Minimal surfaces in flat and curved spacetimes of arbitrary dimensionality

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Abstract

Minimal surfaces and domain walls play important roles in astrophysics, general relativity, and modern particle theories. First, we review the Bernstein theorem, which states that a well-defined single-valued minimal surface in 3-dimensional Euclidean flat spaces should be a plane, and its failure in general dimensions. Then, we show that this failure of Bernstein theorem in higher dimensions results in interesting consequences in the systems where the branes interact with black holes.

1 Bernstein conjecture, branes, and minimal cones

Minimal surfaces[2] in Euclidean space $E^3$ have been extensively studied since the pioneering work of Thomas Young and of Laplace. In Monge, or non-parametric, gauge the surface is specified by the height function $z = z(x, y)$ above some plane. The non-parametric minimal surface equation governing the function $z(x, y)$ is

$$
\frac{\partial_x}{\sqrt{1 + z_x^2 + z_y^2}} + \frac{\partial_y}{\sqrt{1 + z_x^2 + z_y^2}} = 0.
$$

A famous result of Bernstein asserts that the only single valued solution of Eq. (1) defined for all $(x, y) \in \mathbb{R}^2$ is a plane. It may also be shown that the planar solution is a minimizer of the area functional among compactly supported variations of the surface. In terms of brane theory, this means that the “classical ground state”, i.e., the static minimum of the energy functional for a membrane in three dimensional Euclidean space $E^3$, which may be thought of as a static configuration in 4-dimensional Minkowski spacetime $E^{3,1}$, is smooth and indeed planar. From the world volume point of view the classical ground state of the membrane preserves $(2+1)$-dimensional Poincaré invariance and may be thought of as a copy of $(2+1)$-dimensional Minkowski spacetime $E^{2,1}$.

It is natural to conjecture that Bernstein’s theorem remains valid for a minimal $p$-dimensional hypersurface in $(p+1)$-dimensional Euclidean space $E^{p+1}$. In other words, the classical ground state of a $p$-brane in $(p+1)$-dimensional Minkowski spacetime $E^{p+1,1}$ should be flat and invariant under the action of the $(p+1)$-dimensional Poincaré group $E(p, 1)$. Remarkably, although true for $p \leq 7$ it fails for $p+1 = 9$ [2]. In other words, the classical ground state of an 8-brane in 10-dimensional Minkowski spacetime spontaneously breaks 10-dimensional Poincaré invariance. The proof [2] rests on the fact that in $E^8$ and above, a minimal hypersurface which is a minimizer of the $p$-volume functional among compactly supported variations need not be smooth. There are rather explicit counterexamples called minimal cones. Their existence leads to the conclusion that Bernstein’s theorem fails in $E^9$ [2].

As far as we aware, there has been very little discussion of the significance of this fact in the M/String theory literature. The breakdown of regularity of minimal hypersurfaces of flat space extends to minimal hypersurfaces of curved Riemannian manifolds and this has consequences for proofs of the positive energy theorem of general relativity which make essential use of minimal surfaces as a technical tool (e.g., see [3] for the application to the positive mass theorem). It seems worthwhile therefore to examine the behavior of minimal surfaces in higher dimensions and in curved spaces in some explicit detail in order to understand better the situation and its possible physical implications. In particular, it is interesting to see whether the existence of various critical dimensions which has been noted in related contexts is of

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\[2\] This article is based on paper [1].
a universal nature and related to the the breakdown of Bernstein’s theorem and the existence of minimal cones.

To make progress it is helpful to assume that the relevant surfaces have sufficient symmetries with which the problem may be reduced to one involving ordinary differential equations in an appropriate quotient space \( X \), a ploy known to mathematicians as equivariant variational theory. Typically the brane equations of motion reduce to finding geodesics in \( X \) with respect to a suitable metric \( g \) on \( X \), induced by the \( p \)-volume functional. The \( p \)-brane will be \( p \)-volume minimizing if the corresponding geodesic \( \gamma \) is length minimizing. A necessary condition that a geodesic joining points \( a \) and \( b \) be length minimizing is that \( \gamma \) contains no points between \( a \) and \( b \) conjugate to either. The existence of such conjugate points is governed by the Jacobi or geodesic deviation equation, solutions of which depend on the curvature of \( X \). For example, the geodesic deviation and Gauss curvature of metric \( g \) are positive and lead to a second variation or Hessian of indefinite sign. However, the situation is more subtle since the relation is not clearly a minimizer or not. Readers are directed to [2] for an introduction to this topic. The volume function on \( \Sigma \) is to be found. The metric \( g \) of ambient flat space is written in the form

\[
\eta = \tau^2 + \sqrt{y^2 + \tau^2},
\]

(2)

A projected metric \( \tilde{h} \) on \( \Sigma/G \) is defined via a transformation of an inner product, \( \tilde{h}(u_1, u_2) = h(u_1, u_2) \), where \( (u_1, u_2) \) is a pair of vectors tangent to the orbit \( G(s), s \in S \). Thus, in the present case \( \tilde{h} = \tau^2 + dy^2 \).

The volume function on \( \Sigma/G \), which is the volume of the orbit \( G(s) \), is given by \( v(x, y) := Vol\ G(s) = \Omega_m x_m \Omega_n y^n \), where \( \Omega_n \) is the volume of unit \( n \)-sphere. Now, we are ready to define an effective space where the geodesic \( \gamma \) is to be found. The metric \( g \) of the effective space is defined as

\[
g = d\ell^2 := v^{2/\lambda}(x, y)\tilde{h} = \Omega^2 m \Omega^2 n x_m y^n (dx^2 + dy^2),
\]

(3)

where \( \lambda \) is the co-dimension of the \( G \)-invariant surface \( S \). For the present example, \( \lambda \) is given by \( \lambda := \dim S - \dim G(x) = (m + n + 1) - (m + n) = 1 \). The problem to find the minimal surface has been reduced to find the geodesic \( \gamma \) with \( g = d\ell^2 \). The action to be minimized is \( \ell = \int \tau^m y^n \sqrt{dx^2 + dy^2} \). If one denotes the geodesic by \( y = y(x) \) and varies the action with respect to it, one has

\[
xyy'' + (myy' - nx)(1 + y'^2) = 0.
\]

(4)

One can easily see that a cone \( y = \sqrt{n/m} x \) solves Eq. (4).

As noted before, one cannot know whether the above cone is indeed a minimizer or not until examining the second variation. Here, we introduce an alternative criterion to examine whether a geodesic is a minimizer or not. Readers are directed to [4] for an introduction to this topic. The Jacobi equation or equation of geodesic deviation is written as

\[
\frac{d^2 \eta}{d\ell^2} + K\eta = 0,
\]

(5)

where \( \eta \) and \( K \) are the geodesic deviation and Gauss curvature of metric \( g \). For a general metric of form \( g = v(x, y)^{2/\lambda}(dx^2 + dy^2) \), the Gauss curvature is given by \( K = (1/2)(\text{Ricci scalar}) = -2\lambda^{-1}v^{-2/\lambda}(\partial_x^2 + \partial_y^2) \).
\( \partial^2 \ln v \). For the present case, where \( v(x, y) = x^m y^n \) and \( \lambda = 1 \), we have

\[
K = \frac{1}{x^{2m} y^{2n}} \left( \frac{m}{x^2} + \frac{n}{y^2} \right),
\]

which is positive definite. One can calculate the Gauss curvature and proper distance along the geodesic cone \( y = \sqrt{n/m} x \),

\[
K = \frac{2m^{n+1}}{n^m x^{2(m+n+1)}}, \quad \ell = \frac{n^{n/2}(m+n)^{1/2}}{m^{(n+1)/2}(m+n+1)} x^{m+n+1}.
\]

Combining Eqs. (5) and (7), we have

\[
\frac{d^2 \eta}{d\ell^2} + c \frac{\ell^2}{\eta} = 0, \quad c = \frac{2(m+n)}{(m+n+1)^2}.
\]

It is well known that the behavior of solution to this equation changes at \( \ell = 1/4 \). That is, Eq. (8) has a simple power solution \( \eta = \ell^{\beta_{\pm}}, \beta_{\pm} = (1/2) \left( 1 \pm \sqrt{1-4c} \right) \). Thus, the geodesic deviation oscillates (i.e., there exists a conjugate point of the geodesic) for \( 2 \leq m+n \leq 5 \), while not for \( m+n \geq 6 \) (i.e., there exists no conjugate point of the geodesic). These results, and further work by Bombieri et al. imply that the cone as \( SO(m+1) \times SO(n+1) \)-invariant hypersurface \( S \subset \mathbb{E}^{m+n+2} \) is a minimizer for \( m+n+2 \geq 8 \).

3 Application: Branes interacting with a black hole

So far, we have considered the Bernstein theorem and its failure in higher dimensions for minimal surfaces in flat spaces. Although not recognized well in literature, the failure of Berenstein conjecture play important roles in various context of physics, especially in higher dimensional gravitational theories. See the original paper [1] and references therein. Here, let us consider the primal gravitational system which was first investigated by Frolov [5]. Let us write the \( N \)-dimensional Schwarzschild-Tangherlini metric as

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\Omega^2_{N-3}), \quad f(r) = 1 - \left( \frac{r_0}{r} \right)^{N-3}.
\]

Then, we focus on a local region near the south pole of horizon by setting \( r = r_0 + \xi, \theta = \pi - \eta \) with small \( \xi/r_0 \) and \( \eta \). At the leading order, the metric is written as

\[
ds^2 = -\left( \frac{N-3}{r_0} \right)\xi dt^2 + \frac{r_0}{(N-3)\xi} d\xi^2 + r_0^2(d\eta^2 + \eta^2 d\Omega^2_{N-3}).
\]

Furthermore, introducing the following local coordinates \( x = \sqrt{4r_0 \xi}/(N-3), \quad y = r_0 \eta \), the near horizon metric reduces to

\[
ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + dy^2 + y^2 d\Omega^2_{N-3},
\]

where \( \kappa = (N-3)/2r_0 \) is the surface gravity. Thus, the near-horizon effective 2-dimensional metric in which the geodesic is to be found is

\[
g = x^2 y^{2(N-3)} (dx^2 + dy^2).
\]

The problem has been reduced to that of \( (m, n) = (1, N-3) \) in the flat-space case. Note that the factor \( x^2 \) in \( g \) comes from the time component of metric (11). Thus, the cone \( y = \sqrt{N-3} x \) is a geodesic near the horizon, and from the analysis of geodesic deviation, this geodesic corresponds to a minimizer if \( N = p + 2 \geq 8 \).

The work by Frolov was in part motivated by that of Kol [6] in which the “merger transition” from the Kaluza-Klein black holes to a black string was investigated. The black-hole–brane system indeed serves as a toy model of the merger transition and is shown to possess a critical dimension [7]. In addition, this system serves as the simplest (as far as we know) example of critical phenomena in gravitational
systems [8]. The cone solution separates two phases of the brane: one has a Minkowski topology and another a black hole topology. The change of stability nature of the brane appears at $p = 6$ and results in that of mass scaling of the black hole on the brane. It seems that the self similarity of the critical solution changes from discrete one to continuous one. It would be interesting to clarify why the breakdown of Bernstein conjecture is related to this change of self similarity in detail.

In addition, the black-hole–brane system has many applications to the physics of fundamental interactions via the AdS/CFT correspondence. The holographic dual of the phase transition from the Minkowski embedding to the brane embedding corresponds to the meson melting phase transition of matter in the fundamental representation (see, e.g., Refs. [9]). Although the systems investigated in the literature so far correspond to the black-hole–brane systems below the critical dimension (as far as we know), it would be interesting to see in what the failure of the Bernstein conjecture results in the gauge theory side.

Acknowledgments

I would like to thank Valeri P. Frolov, Dan Gorbonos, Barak Kol, and Oleg Lunin for useful discussions and comments. I would like to acknowledge also the hospitality during my stay in Sep. 2008 at DAMTP and the Centre for Theoretical Cosmology, Cambridge University, where the work in paper [1] was started.

References