Black Hole Magnetosphere for Two-fluid Flows

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Abstract
A numerical approach to construct a black hole magnetosphere is given. The stationary axisymmetric structures of electromagnetic fields and plasma flows in Schwarzschild spacetime are assumed. The charge density and current as the source terms of the Maxwell’s equations are calculated by solving motions of positively and negatively charged particles for which electromagnetic forces are determined by global electromagnetic fields. The two-fluid approach without using an ideal MHD condition is discussed.

1 Introduction
Magnetosphere around a black hole has been studied for more than three decades. See e.g.,[1, 2] for a review. The energy extraction from central supermassive black hole may be important as a model of AGNs[3]. The numerical construction of global electromagnetic structure is not easy even if stationary and axially symmetries are assumed. Normally, an ideal MHD condition is also assumed everywhere. There are five constants along the field line, and the global structure of magnetic flux function \( G(r, \theta) \) is determined by the trans-field equation, so-called the Grad-Shafranov equation. It is not easy to obtain a consistent solution, since the equation is highly non-linear and contains unknown functionals, \( \Phi(G), S(G) \).

For example, a poor choice of the functionals leads to a singularity of the function \( G(r, \theta) \). It is therefore necessary to determine \( G(r, \theta) \) and \((\Phi(G), S(G))\). We here consider the problem in term of two-fluid approximation, by relaxing the ideal MHD condition. The method is applied to construction of pulsar magnetosphere[4].

2 Formalism
We consider a stationary, axially symmetric electromagnetic field in the Schwarzschild metric,

\[
ds^2 = -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 d\theta^2 + (r \sin \theta)^2 d\phi^2,
\]

where \( \alpha = 1 - 2M/r \) and \( M \) is the mass of the black hole. It is not difficult to extend the Kerr metric, which is necessary to examine the Blandford-Znajek process[3]. The 3+1 split formalism[5] is used in this paper, and electromagnetic fields \((\vec{E}, \vec{B})\), charge density \( \rho_c \), and current \( \vec{j} \) mean the quantities measured by fiducial observer at each point. Electromagnetic fields are generally expressed by three scalar functions \( \Phi(r, \theta), G(r, \theta), S(r, \theta) \) as

\[
\begin{align*}
\vec{E} &= -\frac{1}{\alpha} \vec{\nabla} \Phi, \\
\vec{B} &= \frac{1}{r \sin \theta} \vec{\nabla} \times \vec{e}_\phi + \frac{S}{\alpha r \sin \theta} \vec{e}_\phi,
\end{align*}
\]

where \( \vec{e}_\phi \) is the azimuthal unit vector. A set of Maxwell equations in stationary, axisymmetric case

\[
\nabla \cdot \vec{E} = 4\pi \rho_c, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times (\alpha \vec{E}) = 0, \quad \nabla \times (\alpha \vec{B}) = 4\pi \vec{j}
\]

are reduced to

\[
\frac{\alpha}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2 \partial \Phi}{\partial r} \right) + \frac{1}{\alpha r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -4\pi \rho_c,
\]
\[
\frac{\partial}{\partial r} \left( \alpha^2 \frac{\partial G}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial G}{\partial \theta} \right) = -4\pi r \sin \theta j_\phi,
\]
and
\[
\frac{1}{r \sin \theta} \nabla S \times \vec{e}_\phi = 4\pi \alpha \vec{j}_p.
\]
The ideal MHD condition, \( \vec{E} \cdot \vec{B} = 0 \) leads to \( \Phi = \Phi(G) \). The electric current should be along the magnetic field line, so that we have \( S = S(G) \). These functional relations are used in the MHD approach, but are not assumed here.

We adopt a treatment in which the plasma is modeled as a two-component fluid. Each component, consisting of positively or negatively charged particles, is described by a number density \( n_\pm \) and velocity \( \vec{v}_\pm \). We assume that the positive particle has mass \( m \) and charge \( q \), while the negative one has mass \( m \) and charge \( -q \). The charge density and current are given as
\[
\rho_e = q(n_+ - n_-),
\]
\[
\vec{j} = q(n_+ \vec{v}_+ - n_- \vec{v}_-).
\]
Continuity equation for each component in the stationary condition is
\[
\nabla \cdot (n_\pm \vec{v}_\pm) = 0.
\]
This equation is satisfied by introducing a stream function \( F_\pm(r, \theta) \) as
\[
n_\pm \vec{v}_\pm = \frac{1}{r \sin \theta} \nabla F_\pm \times \vec{e}_\phi.
\]
The number density is given by
\[
n_\pm = \frac{|\nabla F_\pm|}{\alpha r \sin \theta (v_{\pm x}^2 + v_{\pm \theta}^2)^{1/2}}.
\]
The current function \( S \) in eq.(7) can be solved as
\[
S = 4\pi q(F_+ - F_-).
\]
The interaction between two-component fluids is assumed only through the global electromagnetic fields, i.e, collision and thermal pressure are ignored. The gravity is expressed by the derivative of \( \alpha \) in the Schwarzschild spacetime. The equation of motion for each component with mass \( m \) and charge \( \pm q \) in the stationary state is given by
\[
\left( \vec{v}_\pm \cdot \nabla \right) \gamma_\pm \vec{v}_\pm = -\gamma_\pm \nabla \ln \alpha \pm \frac{q}{m} \left[ \vec{E} + \vec{v}_\pm \times \vec{B} \right].
\]
There are two conserved quantities along each stream line, i.e, generalized angular momentum \( J_\pm \) and Bernoulli integral \( K_\pm \):
\[
J_\pm = \gamma_\pm v_\pm \hat{\phi} r \sin \theta \pm \frac{q}{m} G,
\]
\[
K_\pm = \alpha \gamma_\pm \pm \frac{q}{m} \Phi.
\]
These quantities depend on the stream functions \( F_\pm \) only, and the spatial distributions are therefore determined by \( F_\pm \) which is specified at the injection point in our model. The stream functions are determined by
\[
\frac{\partial}{\partial r} \left( \alpha^2 \frac{\partial F_\pm}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial F_\pm}{\partial \theta} \right) = \nabla\nabla \left( \frac{n_\alpha^2 \gamma_\pm}{\gamma_\pm} \right) \cdot \nabla F_\pm \pm \frac{q}{mc \gamma_\pm} S + \frac{alpha^2 q^2 \sin^2 \theta}{gamma_\pm} \left( K_\pm - \frac{\alpha v_\pm \vec{j}_p'}{r \sin \theta} \right),
\]
where \( J_\pm' \) and \( K_\pm' \) are derivatives of \( J_\pm \) and \( K_\pm \) with respect to \( F_\pm \).
3 Summary

The global structure of electromagnetic fields and plasma flows is determined by four partial differential equations of elliptic type, i.e, eqs. (5),(6) and (17). They should be subject to appropriate boundary conditions. There are two integrals along the stream line for each fluid component. The details of numerical method and results will be given elsewhere.

References