Phantom behaviour and growth index anomalous evolution in viable $f(R)$ gravity models

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Abstract

We present numerical calculation of the evolution of a background space-time metric and sub-horizon matter density perturbations in viable $f(R)$ gravity models of present dark energy and cosmic acceleration. We found that viable models generically exhibit recent crossing of the phantom boundary $w_{DE} = -1$. Moreover, as a consequence of the anomalous growth of density perturbations during the end of the matter-dominated stage, their growth index evolves non-monotonically with time and may even become negative.

1 Introduction

It is one of the most important issues for cosmologists and particle physicists to understand the physical origin of the dark energy (DE) which is responsible for an accelerated expansion of the current Universe. Although the standard spatially flat Λ-Cold-Dark-Matter (ΛCDM) model is consistent with all kinds of current observational data [1], some tentative deviations from it have been reported recently [2, 3]. Furthermore, in the ΛCDM model, the cosmological term is regarded as a new fundamental constant whose observed value is much smaller than any other energy scale known in physics. Hence it is natural to seek for non-stationary models of the current DE. Among them, $f(R)$ gravity which modifies and generalizes the Einstein gravity by incorporating a new phenomenological function of the Ricci scalar $R$, $f(R)$, can provide a self-consistent and non-trivial alternative to the ΛCDM model[4, 5].

In the previous paper [6], we calculated evolution of matter density fluctuations in viable $f(R)$ models [4, 5] for redshifts $z \gg 1$ during the matter-dominated stage and found an analytic expression for them. In this paper we extend the previous analysis and perform numerical calculations of the evolution of both background space-time and density fluctuations for the particular $f(R)$ model of Ref. [5] without such a restriction. As a result, we have found crossing of the phantom boundary $w_{DE} = -1$ at an intermediate redshift $z \lesssim 1$ for the background space-time metric and an anomalous behavior of the growth index of fluctuations.

2 Background

We adopt the following action of $f(R)$ models with model parameters $n$, $\lambda$ and $R_s$ [5]:

$$
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_m, \quad f(R) = R + \lambda R_s \left[ \left( 1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right],
$$

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where $S_m$ is the action of the matter content which is assumed to be minimally coupled to gravity. To make the late-time asymptotic de Sitter regime where $R = $ constant stable, $\lambda$ has to satisfy $f'(R) > Rf''(R)$.

As a result, $\lambda$ has a lower limit $\lambda_{min}$ for each $n$. Numerically we find $(n, \lambda_{min}) = (2, 0.9440), (3, 0.7259),$ and $(4, 0.6081)$. From the action (1), we obtain field equations as

\[ R^{\mu}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} R = -8\pi G \left( T^{\mu}_{\nu(m)} + T^{\mu}_{\nu(\text{DE})} \right), \]

\[ 8\pi GT^{\mu}_{\nu(\text{DE})} \equiv (F - 1)R^{\mu}_{\nu} - \frac{1}{2}(f - R)\delta^{\mu}_{\nu} + (\nabla^{\mu}\nabla_{\nu} - \delta^{\mu}_{\nu}\Box)F, \quad F(R) \equiv f'(R). \]

Working in the spatially flat Friedmann-Robertson-Walker (FRW) space-time with a scale factor $a(t)$,

\[ 3H^2 = 8\pi G\rho - 3(F - 1)H^2 + \frac{1}{2}(FR - f) - 3HF, \]

\[ 2\dot{H} = -8\pi G\rho - 2(F - 1)\dot{H} - \dot{F} + H\ddot{F}, \]

where $H$ is the Hubble parameter and $\rho$ is the energy density of the material content which we assume to consist of non-relativistic matter. From (1), we can determine the DE equation of state parameter $w_{\text{DE}}$,

\[ w_{\text{DE}} \equiv \frac{p_{\text{DE}}}{\rho_{\text{DE}}} = -1 + \frac{2\dot{H}(F - 1) - H\ddot{F} + \dot{F}}{-3HRF' + 3(H^2 + H)(F - 1) - (f - R)/2}. \]

We solve the evolution equation (5) numerically using (4) to check numerical accuracy. The moment when the matter density parameter $\Omega(t) = 16\pi G\rho/(16\pi G\rho + \Lambda R_s)$ equals to 0.998 is chosen as the initial time $t_i$. We determine the current epoch $t = t_0$ by the requirement that the value of $\Omega$ takes the observed central value $\Omega_0 = 0.27$. $R_s$ is fixed in such a way as to reproduce the current Hubble parameter $H_0 = 72$km/s/Mpc. We find the ratio $R_s/H_0^2$ is well fit by a simple power-law $R_s/H_0^2 = c_n\lambda^{-p_n}$ with $(n, c_n, p_n) = (2, 4.16, 0.953), (3, 4.12, 0.837),$ and $(4, 4.74, 0.702),$ respectively, whereas in the $\Lambda \text{CDM}$ limit it would behave as $R_s/H_0^2 = 6(1 - \Omega_0)/\lambda \simeq 4.38\lambda^{-1}$.

Figures 1 depict evolution of $w_{\text{DE}}$ as a function of redshift $z$ where phantom crossing is manifest. As expected, it approaches $w_{\text{DE}} = -1$ as constant as we increase $\lambda$ for fixed $n$. For the minimal allowed values of $\lambda$, deviations from $w_{\text{DE}} = -1$ are observed at 5% level on both directions in $z \lesssim 2$ independently of $n$. From (6), we can read off that this phantom crossing behavior is not peculiar to the specific choice of the function (1) but a generic one for models which satisfy the stability condition $F' > 0$.

### 3 Perturbations

We now turn to the evolution of density fluctuations. In $f(R)$ gravity, the evolution equation of density fluctuations, $\delta$, deeply in the sub-horizon regime is given by[7]

\[ \ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho \delta = 0, \quad G_{\text{eff}} = \frac{G}{1 + 4\frac{\xi R}{F} F'}. \]
In the previous paper\cite{6} we obtained an analytic solution for the high-curvature regime when the scale factor evolves as $a(t) \propto t^{2/3}$ and $F$ takes an asymptotic form $F \simeq 1 - 2n\lambda (R/R_0)^{-2n-1}$. In the present paper, we numerically integrate (7) up to $z = 0$ without using the approximation $|F - 1| \ll 1$.

The wavenumber of our particular interest is the scale corresponding to $\sigma_8$ normalization, for which we find $k_{\text{eff}}(r = 8h^{-1}\text{Mpc}) = 0.174h\text{Mpc}^{-1}$. Since the standard ΛCDM model normalized by CMB data explains galaxy clustering at small scales well, $\delta_{\text{RG}}$ should not be too much larger than $\delta_{\text{CDM}}$ on these scales. We may typically require $(\delta_{\text{RG}}/\delta_{\text{CDM}})^2 \lesssim 1.1$. Although we neglect non-linear effects here, the difference between linear calculation and non-linear N-body simulation remained smaller than 5\% at wavenumber 0.174hMpc$^{-1}$\cite{8}.

The left panel of Fig. 2 represents $(\delta_{\text{RG}}/\delta_{\text{CDM}})^2$ as a function of $\lambda$ for $n = 2$ together with two fitting functions. The solid line is from the analytic formula obtained in Ref. \cite{6}, and the broken line is numerical fitting using an exponential function $1 + b_ne^{-\gamma\lambda}$. From these analysis, we can constrain the parameter space as the right panel of Fig. 2. The region which satisfy $(\delta_{\text{RG}}/\delta_{\text{CDM}})^2 < 1.1$ lies above the solid line. The region below the dotted line is forbidden from instability of de Sitter regime.

Next we turn to another important quantity used to distinguish different theories of gravity, namely, the gravitational growth index, $\gamma(z)$, of density fluctuations. It is defined through

$$\frac{d \ln \delta}{d \ln a} = \Omega_m(z)^{\gamma(z)}, \quad \text{or} \quad \gamma(z) = \frac{\log \left( \frac{\delta}{\delta_0} \right)}{\log \Omega_m}. \tag{8}$$

It takes a practically constant value $\gamma \cong 0.55$ in the standard ΛCDM model, but it evolves with time in modified gravity theories in general. We also note that $\gamma(z)$ has a nontrivial $k$-dependence in $f(R)$ gravity since density fluctuations with different wavenumbers evolve differently. Therefore, this quantity is a useful measure to distinguish modified gravity from ΛCDM model in the Einstein gravity.

Figures 3 show the evolution of $\gamma(z)$ together with that of $G_{\text{eff}}/G$ for different values of $k$. In the early high-redshift regime, $\gamma(z)$ takes a constant value identical to the ΛCDM model. It gradually decreases
in time, reaches a minimum which may be even negative, and then increase again towards the present epoch. We note that recently Narikawa and Yamamoto[9] numerically calculated time evolution of $\gamma(z)$ in a simplified model which we had used in the previous paper and also obtained some analytic expansion, which behaves qualitatively the same as the numerical result but with much more exaggerated amplitudes. Our results, which satisfy all viability conditions, exhibit milder deviation from $\Lambda$CDM model than they found. At present, the constraints for the growth index is not so strict to distinguish the deviation from the $\Lambda$CDM model[10], but observations may reveal its time and wave number dependence in future.

4 Conclusion

In the present paper we have numerically calculated the evolution of both homogeneous background and density fluctuations in a viable $f(R)$ model of accelerated expansion based on the specific functional form proposed in Ref. [5]. We have found that viable $f(R)$ gravity models of accelerated expansion generically exhibit phantom behavior during the matter-dominated stage with crossing of the phantom boundary $w_{DE} = -1$ at redshifts $z \lesssim 1$. The predicted time evolution of $w_{DE}$ has qualitatively the same behaviour as that was recently obtained from observational data in [2].

As for density fluctuations, we have numerically confirmed our previous analytic results on a shift in the power spectrum index for large wavenumbers which exceed the scalaron mass during the matter dominated epoch[6], while for smaller wavenumbers fluctuations have the same amplitude as in the $\Lambda$CDM model.

We have also investigated the growth index $\gamma(k, z)$ of density fluctuations and have given an explanation of its anomalous evolution in terms of time dependence of $G_{\text{eff}}$. Since $\gamma$ has characteristic time and wavenumber dependence, future detailed observations may yield useful information on the validity of $f(R)$ gravity through this quantity, although current constraints have been obtained assuming that it is constant both in time and in wavenumber[3, 10].

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