THE AFFINE PART OF THE PICARD SCHEME (CORRECTED).

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ABSTRACT. We describe the maximal torus and maximal unipotent subgroup of the Picard variety of a proper scheme over a perfect field. (This is a corrected and improved version of the article originally published in Comp. Math. 145 (2009)).

1. INTRODUCTION

For a proper scheme $p : X \to k$ over a perfect field, the Picard scheme Pic_X representing the functor $T \mapsto H^0(T_{\mathrm{et}}, R^1p_*\mathbb{G}_m)$ exists, and its connected component Pic^0_X is separated and of finite type [Mu64, II 15]. By Chevalley's structure theorem [Chev60], the reduced connected component $\mathrm{Pic}^{0,\mathrm{red}}_X$ is an extension of an abelian variety A_X by a linear algebraic group L_X:

$$0 \to L_X \to \mathrm{Pic}^{0,\mathrm{red}}_X \to A_X \to 0. \tag{1}$$

The commutative, smooth affine group scheme L_X is the direct product of a torus T_X and a unipotent group U_X. The following theorem completely characterizes T_X:

Theorem 1. *If X is proper over a perfect field, then the cocharactermodule $\mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T_X)$ of the maximal torus of Pic_X is isomorphic to $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z})$ as a Galois-module.*

To analyze the unipotent part, we let $\mathrm{Pic}(X[t])_{[1]}$ be the typical part, i.e. the subgroup of elements x of $\mathrm{Pic}(X[t])$ such that the map $X[t] \to X[t]$, $t \mapsto nt$ sends x to nx.

Theorem 2. *Let X be proper over a perfect field. Then $\mathrm{Pic}(X[t])_{[1]}$ is isomorphic to the group of morphisms of schemes $f : \mathbb{G}_a \to U_X$ satisfying $f(nx) = nf(x)$ for every $n \in \mathbb{Z}$. In particular, $\mathrm{Hom}_k(\mathbb{G}_a, U_X) \subseteq \mathrm{Pic}(X[t])_{[1]}$, and this is an equality in characteristic 0.*

To get another description of U_X, we assume that X is reduced (the map on the Picard scheme induced by the map $X^{\mathrm{red}} \to X$ is well understood by the work of Oort [Oort62]). The semi-normalization $X^+ \to X$ is the largest scheme between X and its normalization which is strongly universally homeomorphic to X in the sense that the map $X^+ \to X$ induces an isomorphism on all residue fields. A Theorem of Traverso [Tra70] implies that $\mathrm{Pic}(X[t])_{[1]}$, hence U_X, vanishes if X is reduced and seminormal. We use this to show

Theorem 3. *Let X be reduced and proper over a perfect field and let $X' \to X$ be a universal homeomorphism of proper k-schemes with X' seminormal.*

a) We have a short exact sequence

$$0 \to K \to \mathrm{Pic}^{0,\mathrm{red}}_X \to \mathrm{Pic}^{0,\mathrm{red}}_{X'} \to 0, \tag{2}$$

and inclusions of unipotent group schemes

$$U_X \subseteq K \subseteq p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$$

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with quotients finite unipotent group schemes.

b) The group scheme $p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$ *represents the functor*

$$T \mapsto \{\mathcal{O}_{X\times T}\text{-line bundles } \mathcal{L} \subseteq \mathcal{O}_{X'\times T} \text{ which are invertible in } \mathcal{O}_{X'\times T}\}.$$

Notation: For a field k, we denote by $\bar{k}$ its algebraic closure, and for a scheme X over k we let $\bar{X} = X \times_k \bar{k}$. Unless specified otherwise, all extension and homomorphism groups are considered on the fpqc site. [1]

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2. The torus

Proposition 4. *If* $p : X \to k$ *is reduced, geometrically connected, and proper over a perfect field, then* $\mathbb{G}_{m,k} \to p_*\mathbb{G}_{m,X}$ *is an isomorphism. Moreover, if* $f : X' \to X$ *is a universal homeomorphism and* X' *is reduced as well, then* f *induces an isomorphism* $p_*\mathbb{G}_{m,X} \cong p_*\mathbb{G}_{m,X'}$. [2]

Proof. Since any scheme T over k is flat, we have by flat base change $R^j q_*\mathcal{O}_{X_T} = H^j(X, \mathcal{O}_X) \otimes_k \mathcal{O}_T$, where $q : X_T \to T$ is the projection. In particular,

$$p_*\mathbb{G}_{m,X}(T) := \Gamma(X \times T, \mathcal{O}_{X\times T})^\times = (\Gamma(X, \mathcal{O}_X) \otimes \Gamma(T, \mathcal{O}_T))^\times,$$

and it suffices to show that $\Gamma(X, \mathcal{O}_X) \cong \Gamma(X', \mathcal{O}_{X'}) \cong k$. Since $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}})^{Gal(\bar{k}/k)} = \Gamma(X, \mathcal{O}_X)$, we can assume that k is algebraically closed and that X is connected, in which case the statement follows because X and X' are reduced, proper, connected, and have a k-rational point. □

Lemma 5. *For any scheme* X *we have isomorphisms*

$$H^1_{\text{et}}(X, \mathbb{Z}) \cong H^1_{\text{fl}}(X, \mathbb{Z}) \cong \operatorname{Ext}^1_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X}).$$

Proof. The first isomorphism is [Mi80, III Rem. 3.11(b)]. To prove the second isomorphism, we note that $\mathcal{H}\mathfrak{om}_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X}) \cong \mathbb{Z}_X$ by [SGA3, VIII Cor. 1.5], and that $\mathcal{E}\mathfrak{xt}^1_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X})$ is isomorphic to the group of extensions of group schemes [Oort66, Cor. 17.5], which vanishes by [SGA7, VIII Prop. 3.3.1].[3] Hence we obtain the isomorphism from the spectral sequence [Mi80, III Thm.1.22]

$$E_2^{s,t} = H^s_{\text{fl}}(X, \mathcal{E}\mathfrak{xt}^t_X(\mathbb{G}_{m,X}, \mathbb{G}_{mX})) \Rightarrow \operatorname{Ext}^{s+t}_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X}).$$

□

Proof. (Theorem 1) Since the maps defined below are natural, we can assume that k is algebraically closed and X is connected. We can also assume that X is reduced, because $H^1_{\text{et}}(X, \mathbb{Z}) \xrightarrow{\sim} H^1_{\text{et}}(X^{\text{red}}, \mathbb{Z})$, and the map $\operatorname{Pic}_X \to \operatorname{Pic}_{X^{\text{red}}}$ has unipotent kernel and cokernel [Oort62, Cor. page 9]. It suffices to calculate $\operatorname{Hom}_k(\mathbb{G}_{m,k}, \operatorname{Pic}_X)$, because there are no homomorphisms from $\mathbb{G}_m$ to

[1] In [Gei09] we used the étale topology

[2] This replaces [Gei09, Prop. 9 a)] which is incorrect as stated because the induction step in the proof does not preserve the hypothesis on reducedness.

[3] This was claimed without proof in [Gei09].

commutative group schemes other than tori [Oort66, p. 81]. By Yoneda's Lemma, the latter group is isomorphic to the group of homomorphisms of sheaves on the fpqc site $\mathrm{Hom}_k(\mathbb{G}_{m,k}, R^1p_*\mathbb{G}_{m,X})$. The Leray spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_k(\mathbb{G}_{m,k}, R^tp_*\mathbb{G}_{m,X}) \Rightarrow \mathrm{Ext}^{s+t}_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X}). \tag{3}$$

gives an exact sequence

$$0 \to \mathrm{Ext}^1_k(\mathbb{G}_{m,k}, p_*\mathbb{G}_{mX}) \to \mathrm{Ext}^1_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X}) \to \mathrm{Hom}_k(\mathbb{G}_{m,k}, R^1p_*\mathbb{G}_{m,X}) \xrightarrow{\delta_X} \mathrm{Ext}^2_k(\mathbb{G}_{m,k}, p_*\mathbb{G}_{m,X}).$$

By Proposition 4 the left term agrees with $\mathrm{Ext}^1_k(\mathbb{G}_{m,k}, \mathbb{G}_{m,k})$, and this vanishes by [Oort66, Cor. 17.5]. Thus it suffices to show that δ_X is the zero map [4]. Choose a closed point of X and let $i : Z \to X$ be the corresponding closed subscheme. Since $p_* \circ i_* = \mathrm{id}$ we have $R^sp_*i_* = R^s(p\circ i)_* = 0$ for $s > 0$. Hence we obtain a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_k(\mathbb{G}_{m,k}, R^1p_*\mathbb{G}_{m,X}) & \longrightarrow & \mathrm{Hom}_k(\mathbb{G}_{m,k}, R^1p_*i_*\mathbb{G}_{m,Z}) = 0 \\
\delta_X \downarrow & & \delta_Z \downarrow \\
\mathrm{Ext}^2_k(\mathbb{G}_{m,k}, p_*\mathbb{G}_{m,X}) & \xrightarrow{\sim} & \mathrm{Ext}^2_k(\mathbb{G}_{m,k}, \mathbb{G}_m).
\end{array}$$

By Proposition 4, the lower horizontal map is an isomorphism. □

Remark. The example in [Gei06, Prop. 8.2] shows that the map $H^i_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \to \mathrm{Ext}^i_{\bar{X}}(\mathbb{G}_m, \mathbb{G}_m)$ is not an isomorphism for $i \geq 2$. One can ask if it is an isomorphism if one replaces $H^i_{\mathrm{et}}(\bar{X}, \mathbb{Z})$ by the eh-cohomology group $H^i_{eh}(\bar{X}, \mathbb{Z})$ of [Gei06].

Example. If X is the node over an algebraically closed field, then $H^1_{\mathrm{et}}(X, \mathbb{Z}) \cong \mathbb{Z}$, and $T_X \cong \mathbb{G}_m$. Let X be a node with non-rational tangent slopes at the singular point. Base changing to the algebraic closure, one sees that $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \cong \mathbb{Z}$, with Galois group acting as multiplication by -1, hence T_X is an anisotropic torus.

Using the theorem, we are able to recover the torsion of T_X, A_X and the diagonalizable part of $NS_X := \mathrm{Pic}_X / \mathrm{Pic}^{0,\mathrm{red}}_X$ in terms of etale cohomology:

Corollary 6. *Let X be proper over a perfect field k. Then we have canonical isomorphisms*

$$\begin{aligned}
H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} &\cong \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, T_X);\\
\mathrm{Div}({}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar{X}, \mathbb{Z})) &\cong \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, A_X);\\
{}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar{X}, \mathbb{Z})/\,\mathrm{Div} &\cong \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, NS_X).
\end{aligned}$$

Proof. Taking the colimit of the isomorphism $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}/m) \cong \mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X)$ of [Mi80, Prop.4.16] or [Ray70, §6.2], we obtain $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X)$. Since $\mathrm{Ext}^1_{\bar{k}}(\mathbb{G}_m, T_X) = 0$, Theorem 1 implies that $\mathrm{Hom}_{\bar{k}}(\mu_m, T_X) \cong \mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T_X)/m \cong H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z})/m$. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, T_X) & = & H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} & & \\
\downarrow & & \downarrow & & \\
\mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}^{0,\mathrm{red}}_X) & \longrightarrow & \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X) & \longrightarrow & \mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, NS_X) \\
\downarrow & & \downarrow & & \| \\
\mathrm{colim}\, \mathrm{Hom}_{\bar{k}}(\mu_m, A_X) & \xrightarrow{f} & {}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar{X}, \mathbb{Z}) & \longrightarrow & \mathrm{coker}\, f.
\end{array}$$

The middle column is the short exact coefficient sequence. The left column and middle row are short exact because $\mathrm{Ext}^1_{\bar{k}}(\mu_m, T_X) = \mathrm{Ext}^1_{\bar{k}}(\mu_m, \mathrm{Pic}^{0,\mathrm{red}}_X) = 0$ by [Oort66, Cor. 17.5, II 14.2]. A

[4] The remainder of the proof is a simplification suggested by O. Gabber.

diagram chase shows that f is injective, and the right vertical map is an isomorphism. The Corollary follows because $\operatorname{colim}\operatorname{Hom}_{\bar k}(\mu_m, A_X)$ is divisible and $\operatorname{colim}\operatorname{Hom}_{\bar k}(\mu_m, NS_X)$ is finite. □

The above result should be compared to [Gei10, Prop.6.2], where we show that, for every proper scheme over an algebraically closed field, the higher Chow group of zero-cycles $CH_0(X,1,\mathbb{Z}/m)$ is the Pontrjagin dual of $H^1_{\text{et}}(X,\mathbb{Z}/m)$. This implies a short exact sequence

$$0 \to {}_{\text{tor}}A^t_X(k) \to CH_0(X,1,\mathbb{Q}/\mathbb{Z}) \to \chi(T_X)\otimes\mathbb{Q}/\mathbb{Z} \to 0,$$

for A^t_X the dual abelian variety of A_X, and $\chi(T_X)$ the character module of T_X. However, in this case the contribution from the torus and from the abelian variety are not compatible with the coefficient sequence

$$0 \to CH_0(X,1)\otimes\mathbb{Q}/\mathbb{Z} \to CH_0(X,1,\mathbb{Q}/\mathbb{Z}) \to {}_{\text{tor}}CH_0(X) \to 0$$

as in Corollary 6.

Looking at tangent spaces, the previous Corollary gives a dimension formula:

Corollary 7. *Let l be a prime different from* $\operatorname{char} k$*. Then*

$$\dim_k H^1(X,\mathcal{O}_X) = \dim U_X + \dim_k Lie(NS^0_X) + \operatorname{rank} H^1_{\text{et}}(X,\mathbb{Z}) + \tfrac{1}{2}\operatorname{corank}_l H^1_{\text{et}}(\bar X,\mathbb{Q}_l/\mathbb{Z}_l).$$

3. The unipotent part

Let $N\operatorname{Pic}(X) := \ker\big(\operatorname{Pic}(X[t]) \xrightarrow{0^*} \operatorname{Pic}(X)\big)$. Since $t\mapsto 0t$ induces $x\mapsto 0x$ on the typical part, $\operatorname{Pic}(X[t])_{[1]}$ is a subgroup of $N\operatorname{Pic}(X)$. In [Wei91], Weibel shows that for every scheme there is a direct sum decomposition

$$\operatorname{Pic}(X[t,t^{-1}]) \cong \operatorname{Pic}(X)\oplus N\operatorname{Pic}(X)\oplus N\operatorname{Pic}(X)\oplus H^1_{\text{et}}(X,\mathbb{Z}).$$

Proof. (Theorem 2). We show first that $N\operatorname{Pic}(X) = \ker\big(U_X(\mathbb{A}^1)\to U_X(k)\big)$. Since there are no non-trivial morphisms of schemes from $\mathbb{A}^1_k$ to an abelian variety, a torus, an infinitesimal group, or a discrete group, we see that the kernel of $U_X(\mathbb{A}^1_k)\to U_X(k)$ agrees with the kernel of $\operatorname{Pic}_X(\mathbb{A}^1_k)\to\operatorname{Pic}_X(k)$. Let $p: X\to k$ and $p': X\times\mathbb{A}^1_k\to\mathbb{A}^1_k$ be the structure morphisms. Then the Leray spectral sequence gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\text{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m) & \longrightarrow & \operatorname{Pic}(X\times\mathbb{A}^1_k) & \longrightarrow & \operatorname{Pic}_X(\mathbb{A}^1_k) & \longrightarrow & H^2_{\text{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m)\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & H^1_{\text{et}}(k, p_*\mathbb{G}_m) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & \operatorname{Pic}_X(k) & \longrightarrow & H^2_{\text{et}}(k, p_*\mathbb{G}_m),
\end{array}$$

and it suffices to show that the outer vertical maps are isomorphisms. Let $X\xrightarrow{g} L\to k$ be the Stein factorization of p, such that $\mathcal{O}_L\cong g_*\mathcal{O}_X$ and L is the spectrum of an Artinian k-algebra. Since $\mathbb{A}^1_k\to k$ is flat, $p'_*\mathcal{O}_{X\times\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}\otimes_k p_*\mathcal{O}_X$, and $X\times\mathbb{A}^1_k \xrightarrow{g'} \mathbb{A}^1_L \to \mathbb{A}^1_k$ is the Stein factorization of p'. We obtain

$$H^i_{\text{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m)\cong H^i_{\text{et}}(\mathbb{A}^1_L, g'_*\mathbb{G}_m)\cong H^i_{\text{et}}(\mathbb{A}^1_L,\mathbb{G}_m),$$

and $H^i_{\text{et}}(k,p_*\mathbb{G}_m)\cong H^i_{\text{et}}(L,\mathbb{G}_m)$. Hence the terms on the left vanish because $\operatorname{Pic}(L)=\operatorname{Pic}(\mathbb{A}^1_L)=0$. To show that $H^2_{\text{et}}(\mathbb{A}^1_L,\mathbb{G}_m)\to H^2_{\text{et}}(L,\mathbb{G}_m)$ is an isomorphism, we can assume that L is a local Artinian k-algebra with (perfect) residue field k'. By [Mi80, III Rem.3.11] we are reduced to showing that $H^2_{\text{et}}(\mathbb{A}^1_{k'},\mathbb{G}_m)\to H^2_{\text{et}}(k',\mathbb{G}_m)$ is an isomorphism, and this can be found in [Mi80, IV Ex.2.20].

Given an element x of $N\operatorname{Pic}(X)$, the condition $x\in\operatorname{Pic}(X[t])_{[1]}$ implies that the corresponding $f\in\operatorname{Hom}_{Sch}(\mathbb{A}^1,U_X)$ satisfies $f(nx)=nf(x)$ for all n. If k has characteristic 0, then $U_X\cong\mathbb{G}_a^r$ for some

r, and the map $f : \mathbb{G}_a \to U_X$ corresponds to a morphism of Hopf algebras $f^* : k[x_1, \cdots, x_r] \to k[t]$. If $f^*(x_i) = \sum_j a_j t^j$, then

$$\sum_j a_j (nt)^j = nf^*(x_i) = f^*(nx_i) = n \sum_j a_j t^j$$

only if $n^j = n$ for all n, hence $j = 1$. □

Example. If k has characteristic p, then $t \mapsto t^{2p-1}$ induces a map $\mathbb{G}_a \to \mathbb{G}_a$ which is compatible with multiplication by n, but not a homomorphism of group schemes.

Corollary 8. *We have $U_X = 0$ if and only if $N\operatorname{Pic}(X) = 0$.*

Proof. This follows from $N\operatorname{Pic}(X) = \ker\big(U_X(\mathbb{A}^1) \to U_X(k)\big)$, because any unipotent, connected, smooth affine group is an affine space as a scheme, hence admits a non-trivial morphism from $\mathbb{A}^1$ which sends 0 to 0 if it is non-trivial. □

The kernel and cokernel of $\operatorname{Pic}_X \to \operatorname{Pic}_{X^{\mathrm{red}}}$ has been described in [Oort62], hence we will from now assume that X is reduced. If X^+ is the semi-normalization of X, then the map $\mathcal{O}_X \to \mathcal{O}_{X^+}$ is an injection of sheaves on the same topological space. For X^+ reduced and semi-normal, $N\operatorname{Pic}(X^+) = 0$ by Traverso's theorem [Tra70] together with [Wei91, Thm. 4.7]. Hence the Corollary implies that $U_{X^+} = 0$, and that

$$U_X = \ker(\operatorname{Pic}^{0,\mathrm{red}}_X \to \operatorname{Pic}^{0,\mathrm{red}}_{X^+}).$$

(For curves, this recovers [BLR90, Prop.9.2/10].) Indeed, by Corollary 6, the map $\operatorname{Pic}^{0,\mathrm{red}}_X \to \operatorname{Pic}^{0,\mathrm{red}}_{X^+}$ induces an isomorphism on the torus and abelian variety part, because it induces an isomorphism on etale cohomology.

Lemma 9. *A p-primary finite group scheme F over an algebraically closed field k with $\operatorname{Hom}_k(\mu_p, F) = 0$ is unipotent.*

Proof. It suffices to show that the dual of F is connected. Since F is p-primary, its étale component has connected dual. If the connected-étale component V was non-trivial, then there would be a non-trivial homomorphism $V^D \to \mathbb{Z}/p$, or equivalently there would be a non-trivial homomorphism $\mu_p \to V$, a contradiction. □

Proof. (Theorem 3) a) The isomorphim $H^i_{\mathrm{et}}(X, \mathbb{Z}) \cong H^i_{\mathrm{et}}(X', \mathbb{Z})$ combined with Corollary 6 and the Lemma shows that the canonical map $f^* : \operatorname{Pic}^{0,\mathrm{red}}_X \to \operatorname{Pic}^{0,\mathrm{red}}_{X'}$ induces an isomorphism on the torus components, and is an isogeny with kernel a unipotent group scheme P on the abelian variety parts.[5] Hence f^* is surjective with unipotent kernel K, is an extension of P by U_X. Applying Proposition 4 to the exact sequence of etale sheaves

$$0 \to p_*\mathbb{G}_{m,X} \to p_*\mathbb{G}_{m,X'} \to p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) \to \operatorname{Pic}_X \to \operatorname{Pic}_{X'}$$

[5] O.Gabber [Gab20] showed that, conversely, any finite unipotent commutative group scheme can appear as P_X for $X' = X^+$.

on Spec k, we obtain the diagram with exact columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{Pic}^{0,\mathrm{red}}_X & \longrightarrow & \mathrm{Pic}^{0,\mathrm{red}}_{X'} & \longrightarrow & 0 \\
 & & {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow & & \\
0 & \longrightarrow & p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) & \longrightarrow & \mathrm{Pic}_X & \longrightarrow & \mathrm{Pic}_{X'} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{coker}\, u & \longrightarrow & \mathrm{NS}_X & \longrightarrow & \mathrm{NS}_{X'}\,. & &
\end{array}$$

Since v is injective, so is u. The Neron-Severi group schemes are extensions of finitely generated étale group schemes $\mathrm{NS}^{\mathrm{et}}_X$ and $\mathrm{NS}^{\mathrm{et}}_{X'}$ by a finite connected group schemes NS^0_X and $\mathrm{NS}^0_{X'}$, respectively. The isomorphism $H^2_{\mathrm{et}}(X,\mu_m) \cong H^2_{\mathrm{et}}(X',\mu_m)$ implies that $\mathrm{Pic}_X(\bar{k})/m \to \mathrm{Pic}_{X^+}(\bar{k})/m$ is injective for any m prime to p, and since $\mathrm{Pic}^{0,\mathrm{red}}_X(\bar{k})$ and $\mathrm{Pic}^{0,\mathrm{red}}_{X^+}(\bar{k})$ are m-divisible, the same holds for $\mathrm{NS}_X(\bar{k})/m \to \mathrm{NS}_{X^+}(\bar{k})/m$. Consequently the kernel of $\mathrm{NS}^{\mathrm{et}}_X \to \mathrm{NS}^{\mathrm{et}}_{X'}$, and hence $\mathrm{coker}\, u$ is a finite p-primary group.

It remains to prove unipotence of $\mathrm{coker}\, u$. From $H^i_{\mathrm{et}}(X,\mathbb{Z}) \cong H^i_{\mathrm{et}}(X',\mathbb{Z})$ we conclude that $H^1_{\mathrm{et}}(X,\mathbb{Z}/p) \cong H^1_{\mathrm{et}}(X',\mathbb{Z}/p)$, or equivalently $\mathrm{Hom}_{\bar{k}}(\mu_p,\mathrm{Pic}_X) \cong \mathrm{Hom}_{\bar{k}}(\mu_p,\mathrm{Pic}_{X'})$, and hence we obtain $\mathrm{Hom}_{\bar{k}}(\mu_p, p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})) = 0$. Since there are no non-trivial commutative extensions of the group of multiplicative type μ_p by unipotent groups, we see that $\mathrm{Hom}_{\bar{k}}(\mu_p,\mathrm{coker}\, u) = 0$ and we can conclude by the Lemma.

b) Recall that $q : X_T \to T$, and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X\times T}) & \longrightarrow & \mathrm{Pic}(X\times T) & \longrightarrow & \mathrm{Pic}_{X/k}(T) & \longrightarrow & H^2_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X\times T}) \\
 & & \downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X^+\times T}) & \longrightarrow & \mathrm{Pic}(X^+\times T) & \longrightarrow & \mathrm{Pic}_{X^+/k}(T) & \longrightarrow & H^2_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X^+\times T}).
\end{array}$$

Since $q_*\mathbb{G}_{a,X\times T} = H^0(X,\mathcal{O}_X)\otimes\mathcal{O}_T = H^0(X^+,\mathcal{O}_{X^+})\otimes\mathcal{O}_T = q_*\mathbb{G}_{a,X^+\times T}$ is an isomorphism as in Proposition 4a), the outer maps are isomorphisms, and it suffices to calculate $\ker r$. Let $Y = X\times T$ and $Y' = X^+ \times T$, and consider the tautological map

$$f : \{\mathcal{O}_Y\text{-line bundles } \mathcal{L} \subseteq \mathcal{O}_{Y'} \text{ which are invertible in } \mathcal{O}_{Y'}\} \to \mathrm{Pic}(Y).$$

It suffices to show the following statements:

a) The image of f is contained in $\ker\big(\mathrm{Pic}(Y) \to \mathrm{Pic}(Y')\big)$.
b) f surjects onto $\ker\big(\mathrm{Pic}(Y) \to \mathrm{Pic}(Y')\big)$.
c) f is injective.

a) We claim that the map $\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'} \to \mathcal{O}_{Y'}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'} \xrightarrow{\mu} \mathcal{O}_{Y'}$ is an isomorphism. We can check this on an affine covering, and in this case it is proved in [RS93, Lemma 2.2(4)].

b) Let $\mathcal{L} \in \mathrm{Pic}(Y)$ with $\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'} \cong \mathcal{O}_{Y'}$. Since $\mathcal{L}$ is flat, we get an injection $\mathcal{L} = \mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y \to \mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'} \cong \mathcal{O}_{Y'}$. We claim that the inverse of $\mathcal{L}$ in $\mathcal{O}_{Y'}$ is the sheaf associated to the presheaf $U \mapsto \{x \in \mathcal{O}_{Y'}(U) | x\mathcal{L}(U) \subseteq \mathcal{O}_Y(U)\} \subseteq \mathcal{O}_{Y'}(U)$. This can be checked on an affine covering, and then it is [RS93, Lemma 2.2(2)].

c) Let $\mathcal{L}$ and $\mathcal{L}'$ be subsheaves of $\mathcal{O}_{Y'}$ which are invertible in $\mathcal{O}_{Y'}$ and isomorphic as abstract invertible sheaves. Multiplying with the inverse of $\mathcal{L}'$ inside $\mathcal{O}_{Y'}$, it suffices to show that if $\mathcal{L}$ is a subsheaf of $\mathcal{O}_{Y'}$, and $f : \mathcal{O}_Y \to \mathcal{L}$ an isomorphism, then $\mathcal{L} = \mathcal{O}_Y \subseteq \mathcal{O}_{Y'}$. But $f(1)$ is a global unit of $\mathcal{O}_{Y'}(Y)$, and by Proposition 4a), $\mathcal{O}_Y(Y)^\times = \mathcal{O}_{Y'}(Y)^\times$. Hence $\mathcal{L} = f(1)^{-1}\mathcal{L} = \mathcal{O}_Y$. □

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