \operatorname{Spec} \mathcal{O}'_T \to \operatorname{Spec} \mathcal{O}_K$ is étale outside the set S of prime divisors of $d = [F : K]$ in $\mathcal{O}_K$. Let $(\mathcal{O}'_T)_S$ be the localization of $\mathcal{O}'_T$, then $(\mathcal{O}'_T)_S = (\mathcal{O}_T)_S$, where $\mathcal{O}_T = \prod_e \mathcal{O}_{T_e}$ is the maximal $\mathcal{O}_K$-order of T. In particular for every idempotent e we get: $e \in (\mathcal{O}'_T)_S$, $e \cdot (\mathcal{O}'_T)_S = (\mathcal{O}_{T_e})_S$ and $\mathcal{O}'_T \otimes \mathbb{Z}_p = \mathcal{O}_T \otimes \mathbb{Z}_p$ for $p \not| d$.*

For $p \not| d$ we thus have a decomposition

$$\mathcal{O}'_T \otimes \mathbb{Z}_p = \bigoplus_e \bigoplus_{\mathfrak{P}|p} \mathcal{O}_{T_e,\mathfrak{P}}.$$

Let $M = \mathcal{R}_{F/K}(h_1(E))$ be the Grothendieck restriction of $h_1(E)$ induced by the functor which considers a variety over F as a variety over K. It has multiplication by $\mathcal{O}'_T \otimes \mathbb{Z}_p$ and is thus an object in $\mathcal{M}_{\mathbb{Z}_p}(K, \mathcal{O}'_T \otimes \mathbb{Z}_p)$.

The following proposition clarifies the connection between M and A:

Lemma 5.2. *Let F/K be a Galois extension with Galois group G of degree d prime to p. Then*

$$CH^i(X) \otimes \mathbb{Z}_p \cong \left(CH^i(X \times_K F) \otimes \mathbb{Z}_p\right)^G.$$

Proof: For $p : X \times_K F \to X$ we have $p^* : [Z] \mapsto [Z \times_K F]$ and $p_* : [Y] \mapsto [k(Y) : k(p(Y))][p(Y)]$. Thus $p_* p^*[Z] = d[Z]$ and $p^* p_*[Z] = \sum [Z^\sigma]$. $\square$

Proposition 5.3. *a) For $p \not| [F : K]$ we have the following decomposition in $\mathcal{M}_{\mathbb{Z}_p}(K)$:*

$$h(A) = \bigoplus_{k=0}^{2n} h_k(A),$$

where $h_k(A)$ has the property

$$H^m(\overline{h_k(A)}, \mathbb{Z}/p^n(i)) = 0 \qquad \textit{for} \quad k \neq m.$$

b) There exists an isomorphism $M \cong h_1(A)$ of motives in $\mathcal{M}_{\mathbb{Z}_p}(K)$.

Proof: The base extensions from F to K of A and the Grothendieck restriction of E can be calculated to be

$$A \times_K F = \prod_{\sigma \in G} E^\sigma \qquad \mathcal{R}_{F/K} E \times_K F = \bigcup_{\sigma \in G} E^\sigma.$$

a) If we extend the base field to F, then we have

$$h(A \times_K F) = \prod_\sigma \bigoplus_{i=0}^{2} h_i(E^\sigma) = \bigoplus_{k=0}^{2d} \Big(\bigoplus_{\sum i_\sigma = k} \prod_\sigma h_{i_\sigma}(E^\sigma) \Big) =: \bigoplus_{k=0}^{2d} h_k(A),$$

where σ runs through $Gal(F/K)$. Since the Galois group permutes the E^σ, we see that the $h_k(A)$ are fixed by the Galois group. By the lemma we conclude that the decomposition descends to A/K.

b) The Grothendieck restriction is left adjoint to the base extension functor, so

$$\mathrm{Hom}_F(E, \prod_{\sigma \in G} E^\sigma) = \mathrm{Hom}_K(\mathcal{R}_{F/K} E, A).$$

Let f be the map corresponding to the canonical map $E \xrightarrow{id} E, E \xrightarrow{0} E^\sigma$, and define f^1 to be the composition

$$M \to h(\mathcal{R}_{F/K} E) \xrightarrow{f} h(A) \to h_1(A).$$

We claim that f^1 is an isomorphism. We have

$$M \times_K F = \bigoplus_\sigma h_1(E^\sigma)$$
$$h_1(A) \times_K F = h_1(\prod_\sigma E^\sigma) = \bigoplus_{\sum i_\sigma = 1} \prod_\sigma h_{i_\sigma}(E^\sigma) = \bigoplus_\sigma h_1(E^\sigma).$$

These two motives are isomorphic, the isomorphism being induced by f^1. Thus there exists an inverse map g in the category $\mathcal{M}_{\mathbb{Z}_p}(F)$ of Chow motives over F. Since f^1 and the identity are Galois invariant, we conclude that g is Galois invariant as well. So by the lemma we are done. □

Let $\mathcal{O}_w = \mathcal{O}'_T \otimes \cdots \otimes \mathcal{O}'_T$ (w times), $T_w = \mathcal{O}_w \otimes \mathbb{Q}$, and set $M_w = M \otimes \cdots \otimes M$ (w times), viewed as an object in $\mathcal{M}_{\mathbb{Z}_p}(K, \mathcal{O}_w \otimes \mathbb{Z}_p)$. Furthermore we define $\Lambda = \mathrm{Hom}(T, \mathbb{C})$ and $\Lambda_p = \mathrm{Hom}(\mathcal{O}_T \otimes \mathbb{Z}_p, \mathbb{C}_p)$.

Lemma 5.4. *a) We have $T_w = \oplus_\Theta T_\Theta$, where Θ runs through the* $\mathrm{Aut}(\mathbb{C})$*-orbits of* $\Lambda^{\otimes w} = \mathrm{Hom}(T_w, \mathbb{C})$.

b) We have

$$\mathcal{O}_w \otimes \mathbb{Z}_p = \bigoplus_\Theta \bigoplus_{\mathfrak{P}|p} \mathcal{O}_{\Theta, \mathfrak{P}} = \bigoplus_\Omega \mathcal{O}_\Omega,$$

where Θ is as above, $\mathfrak{P}$ runs through the primes of T_Θ dividing p and Ω runs through the $\mathrm{Aut}(\mathbb{C}_p)$*-orbits of* $\Lambda_p^{\otimes w} = \mathrm{Hom}(\mathcal{O}_w \otimes \mathbb{Z}_p, \mathbb{C}_p)$.

Proof: a) We can distinguish the field components of T_w by embeddings into $\mathbb{C}$, and two embeddings of the same field differ by an automorphism of $\mathbb{C}$.

b) The first equation follows from (a) and the second equation is proven exactly as in (a). Observe that we get a finer decomposition because we only take orbits under the decomposition group of a prime ν of $\bar{\mathbb{Q}}$ above p. $\square$

Let e_Θ be the idempotent corresponding to the field component T_Θ of T_w and define the CM character

$$\varphi_\Theta : I_K \to T_\Theta^*$$

by $\varphi_\Theta = e_\Theta \cdot (\varphi \otimes \cdots \otimes \varphi)$ (w times). The infinity type of φ_Θ can be determined as follows [6, 1.3.2]:

Fix an embedding $K \hookrightarrow \bar{\mathbb{Q}} \subseteq \mathbb{C}$. The embedding $K \subset T$ induces an embedding

$$K \to K \otimes K \cdots \otimes K \to T_w \to T_\Theta.$$

For $\vartheta \in \Theta \cap \mathrm{Hom}_K(T_\Theta, \mathbb{C})$, $\vartheta = (\lambda_1, \cdots, \lambda_w) \in \Lambda^w$ we set

$$a_\vartheta = \#\{i | \lambda_i \in \mathrm{Hom}_K(T, \mathbb{C})\} \quad , \qquad b_\vartheta = w - a_\vartheta.$$

Then we have $\varphi_\Theta((x)) = x^{a_\vartheta} \bar{x}^{b_\vartheta}$ for $x \in K^*$ with $x \equiv 1 \mod \mathfrak{f}$, i.e. $(a_\vartheta, b_\vartheta)$ is the infinity type of φ_Θ. This is independent of $\vartheta \in \Theta \cap \mathrm{Hom}_K(T_\Theta, \mathbb{C})$. For $\bar{\vartheta} \in \Theta \cap \mathrm{Hom}_{K^\sigma}(T_\Theta, \mathbb{C})$ the homomorphisms inducing the conjugate embedding of K, we get $(a_{\bar{\vartheta}}, b_{\bar{\vartheta}}) = (b_\vartheta, a_\vartheta)$. We now have [6, prop. 1.3.1]

Proposition 5.5. *Every Hecke character of K of positive weight w has the form φ_Θ for suitable E/F and Θ.*

$\square$

Let us now consider the p-adic situation.

Writing e_Ω for the idempotent of $\mathcal{O}_w \otimes \mathbb{Z}_p$ corresponding to $\mathcal{O}_\Omega$, we obtain a motive $M_\Omega = (M_w, e_\Omega)$ in $\mathcal{M}_{\mathbb{Z}_p}(K, \mathcal{O}_\Omega)$ for the completion $\mathcal{O}_\Omega$ of one of the $\mathcal{O}_\Theta$ at a place $\mathfrak{P}$ above p.

From the character $\varphi_{\Theta,\mathfrak{P}} : I_K \to T_{\Theta,\mathfrak{P}}^*$ we get the character

$$\varphi_\Omega = \varphi_{\Theta,\mathfrak{P}} : Gal(K^{ab}/K) \to \mathcal{O}_\Omega^*.$$

Notice that the $\mathfrak{P}$-component of φ_Θ agrees with the e_Ω-component of the w-fold tensor product of φ, i.e. the following diagram commutes:

$$\begin{array}{ccccc}
\varphi_{\Theta,\mathfrak{P}} : I_K & \xrightarrow{\varphi_\Theta} & (T_\Theta \otimes \mathbb{Q}_p)^* & \xrightarrow{proj.} & T_{\Theta,\mathfrak{P}}^* \\
\uparrow & & \uparrow & & \uparrow \\
e_\Omega \cdot \varphi_p^{\otimes w} : I_K & \longrightarrow & (T_w \otimes \mathbb{Q}_p)^* & \xrightarrow{proj.} & T_\Omega^*
\end{array}$$

where φ_p denotes the p-component of φ. This shows that φ_Ω factors through $\mathcal{G}$, since φ_p does.

Now let ω be in the orbit Ω. We want to calculate the map φ_Ω on $Gal(F_\infty/F)$. This is analogous to the determination of the infinity type of φ_Θ above. If $\omega = (\lambda_1, \ldots, \lambda_w) \in \Lambda_p^w$, we define

$$a_\Omega = \#\{i | \lambda_i \in \mathrm{Hom}_{\mathcal{O}_{K_\mathfrak{p}}}(\mathcal{O}_T \otimes \mathbb{Z}_p, \mathbb{C}_p)\}$$
$$b_\Omega = \#\{i | \lambda_i \in \mathrm{Hom}_{\mathcal{O}_{K_{\bar{\mathfrak{p}}}}}(\mathcal{O}_T \otimes \mathbb{Z}_p, \mathbb{C}_p)\},$$

i.e. a_Ω counts the λ_i inducing $\mathfrak{p}$ on $\mathcal{O}_K \subseteq \mathcal{O}_T$ and b_Ω counts the λ_i inducing $\bar{\mathfrak{p}}$. Then from the identity $e_\Omega \cdot \varphi_{A,p}^{\otimes w} = \varphi_\Omega$, and the fact that $\varphi_{\mathfrak{P}}$ and $\varphi_{\bar{\mathfrak{P}}}$ agree on $Gal(F_\infty/F)$ with κ_1 respectively κ_2, one easily sees that φ_Ω agrees on $Gal(F_\infty/F)$ with $\kappa_1^{a_\Omega}\kappa_2^{b_\Omega}$.

Notice also that if Ω belongs to Θ and $\mathfrak{P}$, then $(a_\Omega, b_\Omega) = (a_\Theta, b_\Theta)$ if $\mathfrak{P}$ divides $\mathfrak{p}$ and $(a_\Omega, b_\Omega) = (b_\Theta, a_\Theta)$ if $\mathfrak{P}$ divides $\bar{\mathfrak{p}}$.

Let for the rest of the paper

$$\mathfrak{M}_n = H^w(\bar{M}_\Omega, \mathbb{Z}/p^n(w))$$

and

$$\mathfrak{M} = H^w(\bar{M}_\Omega, \mathbb{Z}_p(w)).$$

We will tacitly use the following inclusion

$$\mathfrak{M}_n \subseteq H^w(\bar{M}^{\otimes w}, \mathbb{Z}/p^n(w)) = H^w(\overline{h_1(A)}^{\otimes w}, \mathbb{Z}/p^n(w)) = H^1(\overline{h_1(A)}, \mathbb{Z}/p^n(1))^{\otimes w}.$$

The next proposition explains why we call M_Ω the motive associated to the Hecke character φ_Ω:

Proposition 5.6. *The étale cohomology group $\mathfrak{M} = H^w(\bar{M}_\Omega, \mathbb{Z}_p(w))$ is a free $\mathcal{O}_\Omega$-module of rank 1, and the absolute Galois group G_K of K acts on it via the character $\varphi_\Omega : G_K \to \mathcal{O}_\Omega^*$.*

Proof: Consider the action of G_K on $H^1(\bar{M}, \mathbb{Z}_p(1)) = \mathrm{Hom}(T_pA, \mathbb{Z}_p(1))$. By the Weil pairing this is isomorphic to $T_pA^\vee = T_pA$ (a free $\mathcal{O}_T \otimes \mathbb{Z}_p$-module of rank one) as Galois modules.

By definition of φ, the operation of G_K on the Tate module T_pA is given by $\varphi_p : G_K \to (\mathcal{O}_T \otimes \mathbb{Z}_p)^*$. But then the operation of G_K on

$$\mathfrak{M} = e_\Omega \cdot H^w(\bar{M}^{\otimes w}, \mathbb{Z}_p(w)) = e_\Omega \cdot H^1(\bar{M}, \mathbb{Z}_p(1))^{\otimes w}$$

is given by $e_\Omega \cdot \varphi_p^{\otimes w} = \varphi_\Omega$.

The first claim follows because $e_\Omega \cdot T_pA \otimes \cdots \otimes T_pA$ is a free $e_\Omega \cdot \mathcal{O}_w \otimes \mathbb{Z}_p = \mathcal{O}_\Omega$-module of rank one. □

6. Tools from Iwasawa theory

We need to recall some facts of Iwasawa theory of imaginary quadratic fields from de Shalit [19]. However in our case we have to consider the two variable theory. So we will state the theorems in the two variable setting, as indicated in [19, III 1.14, II 4.17].

Let $\mathfrak{f}$ be as before the lowest common multiple of the conductors of F/K and φ, and let $\mathfrak{f}_n$ be $\mathfrak{f}\bar{\mathfrak{p}}^n$. This gives us the following tower of ray class fields:

$$K \to F \to K(\mathfrak{f}) \xrightarrow{\mathbb{Z}/p^{n-1}\times\Delta_2} K(\mathfrak{f}_n) \xrightarrow{\Gamma_1\times\Delta_1} K(\mathfrak{f}_n\mathfrak{p}^\infty)$$

Let $\mathcal{G}(\mathfrak{f}_n) = Gal(K(\mathfrak{f}_n\mathfrak{p}^\infty)/K)$ be the Galois group of this extension and $H' = Gal(K(\mathfrak{f}p)/K)$ be its prime to p-part. Then H' contains $\Delta_1 \times \Delta_2$ with $Gal(K(\mathfrak{f})/K)$ as quotient. Furthermore we have

$$\mathcal{G}(\mathfrak{f}_n) = H' \times \Gamma_1 \times \mathbb{Z}/p^{n-1}.$$

Let $\mathcal{U}_{\mathfrak{f}_n}$ be the inverse limit with respect to m of the local principal units in $K(\mathfrak{f}_n\mathfrak{p}^m) \otimes_K K_{\mathfrak{p}} = \prod_{\nu|\mathfrak{p}} K(\mathfrak{f}_n\mathfrak{p}^m)_\nu$ (note that this product remains finite as $n, m \to$

∞) and $\mathcal{U}(\mathfrak{f}) = \varprojlim \mathcal{U}_{\mathfrak{f}_n}$. Let $\mathcal{C}_{\mathfrak{f}_n}$ be the elliptic units as in [19, III 1.4] and $\mathcal{C}(\mathfrak{f}) = \varprojlim \mathcal{C}_{\mathfrak{f}_n}$. Finally denote by

$$\Lambda_n = \Lambda(\mathcal{G}(\mathfrak{f}_n), \mathbb{D}) \cong \mathbb{D}[[\mathcal{G}(\mathfrak{f}_n)]]$$

the Iwasawa algebra of $\mathcal{G}(\mathfrak{f}_n)$ and by $\Lambda_\infty = \varprojlim \Lambda_n = \Lambda(\mathcal{G}(\mathfrak{f}\bar{\mathfrak{p}}^\infty), \mathbb{D})$. Here we use $\mathbb{D}$, the ring of integers of the maximal unramified extension of $K_\mathfrak{p}$, to make sure that all characters of finite groups of order prime to p have values in $\mathbb{D}$.

Fix a place ν of $K(\mathfrak{f}p^\infty)$ above p. Let H'_ν be the decomposition group of ν in H'. As in [19, III 1.3] we get for any character χ of H' different from the cyclotomic character on H'_ν

$$\big(\mathcal{U}(\mathfrak{f}) \hat{\otimes} \mathbb{D}\big)^\chi = \varprojlim \big(\mathcal{U}_{\mathfrak{f}_n} \hat{\otimes} \mathbb{D}\big)^\chi = \varprojlim \Lambda_n^\chi = \Lambda_\infty^\chi.$$

Let $\mu(\mathfrak{f}_n)$ be the measure defined in [19, II 4.12] and $\nu(\mathfrak{f}) = \varprojlim \mu(\mathfrak{f}_n)$. Then we have for χ a character of H' different from the cyclotomic character:

$$\big(\mathcal{C}(\mathfrak{f}) \hat{\otimes} \mathbb{D}\big)^\chi = \varprojlim \big(\mathcal{C}_{\mathfrak{f}_n} \hat{\otimes} \mathbb{D}\big)^\chi = \varprojlim \big(\mu(\mathfrak{f}_n)\Lambda_n\big)^\chi = \big(\nu(\mathfrak{f}) \cdot \Lambda_\infty\big)^\chi.$$

So we get as in [19, III 1.5] that

$$\big(\mathcal{U}(\mathfrak{f})/\mathcal{C}(\mathfrak{f}) \hat{\otimes} \mathbb{D}\big)^\chi = \big(\Lambda_\infty/\nu(\mathfrak{f}) \cdot \Lambda_\infty\big)^\chi = \mathbb{D}[[\Gamma_1 \times \Gamma_2]]/\nu(\mathfrak{f})^\chi$$

for $\nu(\mathfrak{f})^\chi$ the χ-component of the measure $\nu(\mathfrak{f}) \in \Lambda$. We thus have

Proposition 6.1. *With the notation as before we have for $\chi \neq \kappa$ on H'_ν:*

$$\big(\mathcal{U}(\mathfrak{f})/\mathcal{C}(\mathfrak{f}) \hat{\otimes} \mathbb{D}\big)^\chi = \mathbb{D}[[\Gamma_1 \times \Gamma_2]]/\nu(\mathfrak{f})^\chi.$$

In the more general situation $F(E_{p^n}) \subseteq K(\mathfrak{f}p^n)$ we have a surjection of $\mathcal{G}(\mathfrak{f}\bar{\mathfrak{p}}^\infty) = Gal(K(\mathfrak{f}p^\infty))/K)$ onto $\mathcal{G} = Gal(F(E_{p^\infty})/K)$, which gives the following diagram

$$\begin{tikzcd}
F(E_{p^\infty}) \arrow[r] & K(\mathfrak{f}p^\infty) \\
F(E_p) \arrow[u, "\Gamma_1\times\Gamma_2"'] \arrow[r] & K(\mathfrak{f}p) \arrow[u, "\Gamma_1\times\Gamma_2"'] \\
K \arrow[u, "H"'] \arrow[r, equal] & K. \arrow[u, "H'"']
\end{tikzcd}$$

We view any character of H as a character of H' via the canonical map $H' \to H$.

Let $\mathcal{U}$ be the inverse limit of the local principal units in the tower $F(E_{p^n})$ and $\mathcal{C} = N\mathcal{C}(\mathfrak{f}) \subseteq \mathcal{U}$. Then one sees easily that $(\mathcal{U}(\mathfrak{f}) \hat{\otimes} \mathbb{D})^\chi \cong (\mathcal{U} \hat{\otimes} \mathbb{D})^\chi$ and that $(\mathcal{C}(\mathfrak{f}) \hat{\otimes} \mathbb{D})^\chi \cong (\mathcal{C} \hat{\otimes} \mathbb{D})^\chi$. So we get

Proposition 6.2. *With the notation as before we have for $\chi \neq \kappa$ on H_ν*

$$(\mathcal{U}/\mathcal{C} \hat{\otimes} \mathbb{D})^\chi \cong (\mathcal{U}(\mathfrak{f})/\mathcal{C}(\mathfrak{f}) \hat{\otimes} \mathbb{D})^\chi \cong \mathbb{D}[[\Gamma_1 \times \Gamma_2]]/\nu(\mathfrak{f})^\chi.$$

Corollary 6.3. *Let $\chi = \varphi_\Omega \kappa^l$ and $p - 1 \not| a + l + 1$. Then we have*

$$(\mathcal{U} \hat{\otimes} \mathbb{D})^{\chi^{-1}} \cong \mathbb{D}[[\Gamma_1 \times \Gamma_2]]$$

$$(\mathcal{U}/\mathcal{C} \hat{\otimes} \mathbb{D})^{\chi^{-1}} \cong \mathbb{D}[[\Gamma_1 \times \Gamma_2]]/\nu(\mathfrak{f})^{\chi^{-1}}.$$

In particular

$$\big((\mathcal{U} \hat{\otimes} \mathbb{D}) \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\mathcal{G}}$$

is a free $\mathbb{D}$-module of rank 1.

Proof: For the the first two isomorphisms it suffices to show that under the conditions of the corollary χ^{-1} is different from κ on H_ν. But $\Delta_1 = Gal(F(E_\mathfrak{p})/F) \cong Gal(F(E_p)/F(E_{\bar{\mathfrak{p}}}))$ is a subgroup of H_ν since $\mathfrak{p}$ is totally ramified in Δ_1. We know that $\varphi_\Omega|_{\Delta_1} = \kappa^a|_{\Delta_1}$, so we get $\varphi_\Omega^{-1} \cdot \kappa^{-l}|_{\Delta_1} = \kappa^{-a-l}|_{\Delta_1}$, and this is different from κ as long as $\#\Delta_1 = p-1 \not| a+l+1$.

For the last claim we know that

$$\begin{aligned}\big((\mathcal{U} \hat{\otimes} \mathbb{D}) \otimes_\mathbb{D} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_\mathcal{G} &\cong \big((\mathcal{U} \hat{\otimes} \mathbb{D})^{\chi^{-1}} \otimes_\mathbb{D} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\Gamma_1 \times \Gamma_2} \\ &\cong \big((\mathbb{D}[[\Gamma_1 \times \Gamma_2]] \otimes_\mathbb{D} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\Gamma_1 \times \Gamma_2} \cong \mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D}.\end{aligned}$$

□

The measures $\mu(\mathfrak{f}_n) = \mu(\mathfrak{f}\bar{\mathfrak{p}}^n)$ have the additional property that they interpolate Hecke L-series of imaginary quadratic fields:

Theorem 6.4. [19, theorem 4.14] *Let $\mathfrak{f}$ be an integral ideal of K prime to p, and $\nu(\mathfrak{f}) = \varprojlim \mu(\mathfrak{f}_n)$ as above. Then there exist periods $\langle \Omega, \Omega_p \rangle \in (\mathbb{C}^* \times \mathbb{C}_p^*)/\bar{\mathbb{Q}}^*$ and a Gauss sum $G(\epsilon)$ defined in* [19, theorem 4.14] *such that the following formula in $\bar{\mathbb{Q}}$ holds for any grossencharacter ϵ of conductor dividing $\mathfrak{f}p^\infty$, and of infinity type (k,j), $0 \le -j \le k$:*

$$\Omega_p^{j-k} \int_\mathcal{G} \epsilon(\sigma) d\mu(\sigma) = \Omega^{j-k} \big(\frac{\sqrt{d_K}}{2\pi}\big)^j G(\epsilon) \big(1 - \frac{\epsilon(\mathfrak{p})}{p}\big) L_{\infty, \mathfrak{f}\bar{\mathfrak{p}}}(\epsilon^{-1}, 0).$$

Here $L_{\infty,\mathfrak{f}\bar{\mathfrak{p}}}$ is the complex L-series with the Euler factors at ∞ but without the Euler factors at primes dividing $\mathfrak{f}\bar{\mathfrak{p}}$.

Let us recall the connections between measures and power series. Let γ_i be a generator of Γ_i, κ_i be the character of Γ_i giving the action on the torsion points of the elliptic curve and u_i be the image of γ_i in $\mathbb{Z}_p$. We then have isomorphisms

$$\Lambda(\Gamma_1 \times \Gamma_2, \mathbb{D}) \cong \mathbb{D}[[\Gamma_1, \Gamma_2]] \cong \mathbb{D}[[T_1, T_2]]$$

mapping a measure μ to the power series

$$G(T_1, T_2) = \int_{\Gamma_1 \times \Gamma_2} (1+T_1)^\alpha (1+T_2)^\beta d\mu(\alpha, \beta).$$

In particular we get

$$G(u_1^a - 1, u_2^b - 1) = \int (u_1^a)^\alpha (u_2^b)^\beta d\mu(\alpha, \beta) = \int \kappa_1^a \kappa_2^b d\mu$$

Now let μ be a measure in $\Lambda(\mathcal{G}, \mathbb{D})$ and χ be a character of H. If we denote the power series associated to the χ-component of μ by $G(\chi^{-1}, T_1, T_2)$, we get

$$\int_\mathcal{G} \kappa_1^a \kappa_2^b \chi \mu = \int_{\Gamma_1 \times \Gamma_2} \kappa_1^a \kappa_2^b d\mu^\chi = G(\chi^{-1}, u_1^a - 1, u_2^b - 1),$$

i.e. if we write an arbitrary character ϵ of $\mathcal{G}$ as $\kappa_1^a \kappa_2^b \chi$, then the power series $G(\chi^{-1}, T_1, T_2)$ calculates the integral of ϵ.

By the interpolation theorem we see that $G(\chi, u_1^a - 1, u_2^b - 1)$ is a p-adic interpolation of $L(\chi \kappa_1^{-a} \kappa_2^{-b}, 0) = L(\chi \kappa_1^{b-a}, -b)$, at least for $0 \le -b \le a$.

7. Special elements in K-groups with coefficients

We follow Soulé [23] for a construction of elements in $K_a(X, \mathbb{Z}_p)$.

In order that all Adams eigenspaces and K-groups for the motives involved be defined, we assume from now on that for $w \geq 1$ the weight of our Hecke character of K, d the degree of the extension F/K and a fixed integer $l \geq 0$, we have

$$p > 3dw + 2l + w + 1.$$

Then we can define $K_{2l+w+1}(X, \mathbb{Z}/p^n)^{(i)}$ for X in $\mathcal{M}_\Lambda^{dw}(k)$.

Let $p_n : X_n \to X_{n-1}$ be a sequence of Galois coverings of finite dimensional regular schemes with $X_0 = X$. Denote the reduction of coefficients from $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$ by ϵ_n. Consider a sequence $\alpha = (\alpha_n)$ of elements $\alpha_n \in K_a(X_n, \mathbb{Z}/p^n)$.

We say the sequence α has property N if it is an element of $\varprojlim K_a(X_n, \mathbb{Z}/p^n)$, i.e. for all n we have

$$\epsilon_n \circ p_{n*}(\alpha_n) = \alpha_{n-1} \quad \text{in} \quad K_a(X_{n-1}, \mathbb{Z}/p^{n-1}).$$

We say the sequence α has property R if for all n

$$p_n^*(\alpha_{n-1}) = \epsilon_n(\alpha_n) \quad \text{in} \quad K_a(X_n, \mathbb{Z}/p^{n-1}).$$

If the sequence $\alpha = (\alpha_n)$ has property N and the sequence $\beta = (\beta_n)$ has property R, then the sequence $\alpha \cup \beta = (\alpha_n \cup \beta_n)$ has property N, as one sees easily with the projection formula.

Let q_n be the projection $X_n \to X$. Then given a sequence $\alpha = (\alpha_n)$ having property N, the sequence $N(\alpha) = (q_{n*}(\alpha_n))$ is an element in $K_m(X, \mathbb{Z}_p)$. As we just saw we can first form the cup product with sequence(s) with property R before taking the norm to X.

Recall that F_n is the field generated by the p^n-torsion points of the elliptic curve E over F. Let A_n be the abelian variety $A \times_K F_n \cong \prod_{\sigma \in G}(E^\sigma \times_F F_n)$. By the universal property of the Weil restriction, the p^n-torsion points of A are defined over F_n, in fact ${}_{p^n}A_n = \prod_{\sigma \in G} {}_{p^n}(E^\sigma \times_F F_n)$.

We construct the following elements in the K-groups with coefficients in $\mathbb{Z}/p^n$:

(1) From the universal coefficient sequence of K-theory we get $K_1(F_n, \mathbb{Z}/p^n) = F_n^*/p^n$. Any sequence $(u_n) \in \varprojlim F_n^*/p^n$ thus gives us a sequence in $K_1(F_n, \mathbb{Z}/p^n)$ with property N.

(2) We similarly get

$$0 \to K_2(F_n)^{(1)}/p^n \to K_2(F_n, \mathbb{Z}/p^n)^{(1)} \to {}_{p^n}K_1(F_n)^{(1)} \to 0.$$

Here the left term vanishes because $K_2(F_n)^{(1)} = 0$ and the right term is isomorphic to the group of p^n-th roots of unity μ_{p^n}. So we get $K_2(F_n, \mathbb{Z}/p^n)^{(1)} \cong \mu_{p^n}$. An element of the Tate module $(\beta_n) \in \mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$ then gives us a sequence in $K_2(F_n, \mathbb{Z}/p^n)^{(1)}$ having property R.

(3) According to Grayson [12], there are Adams operators on ψ^k which are compatible with the universal coefficient sequence. Choose k to be a primitive root of unity mod p. Let $A^{(i)}$ and $A_{(i)}$ be the largest subgroup and quotient of A such that ψ^k acts like k^i, respectively. Note that $K_a(X)^{(i)} = K_a(X)_{(i)}$.

We get an exact sequence

$$0 \to K_1(A_n)^{(1)}/p^n \xrightarrow{\alpha^1} K_1(A_n, \mathbb{Z}/p^n)^{(1)} \to {}_{p^n}K_0(A_n)^{(1)} \xrightarrow{\delta}$$
$$K_1(A_n)_{(1)}/p^n \xrightarrow{\alpha_1} K_1(A_n, \mathbb{Z}/p^n)_{(1)} \to {}_{p^n}K_0(A_n)_{(1)} \to 0.$$

Via the identification

$$K_1(A_n)^{(1)}/p^n = K_1(A_n)_{(1)}/p^n = F_n^*/p^n = K_1(F_n)^{(1)}/p^n = K_1(F_n, \mathbb{Z}/p^n)^{(1)},$$

we can think of α as being induced by the structure morphism $A_n \to F_n$. But this map is split by the point 0, so the sequence breaks up into two split short exact sequences, and we get a map

$$\eta : {}_{p^n} \operatorname{Pic} A_n \cong {}_{p^n}K_0(A_n)^{(1)} \to K_1(A_n, \mathbb{Z}/p^n)^{(1)}.$$

It is easy to see that an element (v_n) of $\varprojlim {}_{p^n} \operatorname{Pic} A_n = H^1(\bar{A}, \mathbb{Z}_p(1))$ gives us a sequence (α_n) with property R.

We now take the exterior products of these elements:

$$\begin{aligned}\phi_n : F_n^*/p^n \otimes H^1(\bar{A}, \mathbb{Z}/p^n(1))^{\otimes w} \otimes \mathbb{Z}/p^n(l) \to \\ K_1(F_n, \mathbb{Z}/p^n)^{(1)} \otimes K_1(A_n, \mathbb{Z}/p^n)^{(1)\otimes w} \otimes K_2(F_n, \mathbb{Z}/p^n)^{(1)\otimes l} \xrightarrow{\cup} \\ K_{2l+w+1}(A_n^w, \mathbb{Z}/p^n)^{(l+w+1)} \xrightarrow{proj} K_{2l+w+1}(h_1(A_n)^{\otimes w}, \mathbb{Z}/p^n)^{(l+w+1)}\end{aligned}$$

Note that by the definition of F_n, the Hochschild-Serre spectral sequence, proposition 5.3 and the Kuenneth formula we have the following equalities:

$$\begin{aligned}&F_n^*/p^n \otimes H^1(\bar{A}, \mathbb{Z}/p^n(1))^{\otimes w} \otimes \mathbb{Z}/p^n(l) \\ =&H^1(F_n, \mathbb{Z}/p^n(1)) \otimes H^0(F_n, H^1(\bar{A}, \mathbb{Z}/p^n(1))^{\otimes w} \otimes H^0(F_n, \mathbb{Z}/p^n(l)) \\ =&H^1\big(F_n, H^1(\bar{A}, \mathbb{Z}/p^n(1))^{\otimes w}(l+1)\big) \\ =&H^1\big(F_n, H^w(\overline{h_1(A_n)}^{\otimes w}, \mathbb{Z}/p^n(w))(l+1)\big) \\ =&H^{w+1}(h_1(A_n)^{\otimes w}, \mathbb{Z}/p^n(l+w+1))\end{aligned}$$

8. The regulator map

We will construct a regulator map from $K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)}$ to a certain Galois cohomology group and show that in the local situation the maps ϕ_n are splittings of this regulator map (modulo some special cases).

Define the following maps using the map ρ from K-theory to étale K-theory and the degeneration of the Dwyer-Friedlander spectral sequence:

$$\begin{aligned}\xi_n :&K_{2l+w+1}(h_1(A_n)^{\otimes w}, \mathbb{Z}/p^n)^{(l+w+1)} \to H^{w+1}(h_1(A_n)^{\otimes w}, \mathbb{Z}/p^n(l+w+1)) \\ \xi_n' :&K_{2l+w+1}(h_1(A)^{\otimes w}, \mathbb{Z}/p^n)^{(l+w+1)} \to H^{w+1}(h_1(A)^{\otimes w}, \mathbb{Z}/p^n(l+w+1)).\end{aligned}$$

Lemma 8.1. *The map ϕ_n is a splitting of ξ_n, i.e. $\xi_n \circ \phi_n$ is the identity map of $H^1\big(F_n, H^1(\bar{A}, \mathbb{Z}/p^n(1))^{\otimes w}(l+1)\big)$.*

Proof: First note that all maps are compatible with Tate twists, i.e. tensoring with $\mathbb{Z}/p^n(1) \cong H^0(F_n, \mathbb{Z}/p^n(1)) \cong K_2(F_n, \mathbb{Z}/p^n)^{(1)}$, so we may assume $l = 0$.

The rational point 0 of A_n gives us a splitting of the sequence

$$0 \to F_n^*/p^n \to H^1(A_n, \mathbb{Z}/p^n(1)) \to {}_{p^n} \operatorname{Pic} A_n \to 0.$$

We will denote this splitting by s. Our splitting ${}_{p^n}Pic\ A_n \to K_1(A_n,\mathbb{Z}/p^n)$ was constructed via the point 0 of A_n, so we get the commutative diagram

$$\begin{array}{ccc} {}_{p^n}Pic\ A_n & \xrightarrow{\sim} & H^1(\bar{A},\mathbb{Z}/p^n(1)) \\ \downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle proj} \\ K_1(A_n,\mathbb{Z}/p^n)^{(1)} & \xrightarrow{\rho} & H^1(A_n,\mathbb{Z}/p^n(1)) \end{array}$$

with $s = \rho\circ\eta$. Taking the w-fold tensor product of this and tensoring with $H^1(F_n,\mathbb{Z}/p^n(1)) = K_1(F_n,\mathbb{Z}/p^n)$, we see that the left hand column of the following commutative diagram is an isomorphism

$$\begin{array}{ccc} H^1(F_n,\mathbb{Z}/p^n(1))\otimes({}_{p^n}\operatorname{Pic}A_n)^{\otimes w} & & \\ \downarrow{\scriptstyle \eta} & & \\ K_1(F_n,\mathbb{Z}/p^n)^{(1)}\otimes K_1(A_n,\mathbb{Z}/p^n)^{(1)\otimes w} & \xrightarrow{\cup} & K_{w+1}(A_n^w,\mathbb{Z}/p^n)^{(w+1)} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ H^1(F_n,\mathbb{Z}/p^n(1))\otimes H^1(A_n,\mathbb{Z}/p^n(1))^{\otimes w} & \xrightarrow{\cup} & H^{w+1}(A_n^w,\mathbb{Z}/p^n(w+1)) \\ \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} \\ H^1(F_n,\mathbb{Z}/p^n(1))\otimes H^1(h_1(A_n),\mathbb{Z}/p^n(1))^{\otimes w} & \xrightarrow{\cup} & H^{w+1}(h_1(A_n)^{\otimes w},\mathbb{Z}/p^n(w+1)) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ H^1(F_n,\mathbb{Z}/p^n(1))\otimes{}_{p^n}\operatorname{Pic}A_n & \xrightarrow{\sim} & H^1\big(F_n,H^1(\bar{A},\mathbb{Z}/p^n(1))^{\otimes w}(1)\big) \end{array}$$

Observing that the right hand column is the map $\xi_n\circ\phi_n$ we get the lemma. $\square$

As ρ is compatible with decomposition into Chow motives, we get induced surjections

$$K_{2l+w+1}(M_\Omega\times_K F_n,\mathbb{Z}/p^n)^{(l+w+1)} \xrightarrow{\xi_{n,\Omega}} H^1(F_n,\mathfrak{M}_n(l+1))$$

which are split by $\phi_{n,\Omega}\subseteq\phi_n$. We thus get a map

$$H^1(F_n,\mathfrak{M}_n(l{+}1)) \xrightarrow{\phi_{n,\Omega}} K_{2l+w+1}(M_\Omega{\times_K}F_n,\mathbb{Z}/p^n)^{(l+w+1)} \xrightarrow{\pi_*} K_{2l+w+1}(M_\Omega,\mathbb{Z}/p^n)^{(l+w+1)}.$$

As the Galois group $\mathcal{G} = Gal(F_\infty/K)$ acts trivially on $K_{2l+w+1}(M_\Omega,\mathbb{Z}_p)^{(l+w+1)}$, the inverse limit of these maps factors through the coinvariants of $\mathcal{G}$, and we get

$$\phi_\Omega : \big(\varprojlim H^1(F_n,\mathfrak{M}_n(l+1))\big)_{\mathcal{G}} \to K_{2l+w+1}(M_\Omega,\mathbb{Z}_p)^{(l+w+1)}.$$

On the other hand, the inverse limit of the maps ξ'_n gives us a map

$$r_\Omega : K_{2l+w+1}(M_\Omega,\mathbb{Z}_p)^{(l+w+1)} \to H^1\big(K,\mathfrak{M}(l+1)\big)\big).$$

Let $\mathcal{G}_\nu = Gal(F_{\infty,\nu}/K_{\mathfrak{p}})$, then we get in the local situation:

$$\phi_{\Omega,\nu} : \big(\varprojlim H^1(F_{n,\nu},\mathfrak{M}_n(l+1))\big)_{\mathcal{G}_\nu} \to K_{2l+w+1}(M_\Omega\times_K K_{\mathfrak{p}},\mathbb{Z}_p)^{(l+w+1)},$$

$$r_{\Omega,\mathfrak{p}} : K_{2l+w+1}(M_\Omega\times_K K_{\mathfrak{p}},\mathbb{Z}_p)^{(l+w+1)} \to H^1(K_{\mathfrak{p}},\mathfrak{M}(l+1)).$$

We obtain the following description of the composition of $\phi_{\mathfrak{p}}$ with the regulator map $r_{\mathfrak{p}}$ as in [23, lemma 4.3]:

Lemma 8.2. *The kernel (respectively cokernel) of the composition*

$$\big(\varprojlim H^1(F_{n,\nu},\mathfrak{M}_n(l+1))\big)_{\mathcal{G}_\nu} \xrightarrow{\phi_{\Omega,\mathfrak{p}}} K_{2l+w+1}(M_\Omega\times_K K_\mathfrak{p},\mathbb{Z}_p)^{(l+w+1)} \xrightarrow{r_{\Omega,\mathfrak{p}}} H^1(K_\mathfrak{p},\mathfrak{M}(l+1))$$

is contained in the Pontrjagin-dual of $H^2(\mathcal{G}_\nu, e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))$ *and* $H^1(\mathcal{G}_\nu,e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))$ *respectively.*

Proof: Let $F_{n,\nu}$ be as above the field obtained by adjoining the p^n-torsion points of E to $K_\mathfrak{p}$. We get a commutative diagram

$$\begin{array}{ccc} H^1\big(F_{n,\nu},\mathfrak{M}_n(l+1)\big) & & \\ \big\downarrow{\scriptstyle \phi_{n,\Omega}} & & \\ K_{2l+w+1}(M_\Omega\times_K F_{n,\nu},\mathbb{Z}/p^n) & \xrightarrow{N} & K_{2l+w+1}(M_\Omega\times_K K_\mathfrak{p},\mathbb{Z}/p^n)^{(l+w+1)} \\ \big\downarrow{\scriptstyle \xi_{n,\Omega}} & & \big\downarrow{\scriptstyle \xi'_{n,\Omega}} \\ H^1\big(F_{n,\nu},\mathfrak{M}_n(l+1)\big) & \xrightarrow{cores} & H^1\big(K_\mathfrak{p},\mathfrak{M}_n(l+1)\big) \end{array}$$

By this diagram and the last lemma, we see that the map $r_\mathfrak{p}\circ\phi_\mathfrak{p}$ is the inverse limit of the composition of the isomorphism $\xi_{n,\Omega}\circ\phi_{n,\Omega}$ with the corestriction map

$$cor: H^1(F_{n,\nu},\mathfrak{M}_n(l+1))\to H^1(K_\mathfrak{p},\mathfrak{M}_n(l+1)).$$

As in the proof of lemma 5.6 we have

$$\mathrm{Hom}\big(H^1(\bar{M},\mathbb{Z}/p^n(1)),\mathbb{Z}/p^n(1)\big)=A_{p^n}$$

and thus

$$\mathrm{Hom}\big(H^1(\bar{M},\mathbb{Z}/p^n(1))^{\otimes w},\mathbb{Z}/p^n(1)\big)=A_{p^n}^{\otimes w}(-w+1)$$

as Galois modules. So we conclude that for

$$\mathfrak{M}_n=e_\Omega\cdot H^w(\bar{M}_n^w,\mathbb{Z}/p^n(w))=e_\Omega\cdot\otimes H^1(\bar{M}_n,\mathbb{Z}/p^n(1))$$

the dual is given by

$$\mathrm{Hom}(\mathfrak{M}_n,\mathbb{Q}_p/\mathbb{Z}_p(1))=e_\Omega\cdot A_{p^n}^{\otimes w}(-w+1)$$

and thus our map is by local duality dual to the restriction map

$$\varinjlim H^1(K_\mathfrak{p},e_\Omega\cdot A_{p^n}^{\otimes w}(-l-w))\to\varinjlim H^1(F_{n,\nu},e_\Omega\cdot A_{p^n}^{\otimes w}(-l-w))$$

which is the same as

$$H^1(K_\mathfrak{p},e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))\to H^1(F_{\infty,\nu},e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w)).$$

But this fits into the Hochschild-Serre spectral sequence of the Galois extension $F_{\infty,\nu}/K_\mathfrak{p}$:

$$0\to H^1(\mathcal{G}_\nu,e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))\to H^1(K_\mathfrak{p},e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))$$
$$\to H^1(F_{\infty,\nu},e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))^{\mathcal{G}_\nu}\to H^2(\mathcal{G}_\nu,e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w)),$$

and the lemma follows. □

Lemma 8.3. *The group* $H^2(\mathcal{G}_\nu,e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))$ *is zero. The group* $H^1(\mathcal{G}_\nu,e_\Omega\cdot A_{p^\infty}^{\otimes w}(-l-w))$ *is finite and zero unless* $l+a=0$ *or* $l+b=0$.

Proof: We have $\mathcal{G}_\nu \cong H_\nu \times \Gamma_{1,\nu} \times \Gamma_{2,\nu}$ where $p \not| \# H_\nu$ and $\Gamma_{1,\nu} \cong \Gamma_{2,\nu} \cong \mathbb{Z}_p$. On the other hand we know that $\mathfrak{N} := e_\Omega \cdot A_{p^\infty}^{\otimes w}(-l-1) \cong T_\Omega/\mathcal{O}_{T_\Omega}$ (as groups). From the Hochschild-Serre spectral sequence we see that $H^i(\mathcal{G}_\nu, \mathfrak{N}) = H^0(H_\nu, H^i(\Gamma_{1,\nu} \times \Gamma_{2,\nu}, \mathfrak{N}))$. Since $\mathbb{Z}_p$ is procyclic,

$$H^0(\Gamma, \mathfrak{N}) = \mathfrak{N}^\Gamma; \qquad H^1(\Gamma, \mathfrak{N}) = \mathfrak{N}_\Gamma; \qquad H^2(\Gamma, \mathfrak{N}) = 0 \quad \text{for} \quad i \geq 2.$$

The spectral sequence for $\Gamma_{1,\nu} \times \Gamma_{2,\nu}$ gives us

$$H^1(\Gamma_{1,\nu} \times \Gamma_{2,\nu}, \mathfrak{N}) = \mathfrak{N}_{\Gamma_{1,\nu}}^{\Gamma_{2,\nu}} \oplus \mathfrak{N}_{\Gamma_{2,\nu}}^{\Gamma_{1,\nu}}$$
$$H^2(\Gamma_{1,\nu} \times \Gamma_{2,\nu}, \mathfrak{N}) = \mathfrak{N}_{\Gamma_{1,\nu} \times \Gamma_{2,\nu}}$$

Lift the operation of $\Gamma_{1,\nu}$ and $\Gamma_{2,\nu}$ on $T_\Omega/\mathcal{O}_{T_\Omega}$ to an operation on T_Ω via ${}^\gamma x = \chi(\gamma) \cdot x$, where $\chi : \Gamma_{i,\nu} \to \mathcal{O}_\Omega^*$ gives the operation of $\Gamma_{i,\nu}$ on $\mathfrak{N}$. If $\Gamma_{i,\nu}$ acts nontrivially on $\mathfrak{N}$, then $\chi(\gamma) \neq 1$ for a generator γ of $\Gamma_{i,\nu}$, so $\gamma - 1$ acts nontrivially(and thus bijectively) on T_Ω. But then it acts surjectively on $T_\Omega/\mathcal{O}_{T_\Omega} \cong \mathfrak{N}$ and it follows $\mathfrak{N}_{\Gamma_{i,\nu}} = 0$.

Observe now that Γ_1 acts on $\mathfrak{N}$ like $\varphi_\Omega \cdot \kappa^{-l-w} = \kappa_1^{-b-l}$ while Γ_2 acts like $\varphi_\Omega \cdot \kappa^{-l-w} = \kappa_2^{-a-l}$,where κ is the cyclotomic character, and κ_1, κ_2 are as in section 4. This is nontrivial for $a + l \neq 0$ and $b + l \neq 0$ respectively.

Since we assumed $w > 0$, either $a > 0$ or $b > 0$ and we conclude $H^2 = 0$. If Γ_1 and Γ_2 act nontrivially (i.e. $l + a > 0$ and $l + b > 0$), then $H^1 = 0$ as well. □

Recall from section 6 that $\mathcal{U}$ and $\mathcal{U}_\nu$ is the inverse limit of the local principal units of $F_{n,\mathfrak{p}} = \prod_{\nu|\mathfrak{p}} F_{n,\nu}$ and $F_{n,\nu}$ respectively, for a fixed place ν dividing $\mathfrak{p}$. Then $\mathcal{U}$ is a compact $\mathbb{Z}_p[[\mathcal{G}]]$-module and we have $\mathcal{U} = Ind_{\mathcal{G}}^{\mathcal{G}_\nu} \mathcal{U}_\nu$ as $\mathcal{G}$-modules.

Lemma 8.4. *The composition*

$$(\mathcal{U} \otimes \mathfrak{M} \otimes \mathbb{Z}_p(l))_{\mathcal{G}} \xrightarrow{\phi_\mathfrak{p}} K_{2l+w+1}(M_\Omega \times_K K_\mathfrak{p}, \mathbb{Z}_p)^{(l+w+1)} \xrightarrow{r_\mathfrak{p}} H^1(K_\mathfrak{p}, \mathfrak{M}(l+1))$$

is a monomorphism and an isomorphism unless $a + l = 0$ or $b + l = 0$.

Proof: This is a reformulation of lemmas 8.2 and 8.3, see [23]. □

9. The main theorem

Recall from section 5 that we associated to the Hecke character φ of weight w of the imaginary quadratic field K a motive M_Ω with multiplication by $\mathcal{O}_\Omega$. The motive M_Ω was a submotive of $h_1(A)^{\otimes w}$, A an abelian variety of dimension d, and $\mathfrak{M} = H^w(\bar{M}_\Omega, \mathbb{Z}_p(w))$ was a free $\mathcal{O}_\Omega$-module of rank 1 on which the Galois group of K acts like $\varphi_\Omega : G_K \to \mathcal{O}_\Omega^*$.

Let R be the regulator map obtained by localization

$$K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)} \to K_{2l+w+1}(M_\Omega \times_K K_\mathfrak{p}, \mathbb{Z}_p)^{(l+w+1)} \xrightarrow{r_{\Omega,\mathfrak{p}}} H^1(K_\mathfrak{p}, \mathfrak{M}(l+1)).$$

Theorem 9.1. *Let $p > 3dw + 2l + w + 1$, $a + l > 0$ and $b + l > 0$. There exists a submodule*

$$\mathcal{V} \subseteq K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)}$$

such that the length as an $\mathcal{O}_\Omega$-module of the cokernel of the regulator $R|_\mathcal{V}$ restricted to $\mathcal{V}$ equals the p-adic valuation of the p-adic L-function

$$G(\varphi_\Omega \kappa^l, u_1^{-a-l} - 1, u_2^{-b-l} - 1).$$

Proof:

Let $\mathcal{C}$ be the elliptic units of section 6 and $\mathcal{V}$ be the image of $\mathcal{C}$ under ϕ in $K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)}$. Consider the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{C} \otimes \mathfrak{M} \otimes \mathbb{Z}_p(l))_{\mathcal{G}} & \xrightarrow{\alpha} & (\mathcal{U} \otimes \mathfrak{M} \otimes \mathbb{Z}_p(l))_{\mathcal{G}} \\
\downarrow \phi & & \downarrow \phi_{\mathfrak{p}} \\
K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)} & \xrightarrow{\beta} & K_{2l+w+1}(M_\Omega \times_K K_{\mathfrak{p}}, \mathbb{Z}_p)^{(l+w+1)} \\
 & & \downarrow r_{\mathfrak{p}} \\
 & & H^1(K_{\mathfrak{p}}, \mathfrak{M}(l+1))
\end{array}$$

Here the map α is induced by the diagonal embedding of $\mathcal{C}$ into $\mathcal{U}$. We proved that the composition $r_{\mathfrak{p}} \circ \phi_{\mathfrak{p}}$ is an isomorphism of free $\mathcal{O}_\Omega$-modules of rank 1 as long as $a + l > 0$ and $b + l > 0$. So it suffices to calculate the cokernel of α. If we want to use the results of section 6, we have to tensor over $\mathcal{O}_\Omega$ with $\mathbb{D}$, the ring of integers of the completion of the maximal unramified extension of T_Ω, and complete this. But since tensoring with $\mathbb{D}$ and completion is exact, the length of the cokernel of α remains unchanged. So we want to calculate the cokernel of the following map

$$\big((\mathcal{C} \hat{\otimes} \mathbb{D}) \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\mathcal{G}} \to \big((\mathcal{U} \hat{\otimes} \mathbb{D}) \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\mathcal{G}}.$$

Now recall the decomposition $\mathcal{G} = H \times \Gamma_1 \times \Gamma_2$ and take H-coinvariants first (which agree with H-invariants, as $p \not| \# H$). Since H acts on $\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D}$ via $\chi := \varphi_\Omega \cdot \kappa^l|_H$, the cokernel equals

$$\begin{aligned}
&\big((\mathcal{U}/\mathcal{C} \hat{\otimes} \mathbb{D})^{\chi^{-1}} \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\Gamma_1 \times \Gamma_2} \\
&= \big(\mathbb{D}[[\Gamma_1, \Gamma_2]]/\mu(\mathfrak{f})^{\chi^{-1}} \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\Gamma_1 \times \Gamma_2} \\
&= \big(\mathbb{D}[[T_1, T_1]]/G(\chi, T_1, T_2) \otimes_{\mathbb{D}} (\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D})\big)_{\Gamma_1 \times \Gamma_2} \\
&= \mathbb{D}/G(\chi, u_1^{-a-l} - 1, u_2^{-b-l} - 1)
\end{aligned}$$

Here the first equality follows by corollary 6.3, because the hypothesis implies $p-1 > a + l + 1$. The second equality is the transformation from the measure $\mu(\mathfrak{f})^{\chi^{-1}}$ to its associated power series. The last equation follows because $\Gamma_1 \times \Gamma_2$ acts via $\varphi_\Omega \cdot \kappa^l = \kappa_1^{a+l} \kappa_2^{b+l}$ on the free rank-1 $\mathbb{D}$-module $\mathfrak{M}(l) \hat{\otimes}_{\mathcal{O}_\Omega} \mathbb{D}$.

But the length of the last module is $v_p(G(\chi, u_1^{-a-l} - 1, u_2^{-b-l} - 1))$. □

Remark: 1) If $a + l = 0$ or $b + l = 0$, then the length of the cokernel of $R_{\mathcal{V}}$ equals $\# \operatorname{coker} r_{\Omega,\mathfrak{p}} \circ \phi_{\Omega,\mathfrak{p}}$ times the valuation of the p-adic L-series in the theorem. This follows because by 8.2 and 8.3 the composition $r_{\Omega,\mathfrak{p}} \circ \phi_{\Omega,\mathfrak{p}}$ is injective, so we have

$$\# \operatorname{coker} r_{\Omega,\mathfrak{p}} \circ \beta \circ \phi_\Omega = \# \operatorname{coker} \alpha \cdot \# \operatorname{coker} r_{\Omega,\mathfrak{p}} \circ \phi_{\Omega,\mathfrak{p}} = \# \operatorname{coker} r_{\Omega,\mathfrak{p}} \circ \phi_{\Omega,\mathfrak{p}} \circ \alpha.$$

2) If G vanishes, the cokernel is not torsion.

3) We used the fact that $\mathfrak{M}$ is an $\mathcal{O}_\Omega$-module to tensor it with $\mathbb{D}$ over $\mathcal{O}_\Omega$. If we had tensored over $\mathbb{Z}_p$, i.e. calculated the length of the cokernel as a $\mathbb{Z}_p$-module, then we would have obtained an extra multiplicity $f = deg(\mathcal{O}_\Omega/\mathbb{Z}_p)$. The formulation here looks more natural. However we will need this multiplicity in a later application. In any case the cardinality of the cokernel is $p^{v_p(G)f}$.

Corollary 9.2. *a) If $p > 3dw + 2l + w + 1$, then*

$$\mathrm{rank}_{\mathbb{Z}_p} K_{2l+w+1}(M_\Omega \times_K K_{\mathfrak{p}}, \mathbb{Z}_p)^{(l+w+1)} \geq f = [T_\Omega : \mathbb{Q}_p].$$

b) If in addition $G(\chi, T_1, T_2)$ does not have a zero at $(u_1^{-a-l} - 1, u_2^{-b-l} - 1)$ then

$$\mathrm{rank}_{\mathbb{Z}_p} K_{2l+w+1}(M_\Omega, \mathbb{Z}_p)^{(l+w+1)} \geq f = [T_\Omega : \mathbb{Q}_p].$$

Proof: a) Consider the commutative diagram of the previous theorem. Then the claim follows from the fact that $\phi_{\Omega,\mathfrak{p}}$ is injective and that $(\mathcal{U} \otimes \mathfrak{M} \otimes \mathbb{Z}_p(l))_{\mathcal{G}}$ is a free $\mathcal{O}_\Omega$-module of rank 1.

b) Follows because α is injective with finite cokernel (so $(\mathcal{C} \otimes \mathfrak{M} \otimes \mathbb{Z}_p(l))_{\mathcal{G}}$ is a free $\mathcal{O}_\Omega$-module of rank 1) and ϕ is injective because α and $\phi_{\Omega,\mathfrak{p}}$ are injective. □

Remark: It is expected that G does not vanish and that we have equality in the corollary.

10. The case of an elliptic curve

We want to recover from the results of the previous sections the case of an elliptic curve E over a field F/K with complex multiplication by $\mathcal{O}_K$, i.e. $w = 1$.

In section 5 we constructed a decomposition $\mathcal{R}_{F/K} h_1(E) = \oplus_\Omega M_\Omega$ and proved the main theorem for the motives M_Ω. In the case where $w = 1$ we have $(a_\Omega, b_\Omega) = (1, 0)$ or $(0, 1)$. Furthermore we see

$$\bigoplus_\Omega K_{2l+2}(M_\Omega, \mathbb{Z}_p)^{(l+2)} = K_{2l+2}(h_1(E), \mathbb{Z}_p)^{(l+2)} \subseteq K_{2l+2}(E, \mathbb{Z}_p)$$

and

$$\bigoplus_\Omega H^1(K_{\mathfrak{p}}, \mathfrak{M}_\Omega(l+1)) = H^1\big(K_{\mathfrak{p}}, H^1(\bar{M}, \mathbb{Z}_p(1))(l)\big) = H^1(K_{\mathfrak{p}}, T_p A(l)),$$

which is a free $\mathcal{O}_T \otimes \mathbb{Z}_p$-module of rank 1 if $p - 1 \not| l + 1$ and $p - 1 \not| l + 2$.

By taking the direct sum of theorem 9.1 for all Ω and observing that a and b run through the values 0 and 1, we get the following:

Corollary 10.1. *Let E be an elliptic curve over F, $l > 0$, $p > 3d + 2l + 2$. Then there exists a submodule $\mathcal{V} \subseteq K_{2l+2}(h_1(E), \mathbb{Z}_p)^{(l+2)} \subseteq K_{2l+2}(E, \mathbb{Z}_p)$ such that the index of the regulator map $R|_{\mathcal{V}}$ restricted to $\mathcal{V}$ equals $p^{\sum v_\Omega f_\Omega}$, where v_Ω is the p-adic valuation of $G(\varphi_\Omega \kappa^l, u_1^{-a-l} - 1, u_2^{-b-l} - 1)$ and f_Ω is the degree of $\mathcal{O}_\Omega/\mathbb{Z}_p$.*

Let us discuss the decomposition $\mathcal{R}_{F/K} h_1(E) = \oplus_\Omega M_\Omega$ in more detail:

If $A = R_{F/K} E$ has complex multiplication by T, and T splits into a direct product $T = \prod T_e$, then each Ω corresponds to a fixed place $\mathfrak{P}$ above p of one of the fields T_e and M_Ω has multiplication by $\mathcal{O}_{T_e,\mathfrak{P}} = \mathcal{O}_\Omega$. On the other hand, we have the $2d = [F : \mathbb{Q}]$ p-adic grossencharacters

$$\varphi_\lambda : I_K \to I_T \to (T \otimes \mathbb{Q}_p)^* \xrightarrow{\lambda} \mathbb{C}_p^*.$$

If we restrict ourselves to the places above $\mathfrak{p}$ of T, then we get d characters inducing κ_1 on $Gal(F_\infty/F)$ and we similarly get d characters inducing κ_2 on $Gal(F_\infty/F)$ arising from the complex conjugate Hecke character $\bar{\varphi}$, see section 4.

Note that this is in analogy with the complex situation, see [11, par.4]: There we get d grossencharacters for each of the embeddings $T \otimes \mathbb{R} \xrightarrow{\epsilon} \mathbb{C}$ inducing the fixed embedding of K and d characters inducing the complex conjugate embedding.

In the previous sections we fixed an embedding λ for each factor $T_{e,\mathfrak{P}}$, but as noted after the main theorem, we obtain a multiplicity $f = [T_\Omega : \mathbb{Q}_p]$ to correct this. Observe also that two different φ_λ coming from the same T_Ω differ by an automorphism σ of $\mathbb{C}_p$. But we have $\int \sigma(\varphi_\lambda)d\mu = \sigma\big(\int \varphi_\lambda d\mu\big)$ and $v_p\big(\sigma(\int \varphi_\lambda d\mu)\big) = v_p\big(\int \varphi_\lambda d\mu\big)$. Thus the v_Ω agree for different choices of λ and the multiplicity f_Ω corresponds to the different choices of λ for a given Ω.

We see that $p^{\sum v_\Omega f_\Omega}$ equals $p^{\sum v_{\Omega,\lambda}}$, where $v_{\Omega,\lambda}$ now runs through all p-adic grossencharacters induced by φ.

Theorem 10.2. *Let E be an elliptic curve over F, $l > 0$, $p > 3d + 2l + 2$. Then there exists a submodule*

$$\mathcal{V} \subseteq K_{2l+2}(h_1(E), \mathbb{Z}_p)^{(l+2)} \subseteq K_{2l+2}(E, \mathbb{Z}_p)$$

such that the index of the regulator map $R|_{\mathcal{V}}$ restricted to $\mathcal{V}$ equals p^v, where v is the p-adic valuation of

$$\prod_\lambda G(\varphi_\lambda \kappa^l, u_1^{-1-l} - 1, u_2^{-l} - 1) G(\bar{\varphi}_\lambda \kappa^l, u_1^{-l} - 1, u_2^{-1-l} - 1).$$

The product runs over all $\mathbb{C}_p$-valued characters arising from φ inducing $\mathfrak{p}$ of K.

Remark: 1) The product of the p-adic L-series is a p-adic analog of

$$L(E, -l) = \prod_\epsilon L(\varphi_\epsilon, -l) \cdot L(\bar{\varphi}_\epsilon, -l).$$

2) As in the previous section we get the following partial result:

If $l = 0$, then the index of the regulator is greater than or equal to the p-adic valuation of the L-series.

Corollary 10.3. *a) We have for $p > 3d + 2l + 2$*

$$\operatorname{rank}_{\mathbb{Z}_p} K_{2l+2}(E \times_K K_\mathfrak{p}, \mathbb{Z}_p)^{(l+2)} = \sum_{\mathcal{P}|\mathfrak{p}} \operatorname{rank}_{\mathbb{Z}_p} K_{2l+2}(E \times_F F_\mathcal{P}, \mathbb{Z}_p)^{(l+2)} \geq 2d = [F : \mathbb{Q}]$$

b) If in addition none of the L-series in the theorem vanishes, we have

$$\operatorname{rank}_{\mathbb{Z}_p} K_{2l+2}(E, \mathbb{Z}_p)^{(l+2)} \geq 2d = [F : \mathbb{Q}].$$

References

[1] S.Bloch, A.Ogus, Gersten's conjecture and the homology of schemes, Ann. Sci. ENS 7 (1974), 181-202

[2] A.Borel, J.P.Serre, Le Théorème de Riemann-Roch, Bull. Soc. Math. France 86 (1958), 97-136

[3] A.K.Bousfield, D.M.Kan, Homotopy limits, completions and localizations, LNM 304 (1972)

[4] W.Browder, Algebraic K-theory with coefficients in $\mathbb{Z}/p$, LNM 657 (1977), 40-84

[5] C.Deninger, Higher regulators and Hecke L-series of imaginary quadratic fields I, Invent. Math. 96 (1989), 1-69

[6] C.Deninger, Higher regulators and Hecke L-series of imaginary quadratic fields II, Annals of Mathematics 132 (1990), 131-158

[7] W.Dwyer, E.Friedlander, Algebraic and étale K-theory, Transactions AMS 292, No.1 (1985)

[8] W.Fulton, S.Lang, Riemann-Roch-Algebra, Springer Grundlehren 277 (1985)

[9] T.Geisser, A p-adic analogue of Beilinson's conjectures for Hecke characters of imaginary quadratic fields, Schriftenreihe Math. Inst. Univ. Muenster 14 (1995)

[10] H.Gillet, Riemann-Roch theorems for higher algebraic K-theory, Adv. Math. 40, 203-289 (1981)

[11] C.Goldstein, N. Schappacher, Séries d'Eisenstein et fonctions L de courbes elliptiques à multiplication complexe, J. reine angew. Math. 327 (1981), 184-218

[12] D.Grayson, Adams operations on higher K-theory, K-theory 6 (1992), 97-111

[13] F.Hirzebruch, Topological Methods in Algebraic Geometry, Springer (1966)

[14] H.Hiller, λ-rings and algebraic K-theory, J. Pure and Appl. Algebra 20 (1981), 241-266

[15] J.L.Loday, K-théorie algébrique et répresentations des groupes, Ann. Sci. ENS 9 (1976), 309-377

[16] Y.Manin, Correspondences, motives and monoidal transformations, Mat. Sbor. 77, AMS Transl. (1979), 475-507

[17] D.Quillen, Higher algebraic K-theory I, LNM 341 (1973), 85-147

[18] N.Schappacher, Periods of Hecke characters, LNM 1301 (1980)

[19] E.de Shalit, Iwasawa theory of elliptic curves with complex multiplication, Perspectives in Mathematics Vol.3 (1987)

[20] C.Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, Invent. Math. 55 (1979), 251-295

[21] C.Soulé, Operations on étale K-theory, Applications, Lecture Notes in Mathematics 966 (1982), 271-303

[22] C.Soulé, Operations en K-theorie algèbrique, Can. J. Math. Vol.XXXVII No.3 (1985), 488-550

[23] C.Soulé, p-adic K-theory of elliptic curves, Duke Mathematical Journal Vol.54, No.1 (1987), 249-269

[24] G.Tamme, The Theorem of Riemann-Roch, in: Beilinson's conjectures on Special Values of L-functions, Perspectives in Mathematics Vol.4 (1987), 103-168

[25] R.W.Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. ENS 18 (1985), 437-552

[26] F.Waldhausen, Algebraic K-theory of generalized free products, Ann. of Math. 108 (1978), 135-256

Harvard Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138
Email address: geisser@math.harvard.edu