On Quantum Subgroups of Quantum SU(2)

２０１０年１月２８日（木）

RIMS共同研究「作用素環論と特異点の幾何学」

後藤 聡史（上智大学理工学部）

はじめに

The classification of subgroups of quantum $SU(N)$

Adrian Ocneanu

Contemporary Mathematics
Volume **294**, 2002

FIGURE 1. Classification of modules and subgroups of quantum $SU(2)$.

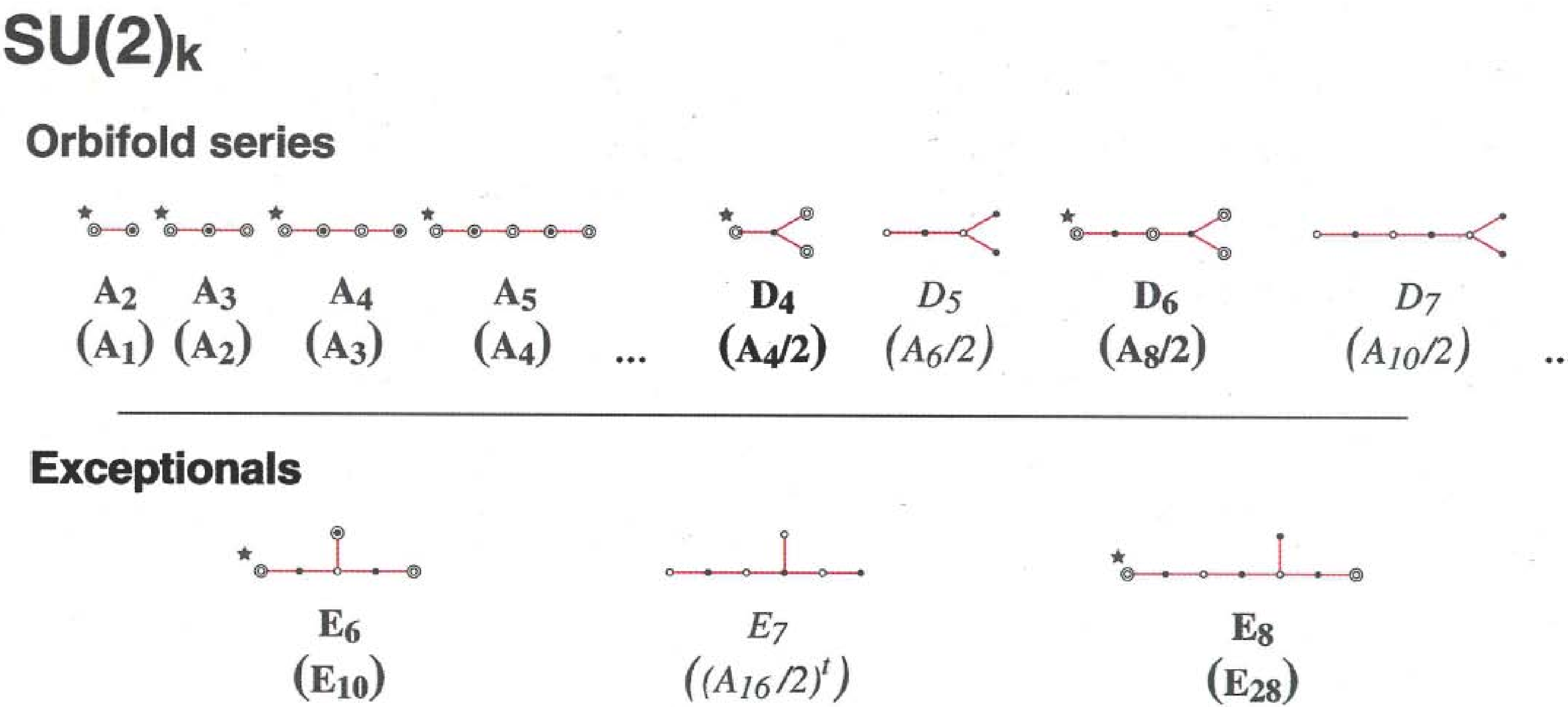


FIGURE 2. Classification of modules and subgroups of quantum $SU(3)$.

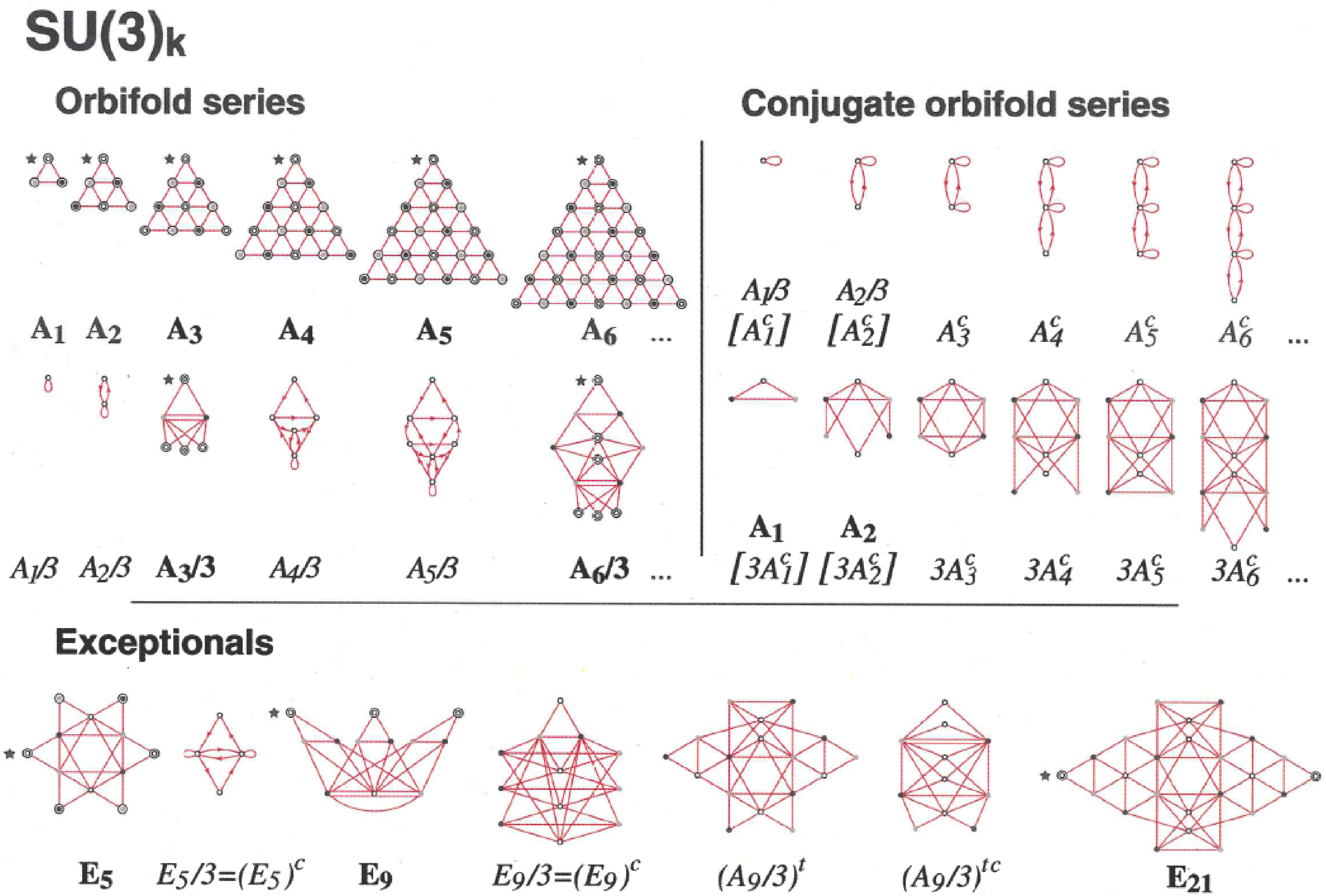


FIGURE 4. Classification of modules and subgroups of quantum $SU(4)$.

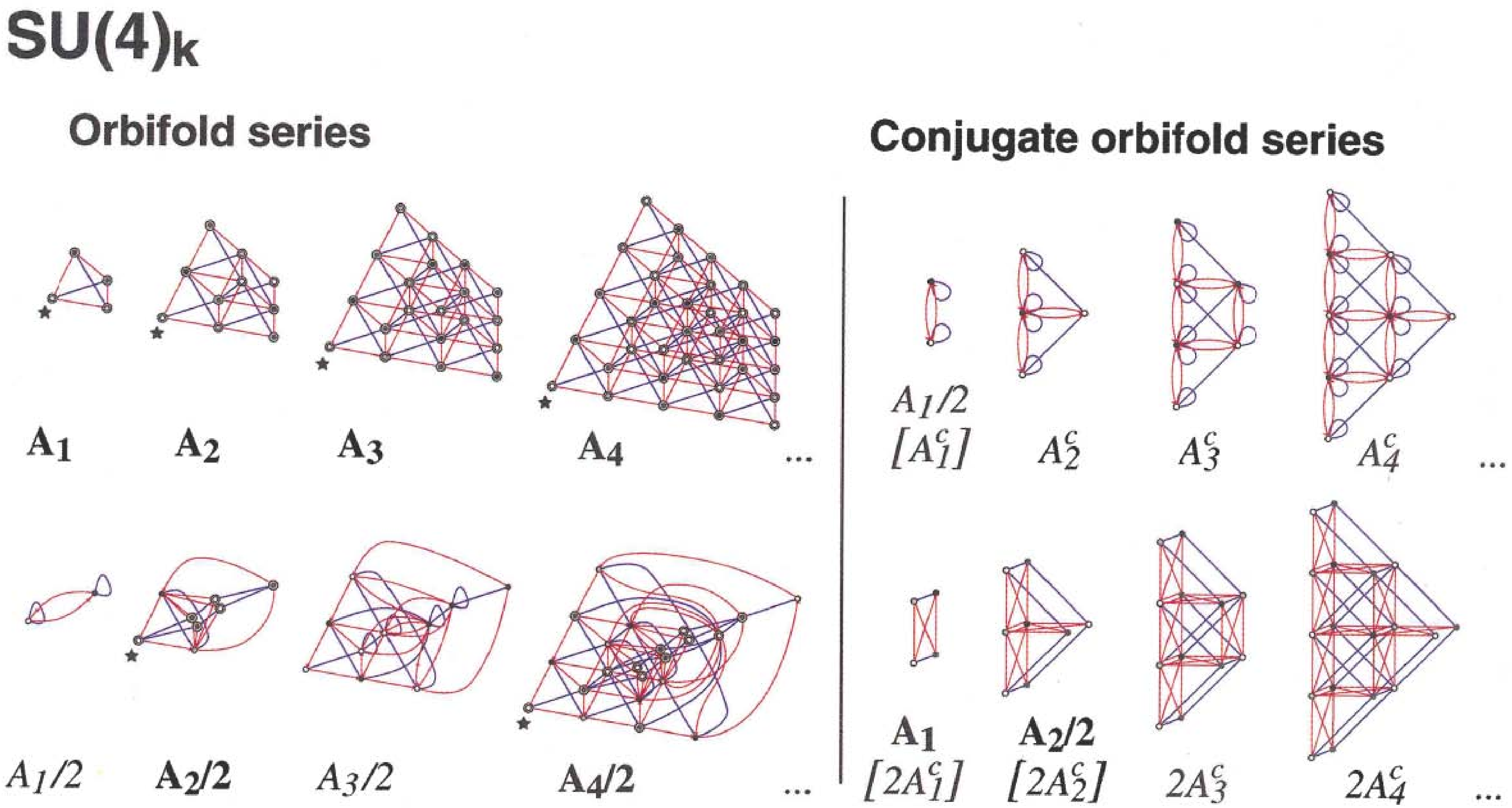
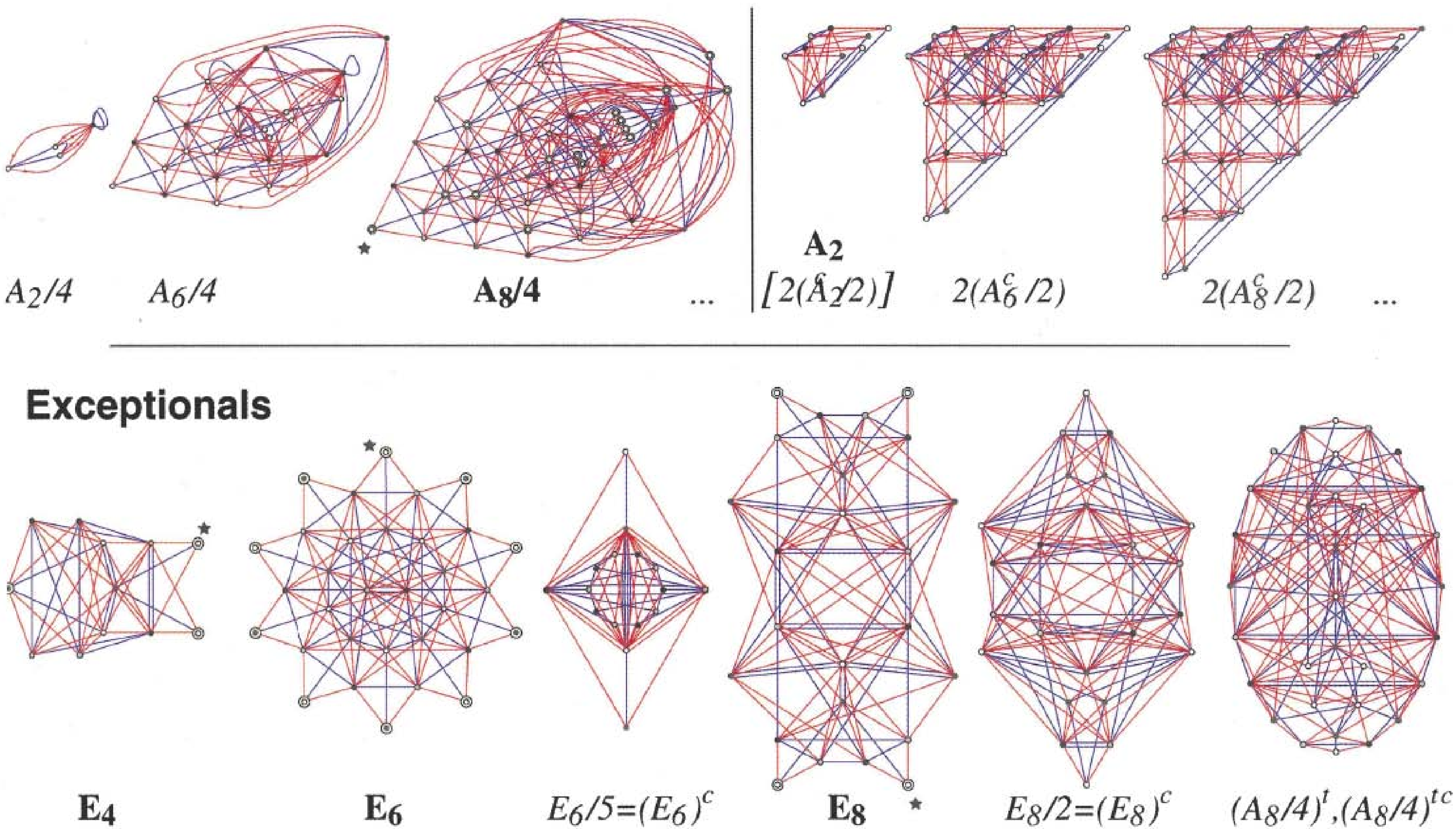
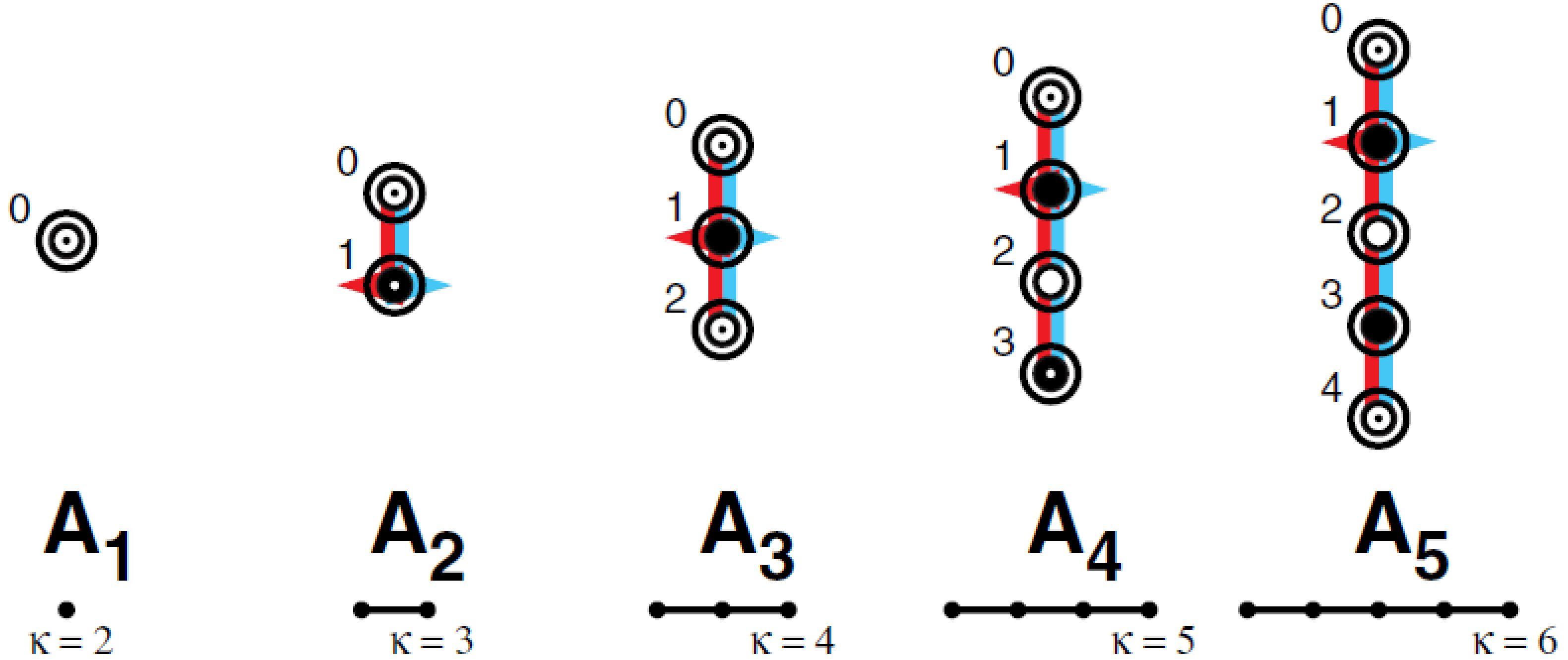


FIGURE 4. Classification of modules and subgroups of quantum $SU(4)$.

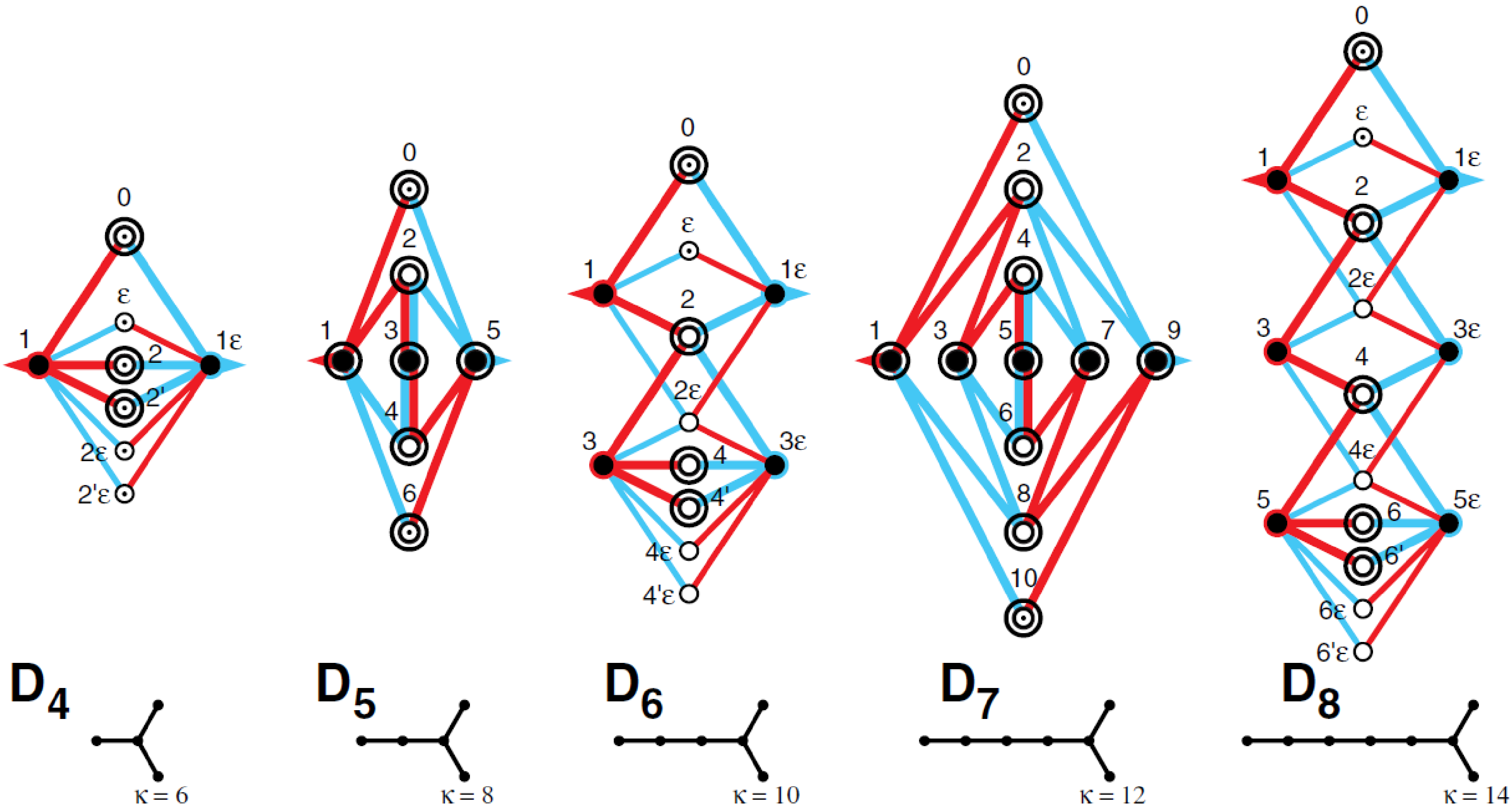


Classification of quantum subgroups of quantum SU(2)

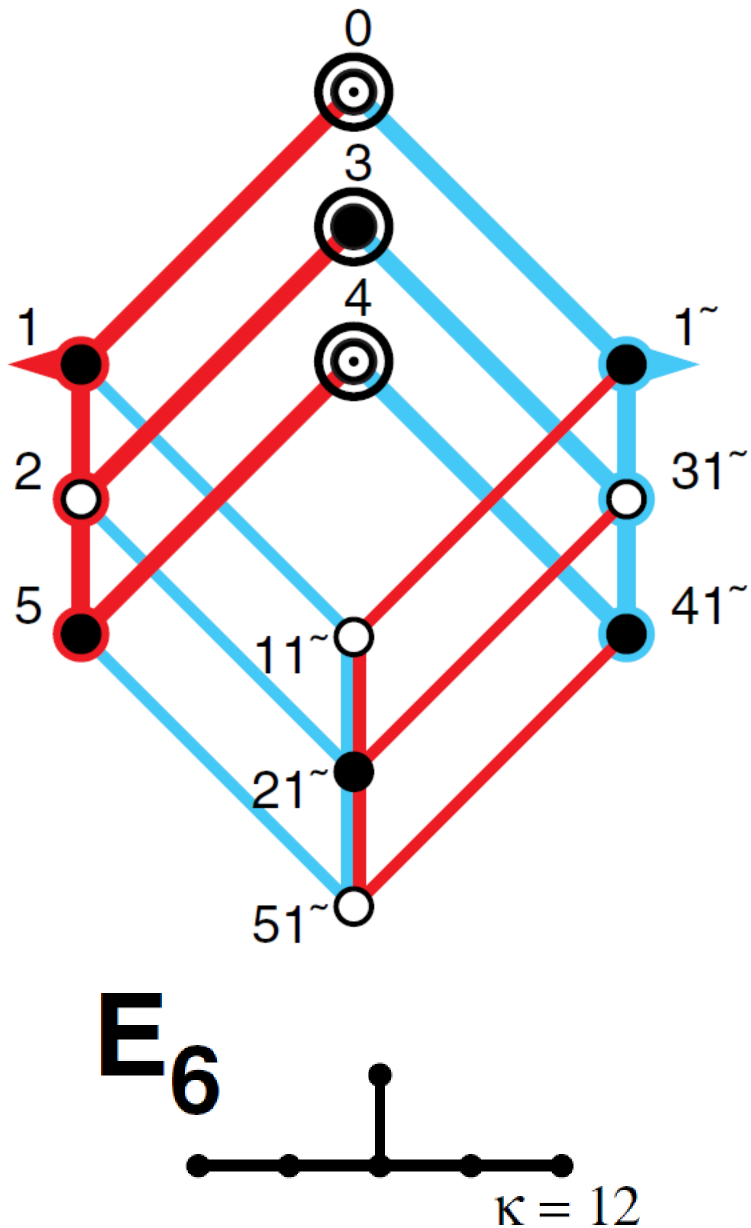
QUANTUM SYMMETRY FOR COXETER GRAPHS



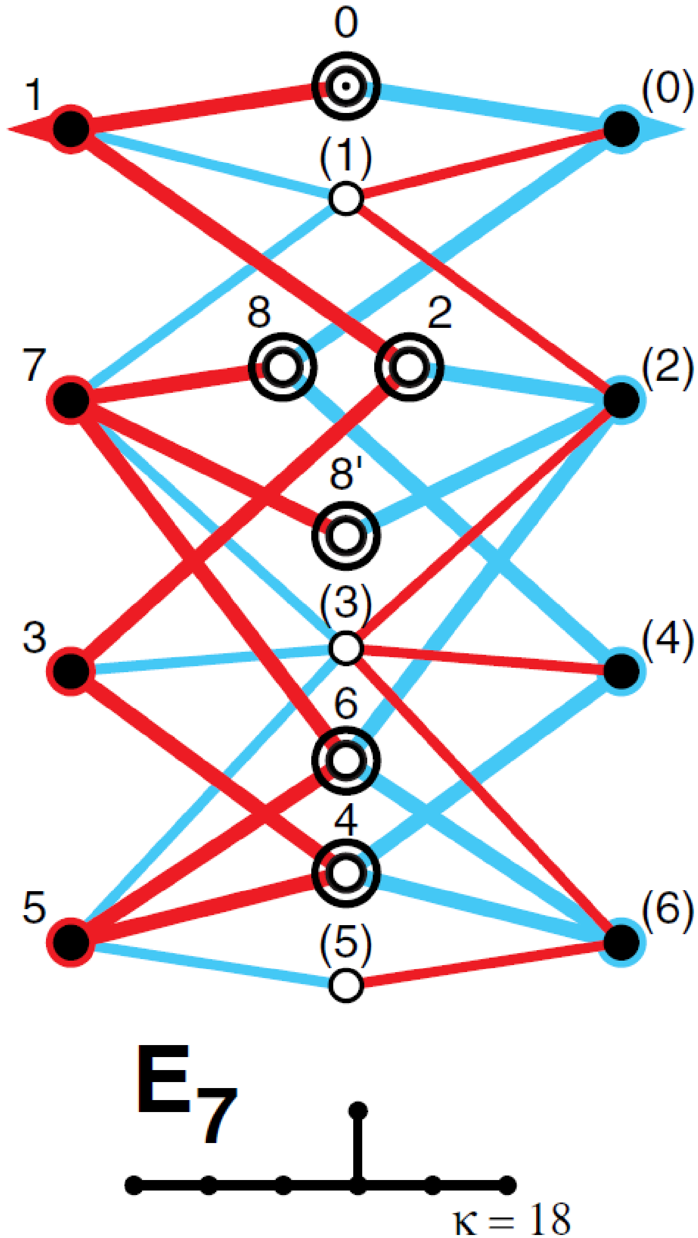
QUANTUM SYMMETRY FOR COXETER GRAPHS



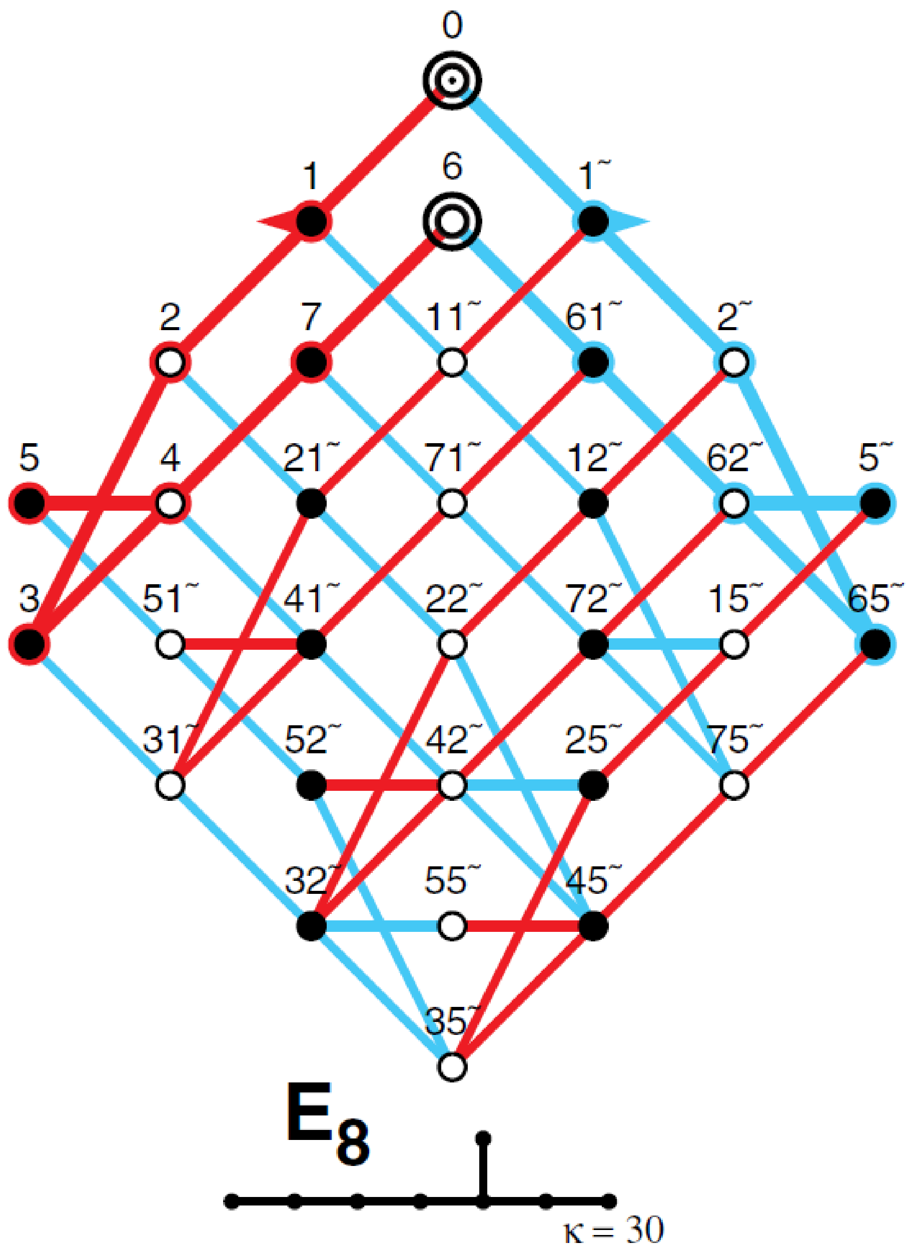
QUANTUM SYMMETRY FOR COXETER GRAPHS

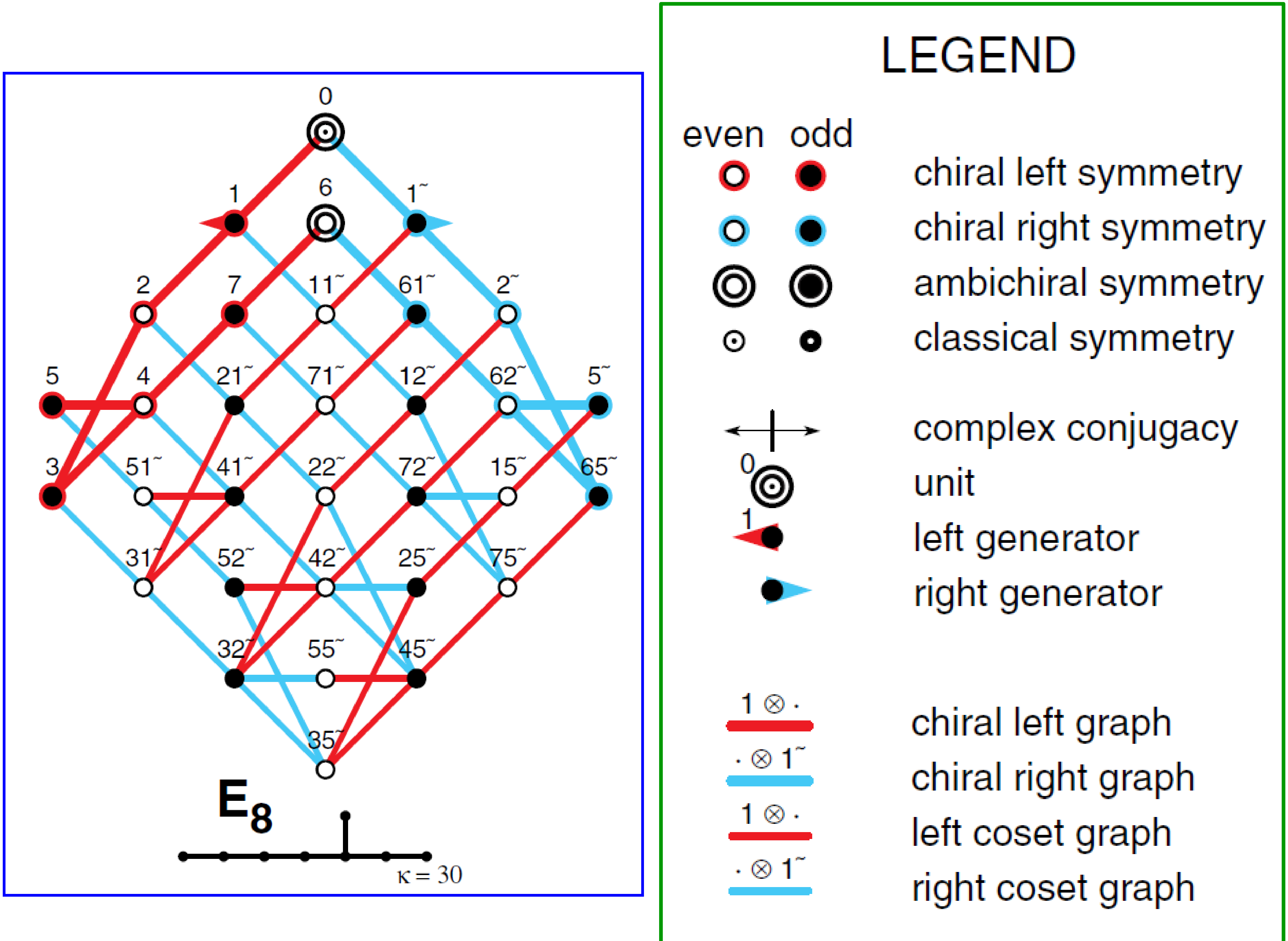


QUANTUM SYMMETRY FOR COXETER GRAPHS



QUANTUM SYMMETRY FOR COXETER GRAPHS

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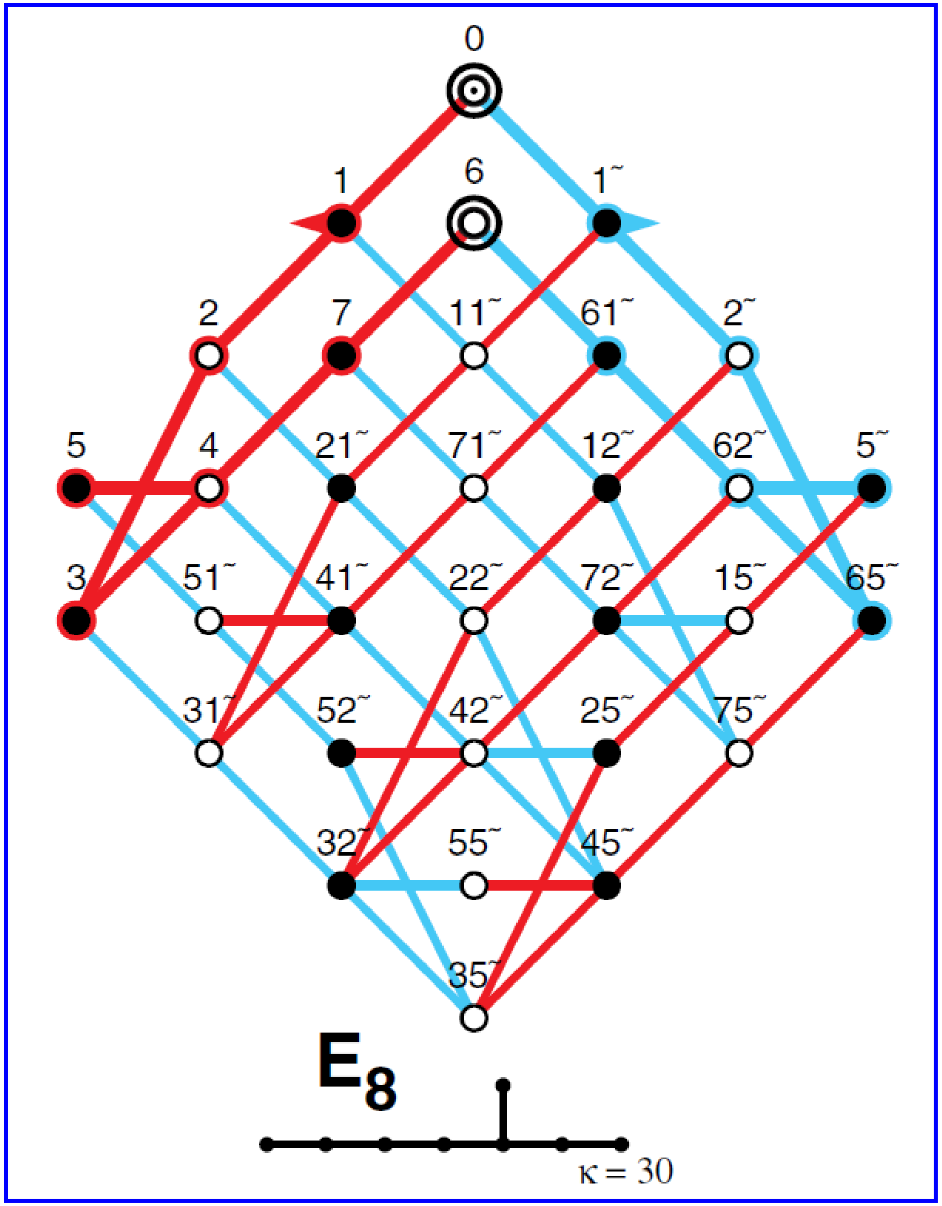
2
7
11˜
61˜
2˜
5
4
21˜
71˜
12˜
62˜
5˜
3
51˜
41˜
22˜
72˜
15˜
65˜
31˜
52˜
42˜
25˜
75˜
32˜
55˜
45˜
35˜
E8
κ = 30
LEGEND
even odd
chiral left symmetry
chiral right symmetry
ambichiral symmetry
classical symmetry
complex conjugacy
0
unit
1
left generator
right generator
1 ⊗ ·
chiral left graph
· ⊗ 1˜
chiral right graph
1 ⊗ ·
left coset graph
· ⊗ 1˜
right coset graph

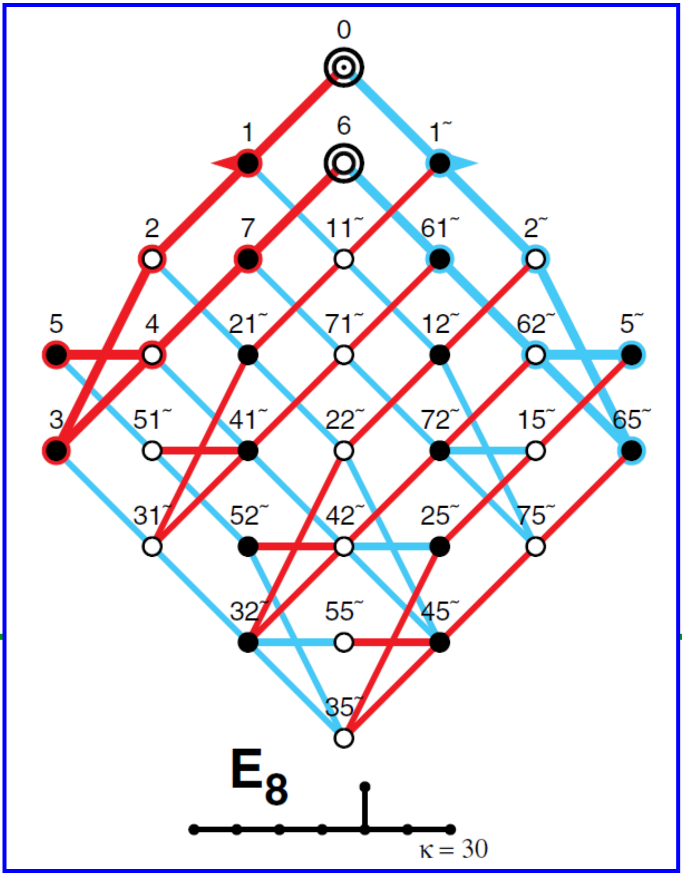


- 各頂点は「既約表現」
- グラフは2つの生成元(generator) による既約分解規則(fusion rule) を表す．（群のケーリーグラフのようなもの）
- 2つの生成元は互いに複素共役

Quantum Symmetry for Coxeter Graphs

- Quantum Symmetry for Coxeter Graphs
- Chiral diagram
- Ocneanu Graph

(Coquereaux, Schieber, Trinchero)



Ocneanu Graph の描き方（1）

- Ocneanuのオリジナルの方法
- Connection (bimodule) の system (fusion rule algebra, 略してFRA) による方法

- SU(2) の場合（modular invariant が不要）
- SU(N) (N≧3) の場合は，SU(2)より複雑で SU(2)とは異なる新しいアイディアが必要.

（modular invariant の分類を使用）

Ocneanu Graph の描き方（2）

- Boeckenhauer -Evans-Kawahigashi による方法
- Sector のFRAによる.
- Conformal Field Theory (CFT), WZW model,
  confomal inclusion
- Braiding, α-induction
- Modular invariant
- SU(N) (N≧３) で実行可能

Ocneanu Graph の描き方（3）

- Coquereaux, Schieber, Trincheroらの方法
- 純代数的な方法（FRA の相対テンソル積，weak Hopf algebra (quantum groupoid)，行列計算)
- Ocneanu Graphに代数的な構造の意味づけがなされる.

- SU(N) (N≧3) の場合にも（SU(2)より複雑だが）実行可能.
- （SU(2)の場合）Modular invariant, toric matrix なども結果的に構成される.

Modular invariants (SU(2) case)

$$\rho_n : SL(2, \mathbb{Z}) \longrightarrow Mat_{n-1}(\mathbb{C})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \left(\frac{-2i}{\sqrt{2n}} \sin \frac{kl\pi}{n} \right)$$

$k, l = 1, 2, \cdots, n-1$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \left(\delta_{kl} \exp\left(\pi i \left(\frac{k^2}{2n} + \frac{1}{4}\right)\right) \right)$$

$k, l = 1, 2, \cdots, n-1$

ρ_n is not irreducible. (has intertwiner)

Problem Find all intertwiners

$$M = (m_{k,l})_{k,l=1,2,\cdots,n-1}$$

$$\rho_n(g) M = M \rho_n(g) \qquad g \in SL(2, \mathbb{Z})$$

$$m_{k,l} \in \{0, 1, 2, \cdots\},$$

$$m_{1,1} = 1$$

Thm (Capelli - Itzykson - Zuber)

The above intertwiners have a one-to-one correspondence to the Dynkin diagram $A_n, D_n, E_{6,7,8}$

$$
A_8\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
\quad
D_6\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
\quad
D_7\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
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\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
$$

Modular invariant matrices of type A_8, D_6, D_7

$$
E_6\begin{pmatrix}
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\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
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\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
\quad
E_7\begin{pmatrix}
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1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
$$

Modular invariant matrices of type E_6 and E_7

$$
E_8 \quad \left(\begin{array}{*{29}{c}}
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\end{array}\right)
$$

Modular invariant matrix of type E_8

Classification of subfactors

- Classification of AFD II_1 subfactors
  with finite index, finite depth.
  (strongly amenable)

  ↓ Popa's deep theorem

  Classification of paragroups (Ocneanu)
  flat bi-unitary connections (on the principal graphs)

- $N \subset M$, $[M:N] < 4$

  The (dual) principal graph is one of the Dynkin diagram A_n. D_n. E_6. E_7. E_8

Classification of subfactors

Classification of subfactors of the AFD II_1 factor

index < 4 ⋯ A, D, E

index = 4 ⋯ extended Dynkin $A^{(1)}, D^{(1)}, E^{(1)}$ etc.

index > 4 ⋯ ?

(Haagerup) list of possible candidates of principal graphs

$4 < \text{index} \leq 3+\sqrt{3}$

- index = 4 ・・・ classical (McKay corresp.)
- index < 4 ・・・ quantum

Classification of connections on the Dynkin diagrams

- NCM, $[M:N]<4$

The (dual) principal graph is one of the Dynkin diagram A_n, D_n, E_6, E_7, E_8

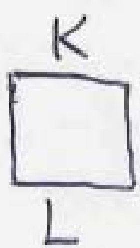
	A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
equiv. connections	1	2	2	2	2	2
isom. connections	1	1	1	2	2	2
flat connections	1	1	0	2	0	2

up to total gauge choice

Problem

① Finer classification of irreducible connections up to vertical gauge choice

②



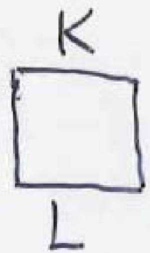
$K, L \in \{A, D, E\}$

We fix only horizontal graphs

	A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
equiv. connections	1	2	2	2	2	2
isom. connections	1	1	1	2	2	2
flat connections	1	1	0	2	0	2

→ up to total gauge choice

① Finer classification of irreducible connections up to vertical gauge choice

② 

$K, L \in \{A, D, E\}$

We fix only horizontal graphs

In the following "connection" always means "bi-unitary connection".

(not necessarily flat)

- 1995. April 19~25, Ocneanu
  Lectures at The Fields Institute.

· essential paths
· extension of Kauffman-Lins' recoupling theory

} double triangle algebra (DTA)

- 1995 June, Ocneanu. talks at Aarhus

Applications
5 problems (3 subfactor, 1 RCFT, 1 TQFT) → 1 solution

Most important problem

classification of irreducible connections
on the Dynkin diagrams.

Other problems

- (dual) principal graph and fusion rule of Goodman-de la Harpe-Jones subfactors
- generalized intermediate subfactors of the Jones' subfactors
- explanation of SU(2) modular invariant matrices.

Conformal inclusion : Xu, Böckenhauer-Evans
B-E-Kawahigashi

Quantum Galois correspondence : Kawahigahi

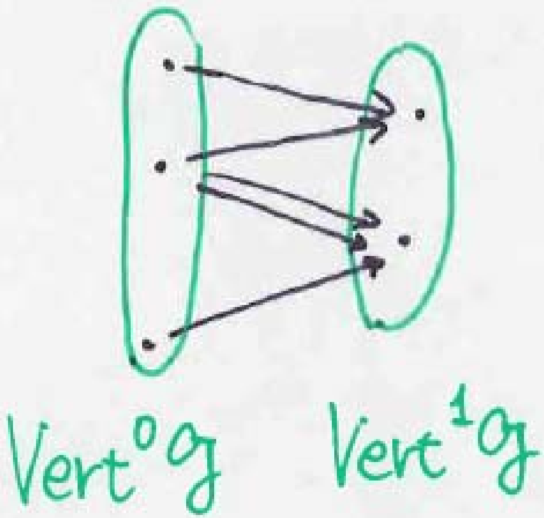
Strategy ： How to find all K-K irreducible connections

- Extend Kauffman–Lins' recoupling theory
- Define double triangle algebra (DTA) with convolution product （⇒ finite dim C*–algebra)
  - minimal central projections
  - irreducible ＊-representations of K-K DTA
  - irreducible K-K connections
- （minimal central projections of DTA, ・product ）＝ a system (FRA) of K-K irreducible connections

Extension of Kauffman–Lins' recoupling theory and the double triangle algebra (DTA)

Wenzl projectors and essential paths

$\mathcal{G}$: finite (oriented) bipartite graph



$\mathrm{Edge}^{(0,1)}\mathcal{G} \quad \to \quad {}_A H_B$

H: Hilbert space with ONB

$e \in \mathrm{Edge}^{(0,1)}\mathcal{G}$

$A = \mathbb{C}^{\mathrm{Vert}^0\mathcal{G}}, \quad B = \mathbb{C}^{\mathrm{Vert}^1\mathcal{G}}$

$$x \cdot \xi \cdot y \;\overset{\text{def}}{=}\; \delta_{x,s(\xi)}\, \delta_{y,r(\xi)}\, \xi$$

μ : P-F eigenvector

annihilation operator

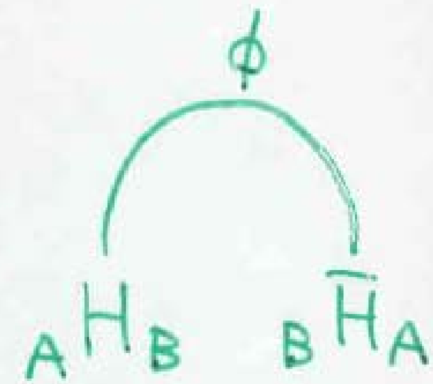
$$c \in \mathrm{Hom}\left({}_A H_B \otimes {}_B \overline{H}_A,\ A\right)$$

$$c(\xi \otimes \overline{\eta}) = \delta_{\xi,\eta} \sqrt{\frac{\mu(r(\xi))}{\mu(s(\xi))}} \cdot s(\xi)$$

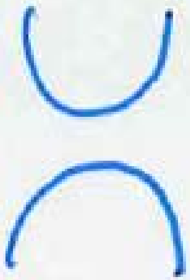
creation operator

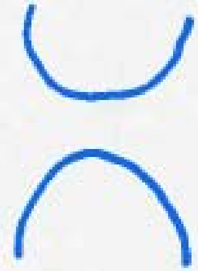
$$c^* \in \mathrm{Hom}\left(A,\ {}_A H_B \otimes {}_B \overline{H}_A\right)$$

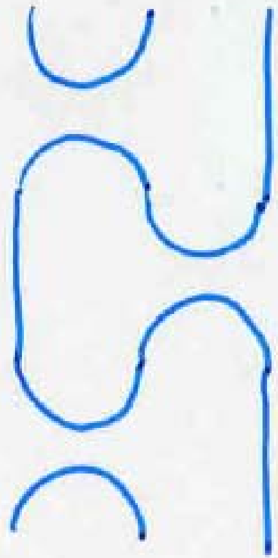
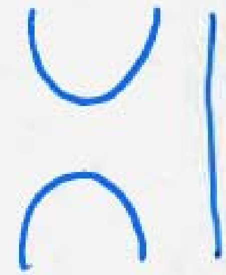
$$c^*(x) = \sum_{\substack{\xi \in \mathrm{Edge}\,\mathcal{G} \\ s(\xi) = x}} \sqrt{\frac{\mu(r(\xi))}{\mu(s(\xi))}}\ \xi \otimes \overline{\xi}$$



 $= C \cdot C^* = \beta$

β^{-1}  $\equiv e$: The Jones projection

$e^2 = \beta^{-2}$  $= \beta^{-1}$  $= e$

$e_1 e_2 e_1 = \beta^{-3}$  $= \beta^{-3}$  $= \beta^{-2} e_1$

tangle representation of Temperley-Lieb algebra.

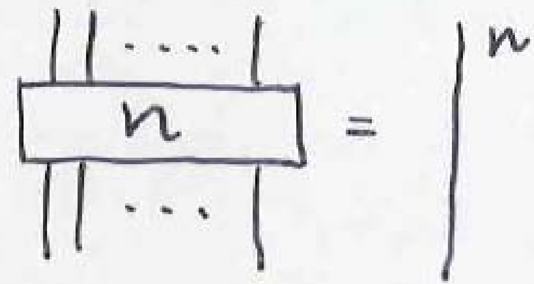
Wenzl projectors and essential paths

$$\mathrm{Path}^{(n)}\mathcal{G} = \{ \xi = (\xi_1, \cdots, \xi_n) \mid \xi_k \in \mathrm{Edge}\,\mathcal{G},\ s(\xi_{k+1}) = r(\xi_k) \}$$

$$\mathrm{HPath}^{(n)}\mathcal{G} = \underbrace{{}_A H_B \otimes {}_B \overline{H}_A \otimes {}_A H_B \otimes \cdots \otimes {}_A H_B \ (\text{or } {}_B \overline{H}_A)}_{n \text{ times}}$$

The Wenzl projector p_n on $\mathrm{HPath}^{(n)}\mathcal{G}$

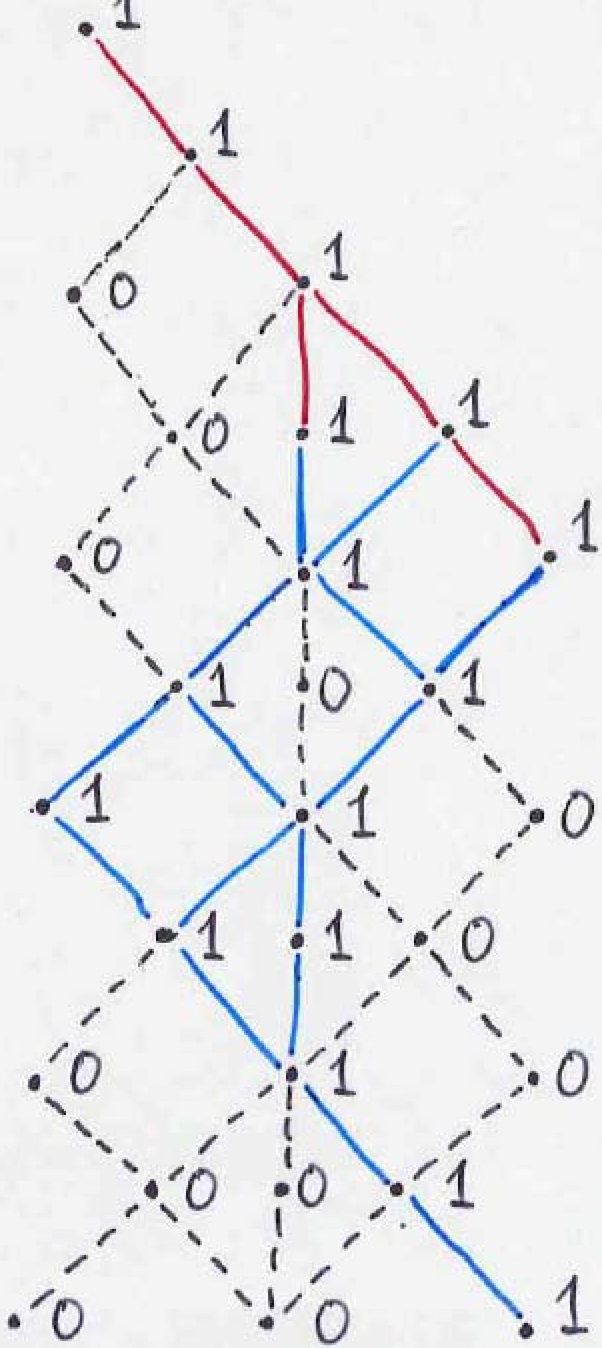
$$p_n \equiv 1 - e_1 \vee e_2 \vee \cdots \vee e_{n-1} =$$



$$\mathrm{EssPath}^{(n)}\mathcal{G} \equiv p_n \cdot \mathrm{HPath}^{(n)}\mathcal{G}$$

$$= \{ \xi \in \mathrm{HPath}^{(n)}\mathcal{G} \mid e_k \xi = 0 \text{ for } k = 0, 1, \cdots, n-1 \}$$

Moderated Pascal rule



$$\dim \mathrm{EssPath}^{(n+1)}_{a.x}\,\mathcal{G} = \sum_{\substack{\xi \in \mathrm{Edge}\,\mathcal{G} \\ r(\xi)=x}} \dim \mathrm{EssPath}^{(n)}_{a.s(\xi)}\,\mathcal{G} - \dim \mathrm{EssPath}^{(n-1)}_{a.x}\,\mathcal{G}$$

Essential paths

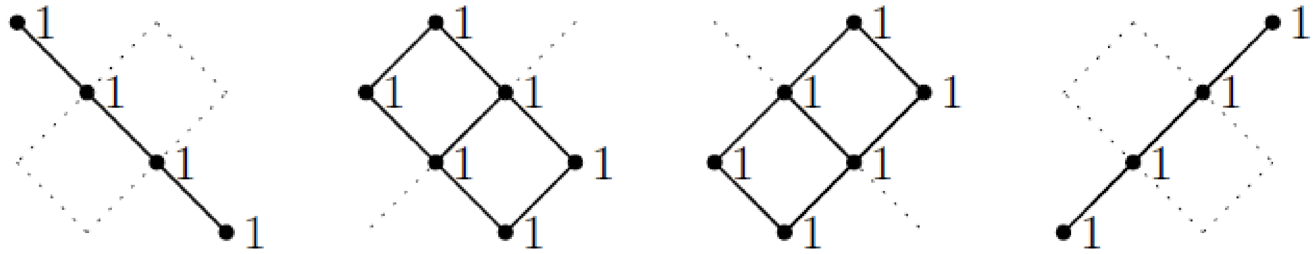


Figure 21: Essential paths on the Coxeter graph A_4

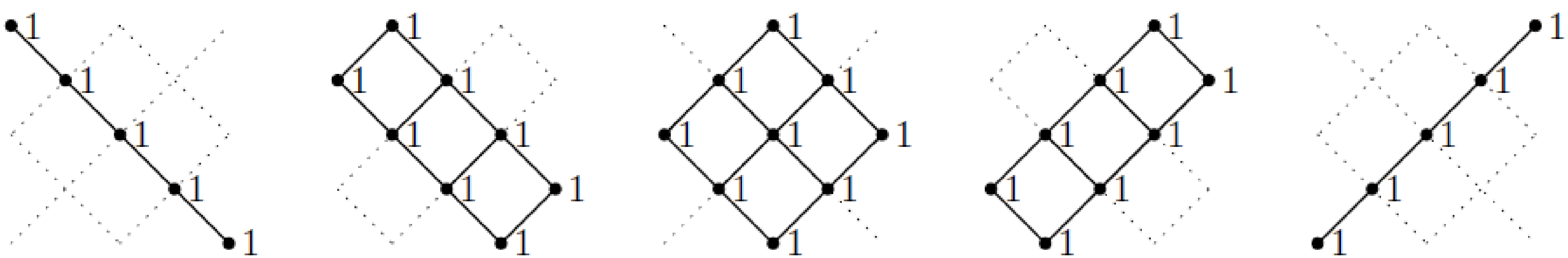


Figure 22: Essential paths on the Coxeter graph A_5

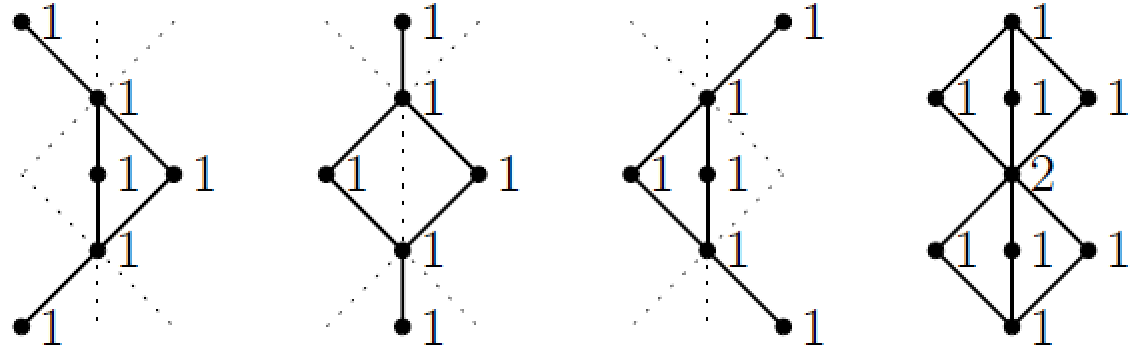


Figure 23: Essential paths on the Coxeter graph D_4

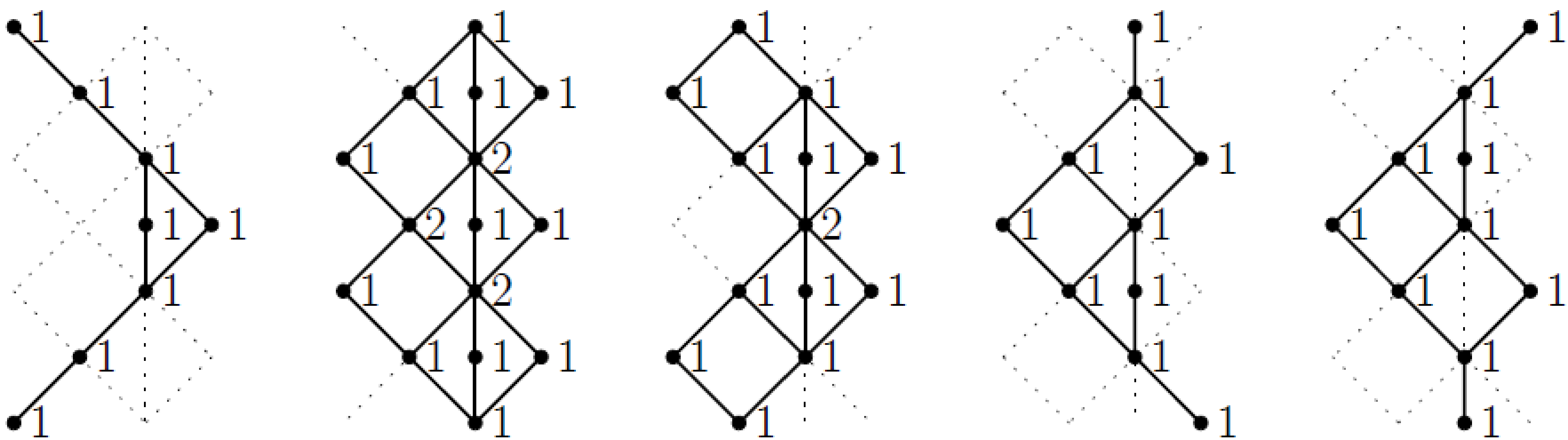


Figure 24: Essential paths on the Coxeter graph D_5

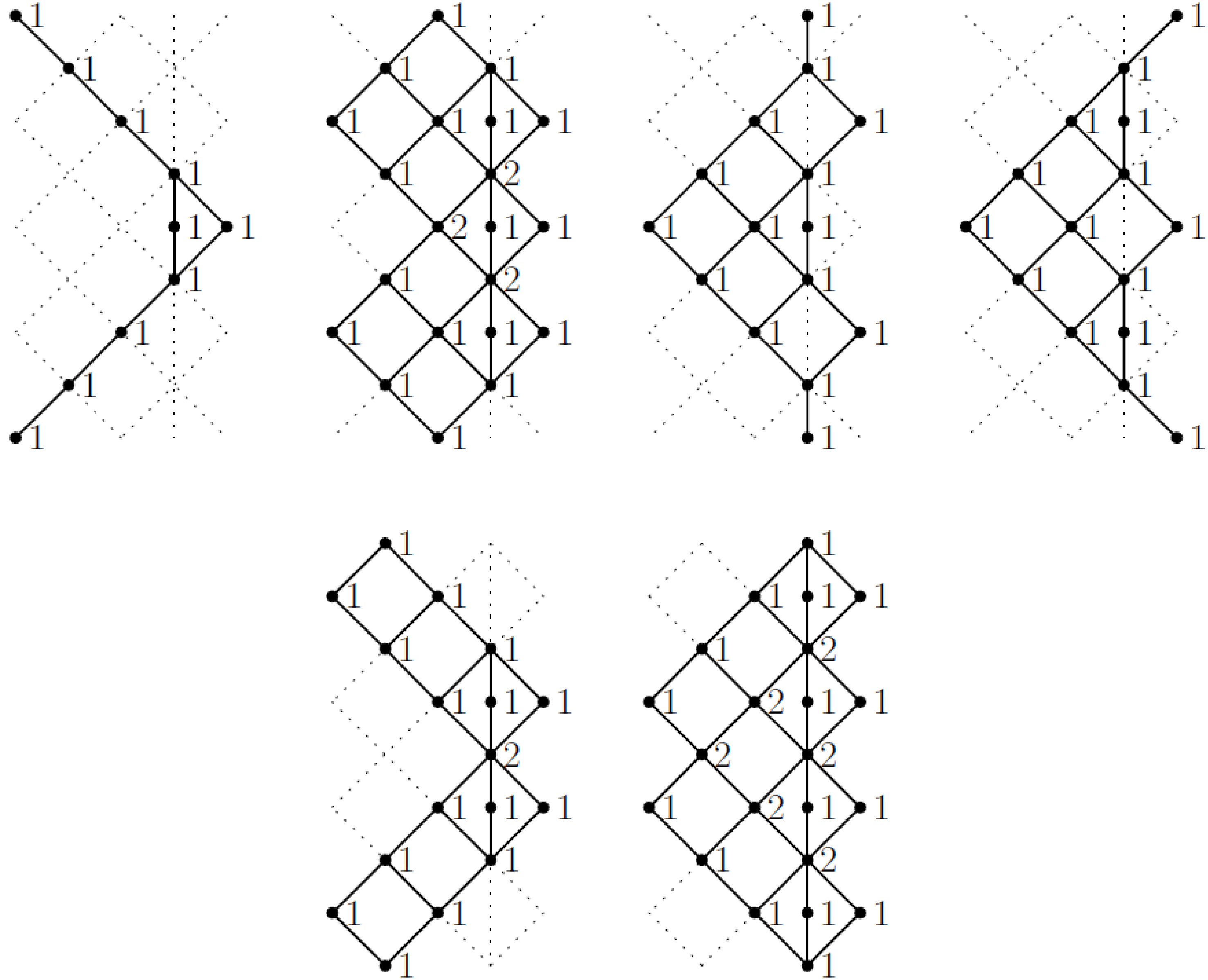


Figure 25: Essential paths on the Coxeter graph D_6

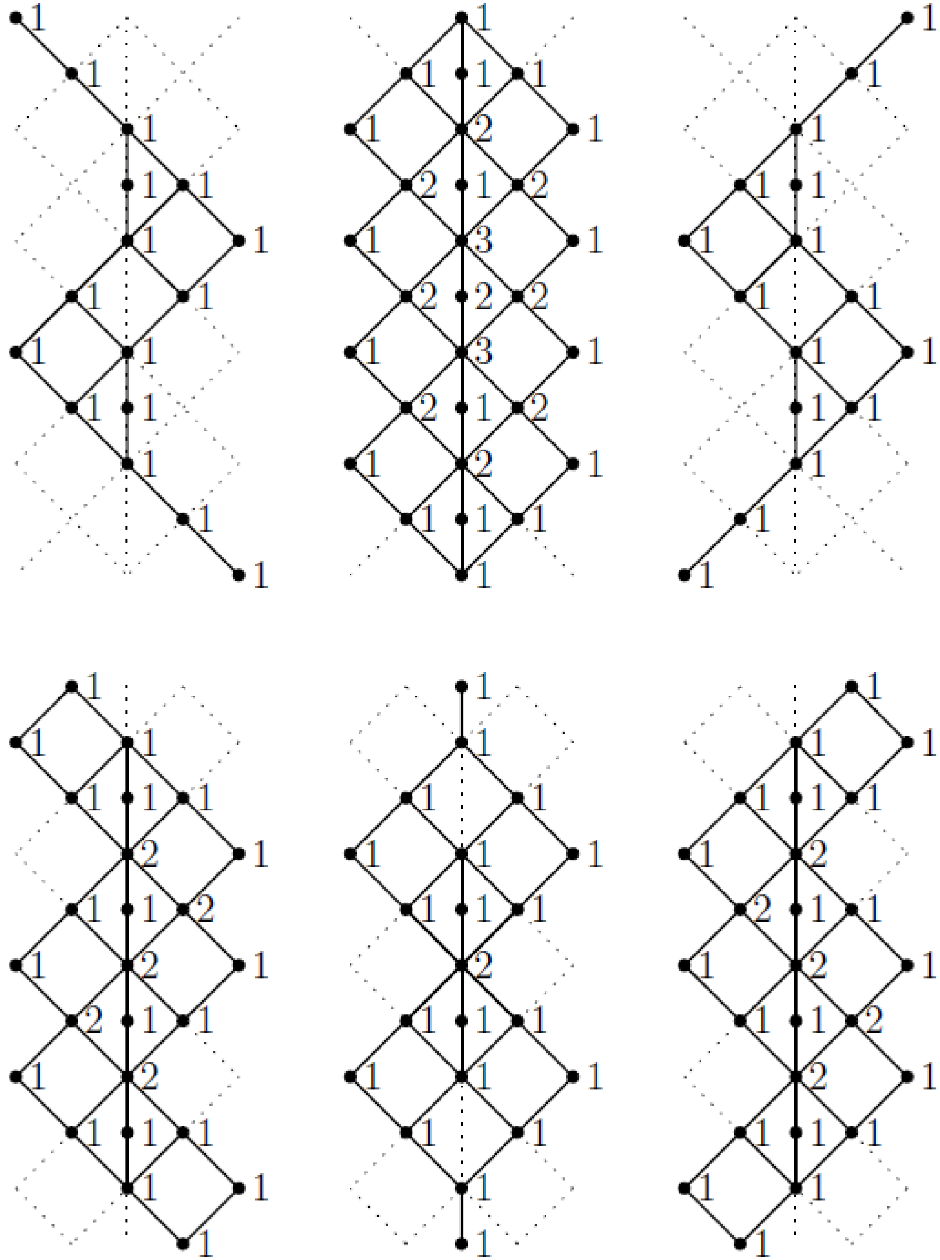


Figure 26: Essential paths on the Coxeter graph E_6

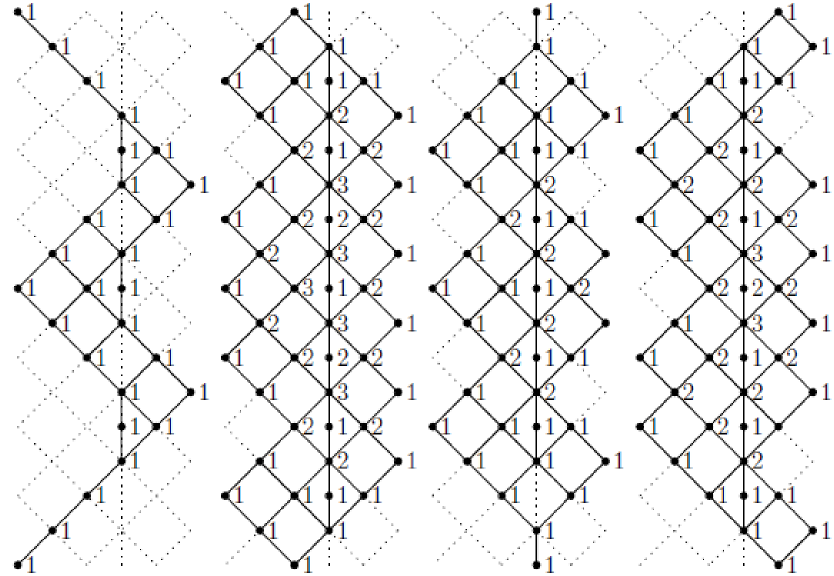


Figure 27: Essential paths on the Coxeter graph E_7 (1)

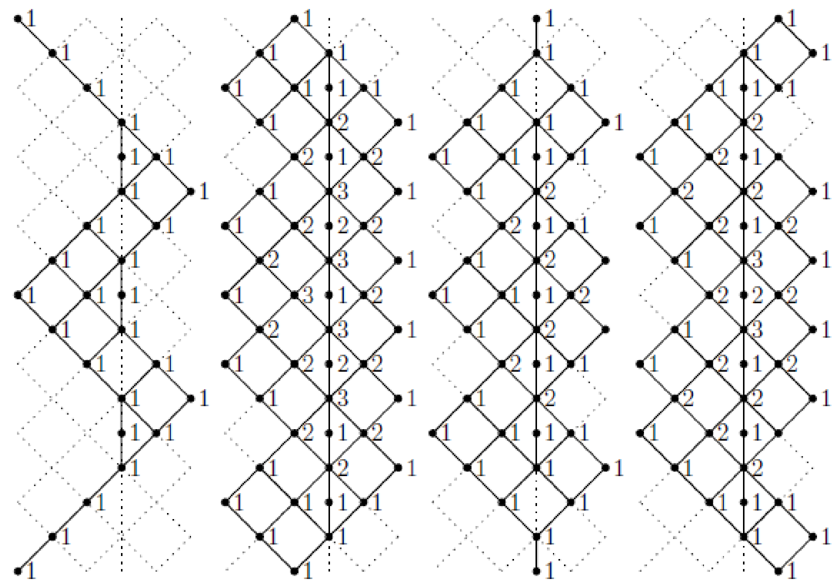


Figure 28: Essential paths on the Coxeter graph E_7 (2)

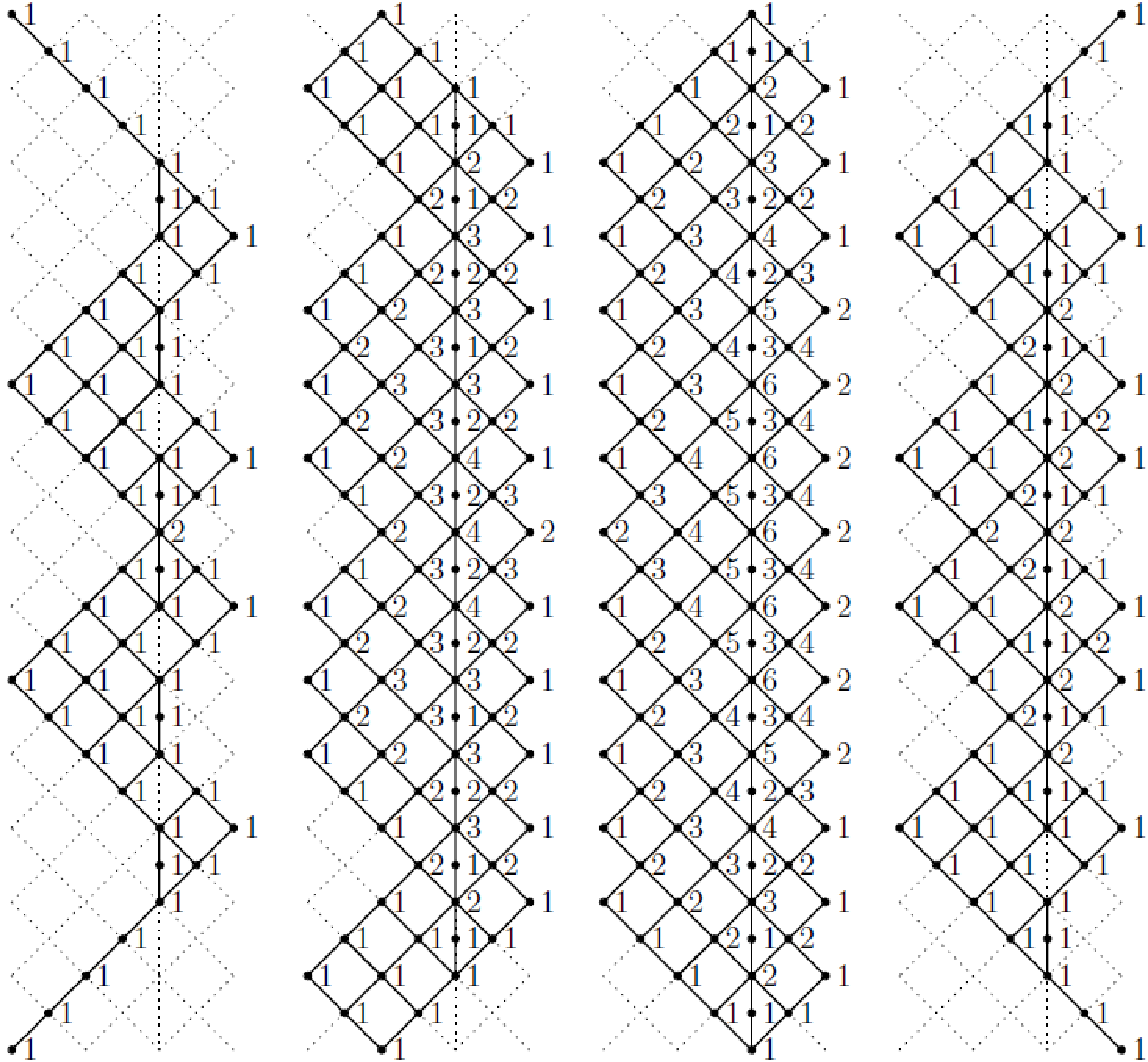


Figure 29: Essential paths on the Coxeter graph E_8 (1)

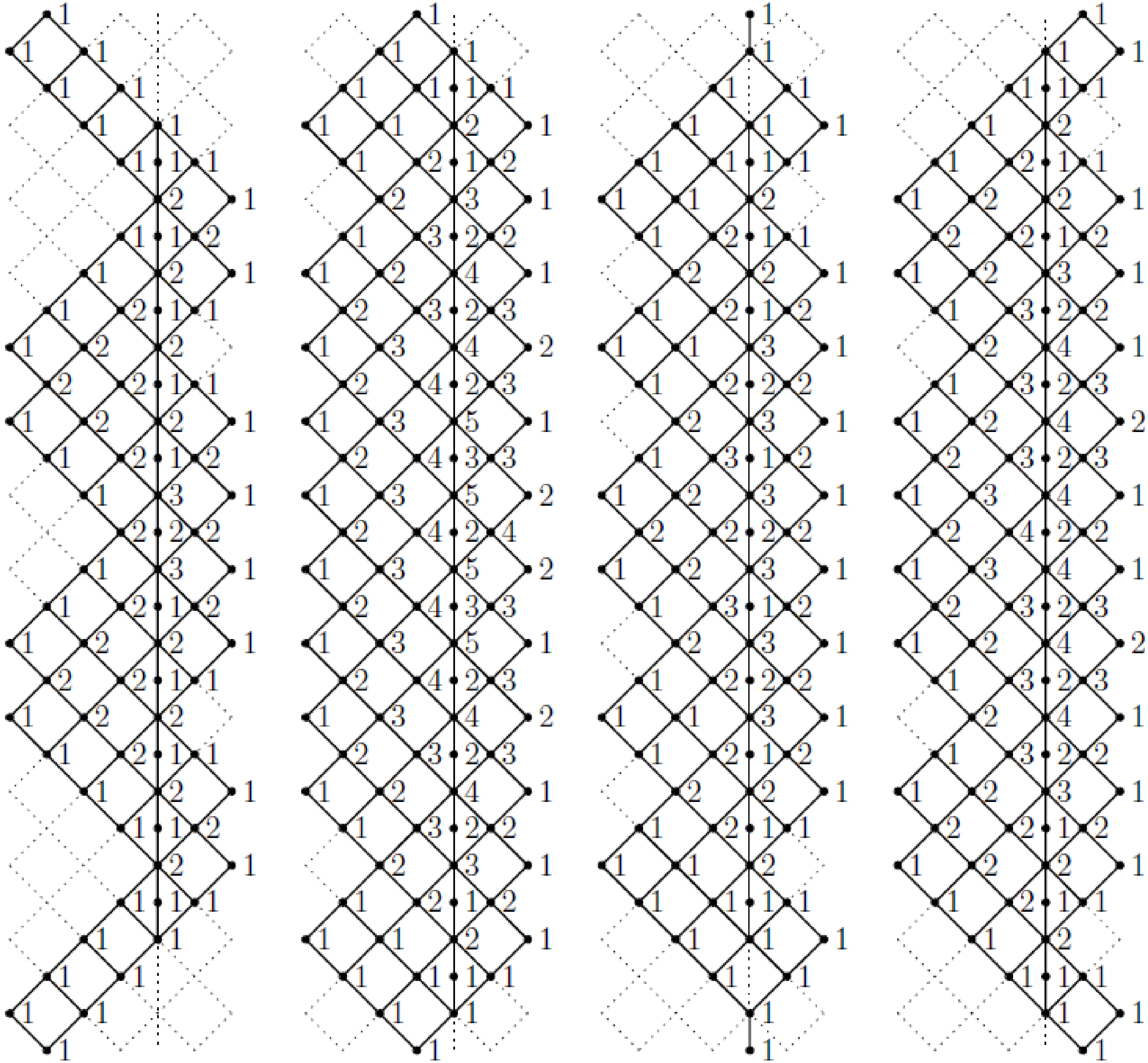
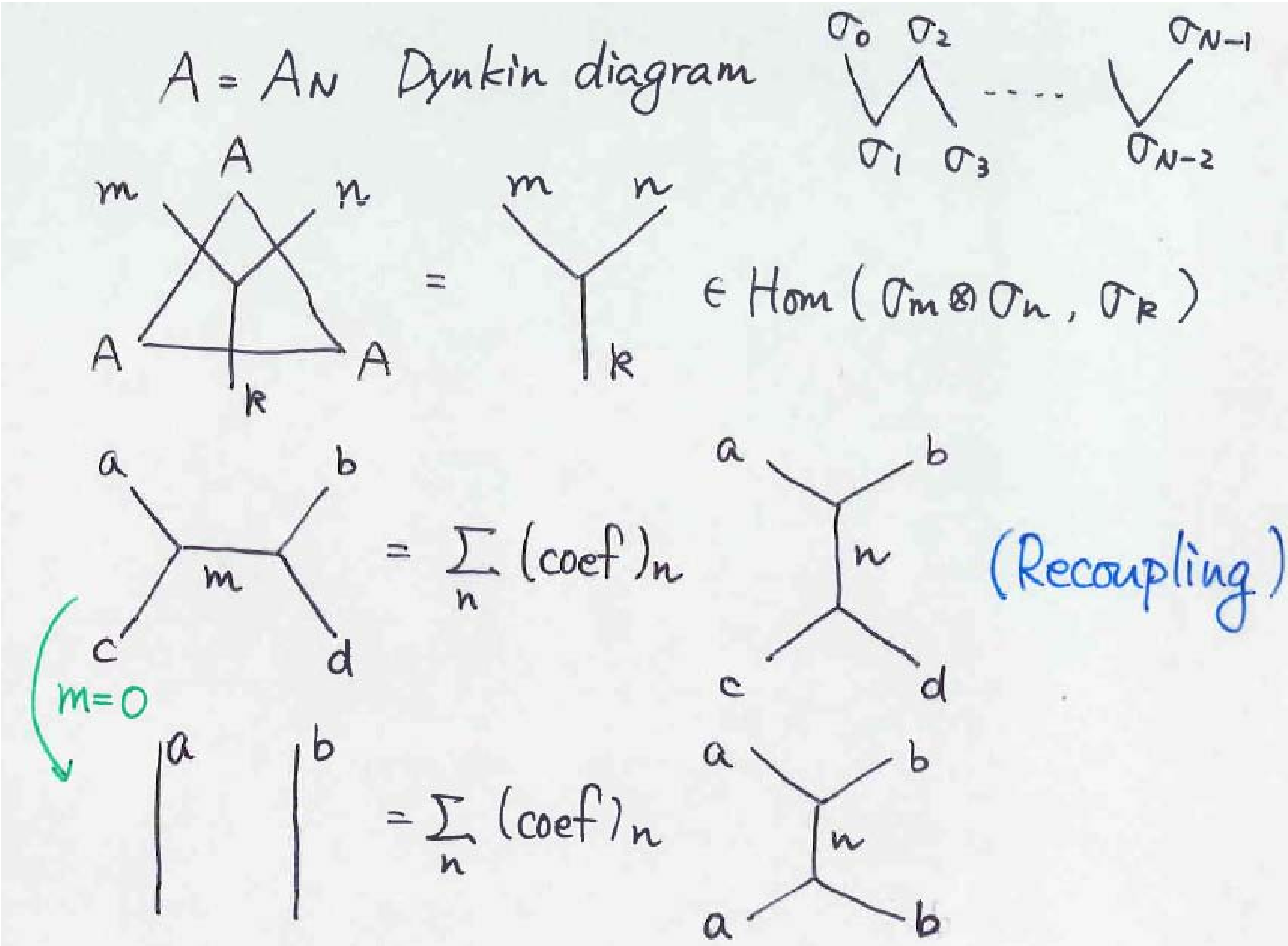
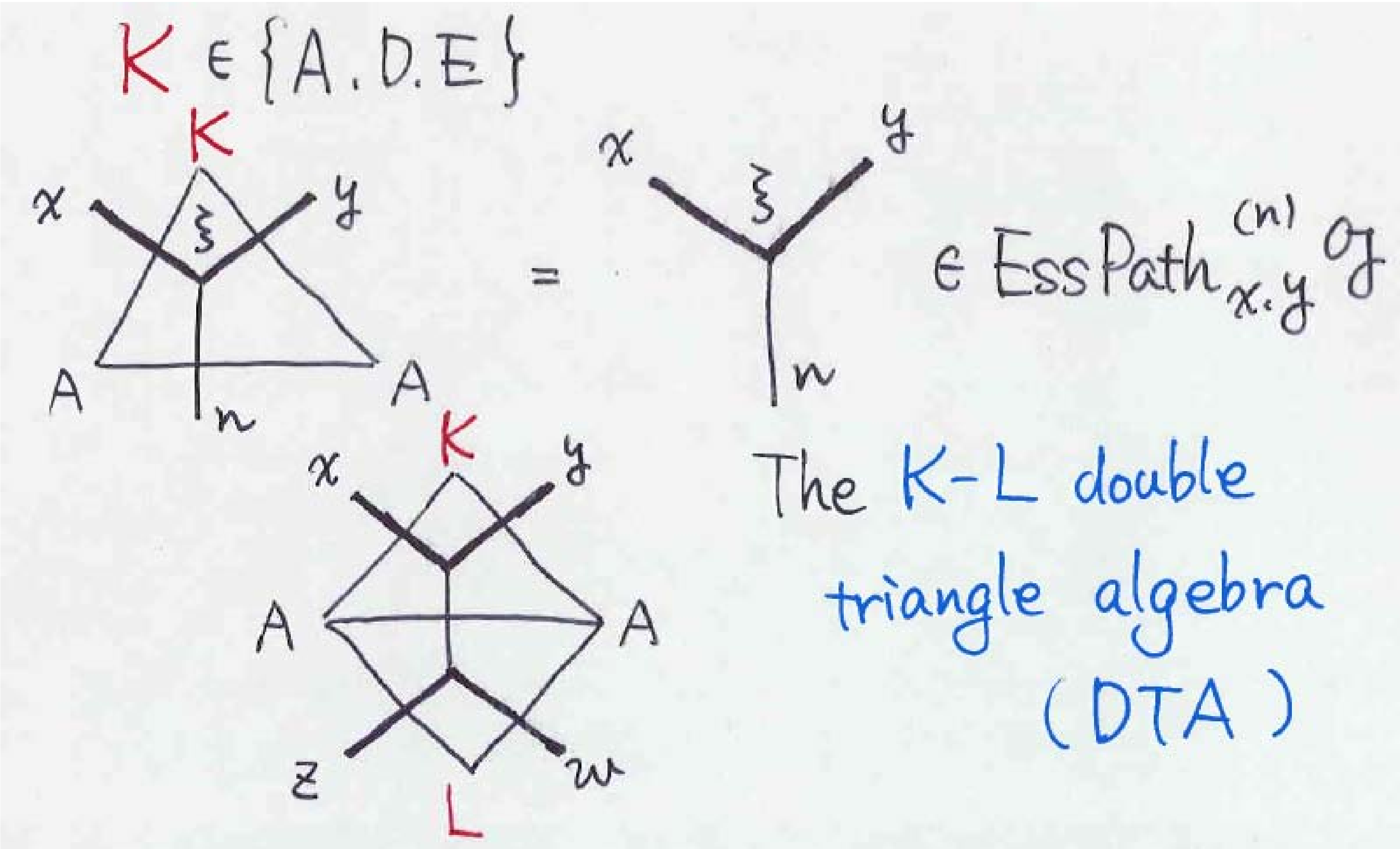


Figure 30: Essential paths on the Coxeter graph E_8 (2)

Extension of recoupling model and the double triangle algebras



double triangle algebras (DTA)



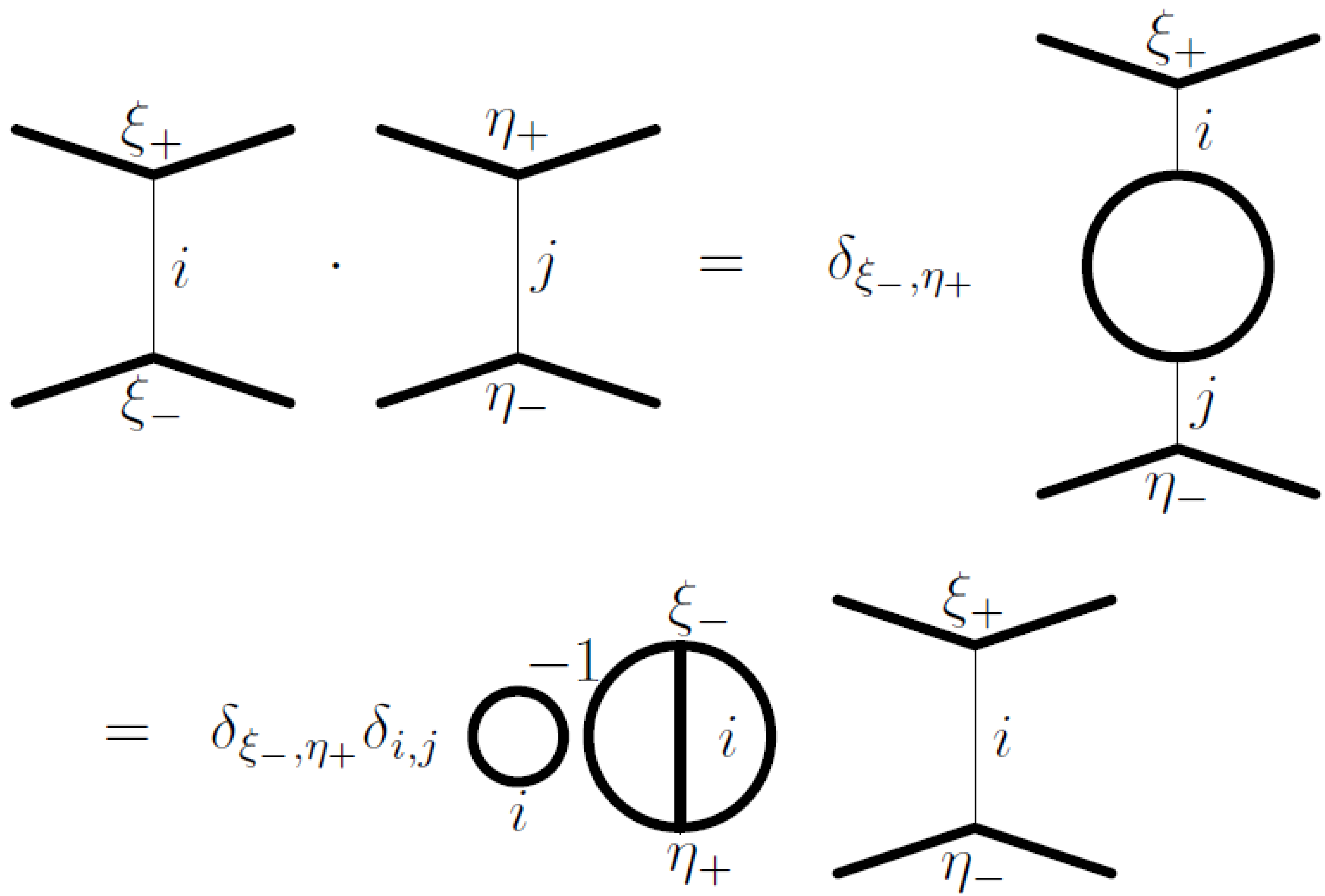


Figure 13: $\cdot$ product on the double triangle algebra

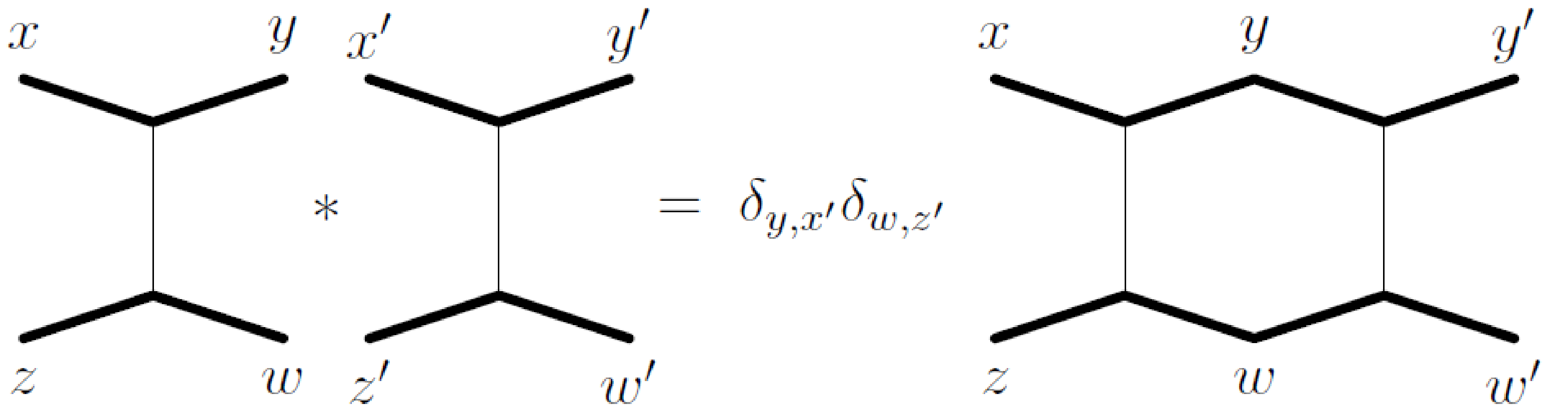
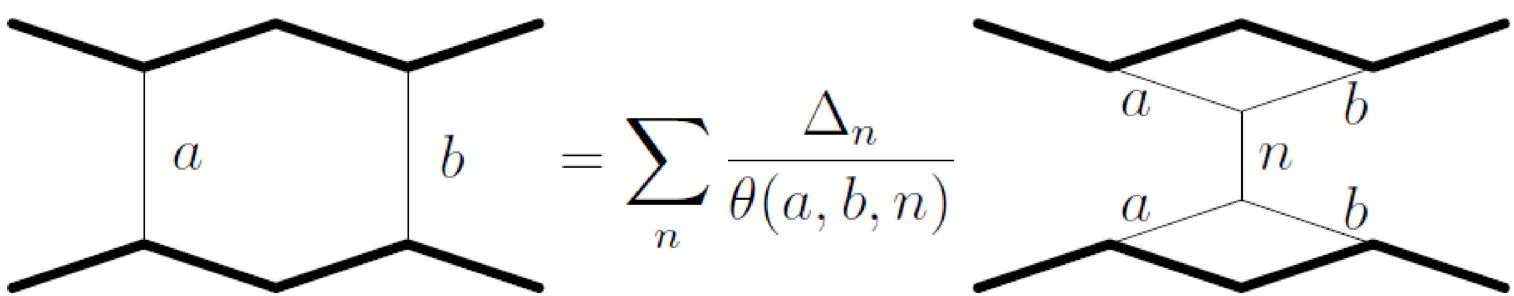
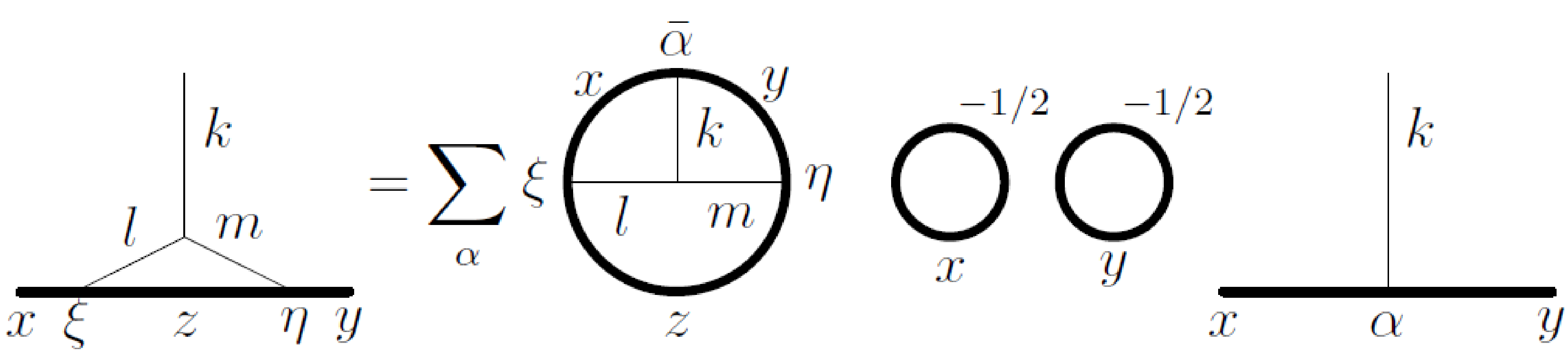
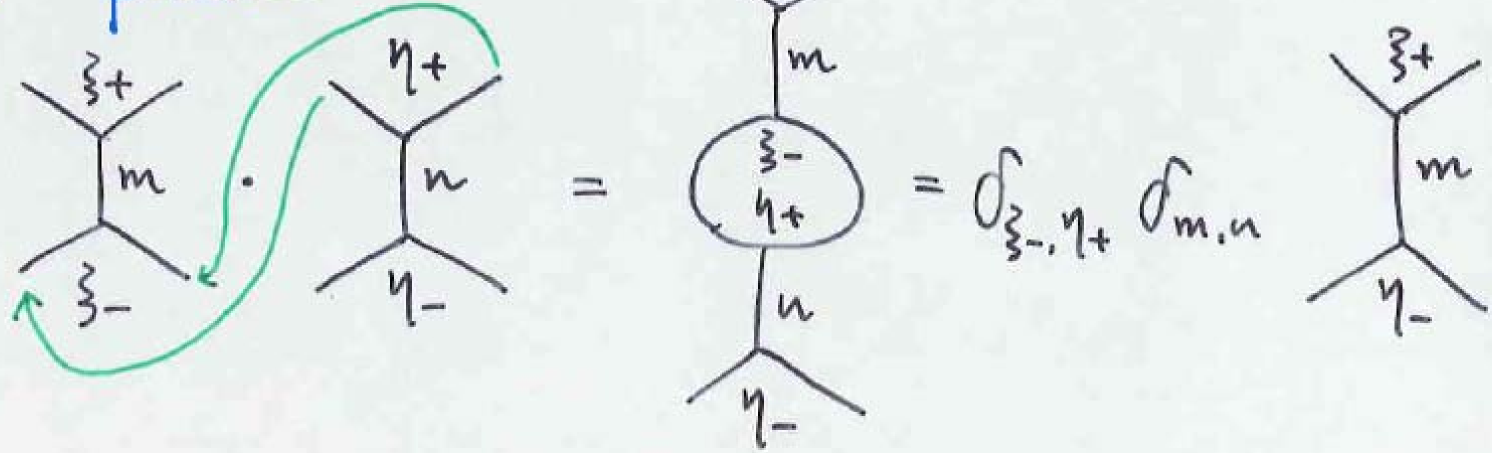


Figure 14: The convolution product on the double triangle algebra

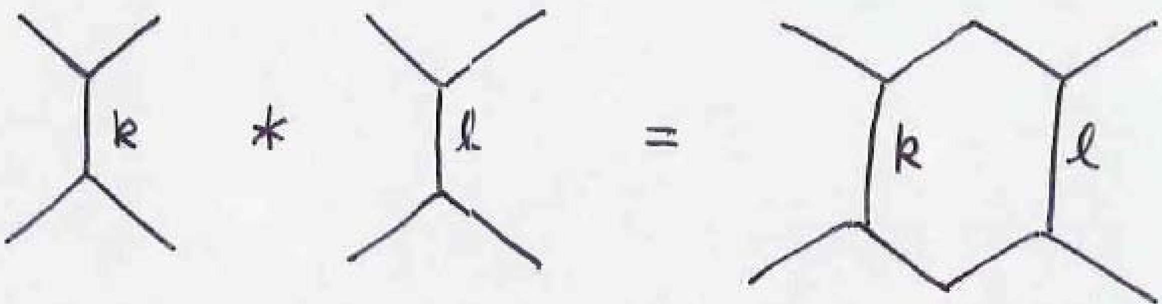


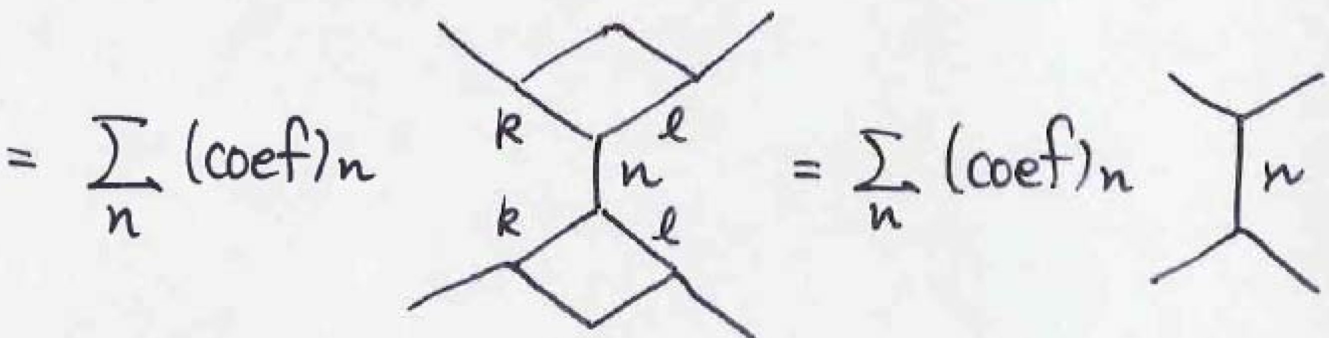


(1) · product



(2) * convolution product





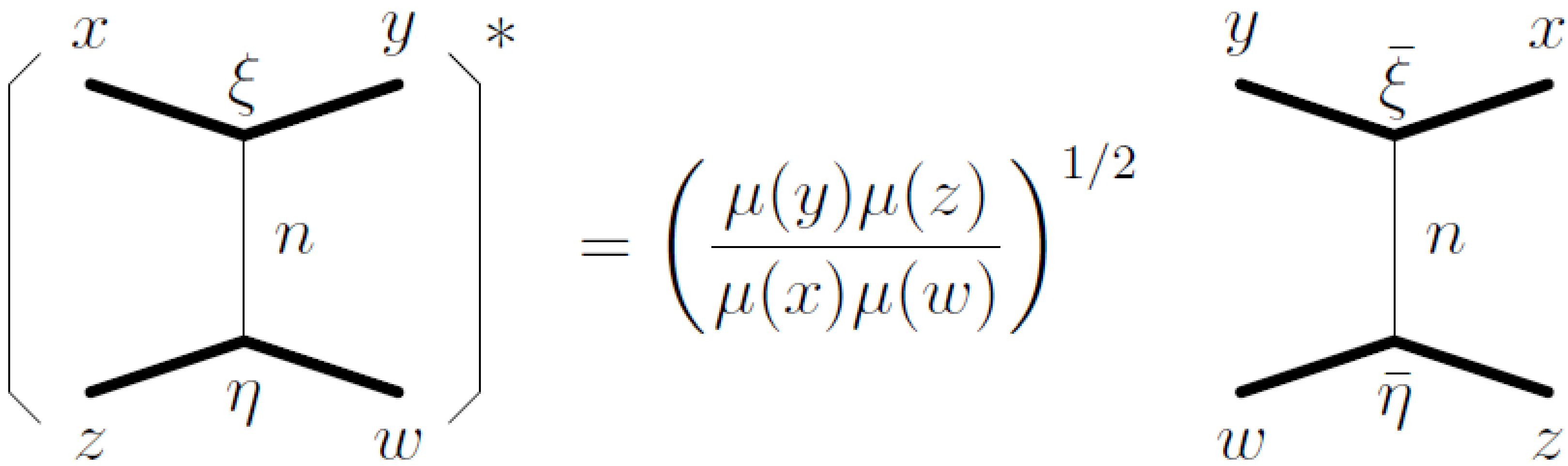


Figure 17: The $*$-operation for the convolution product

$(\mathcal{A}, *)$: K-L DTA. finite dim. C^*-alg

Ocneanu’s chiral projectors

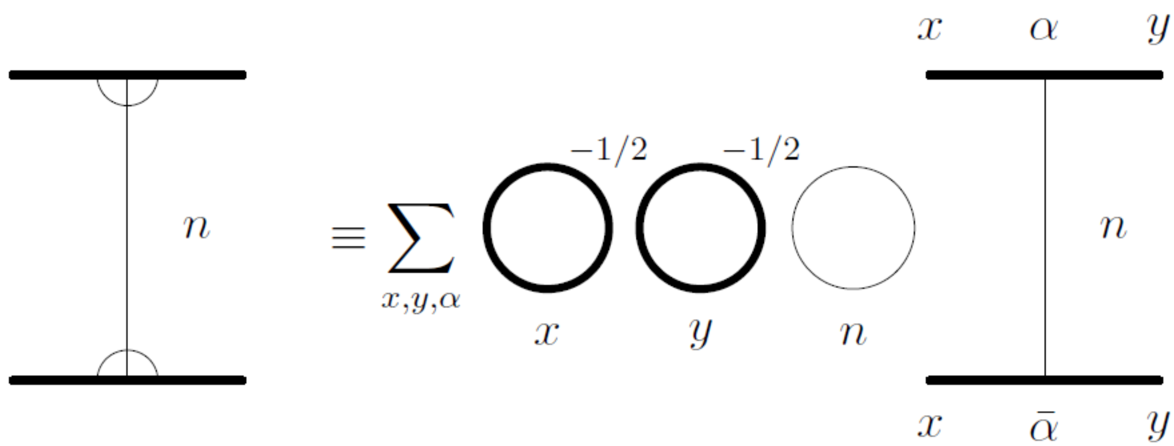


Figure 18:

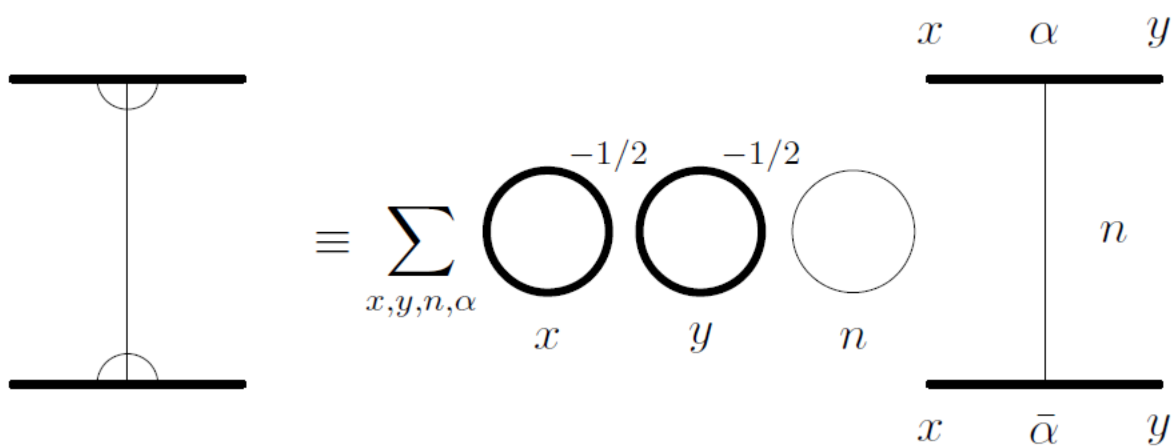
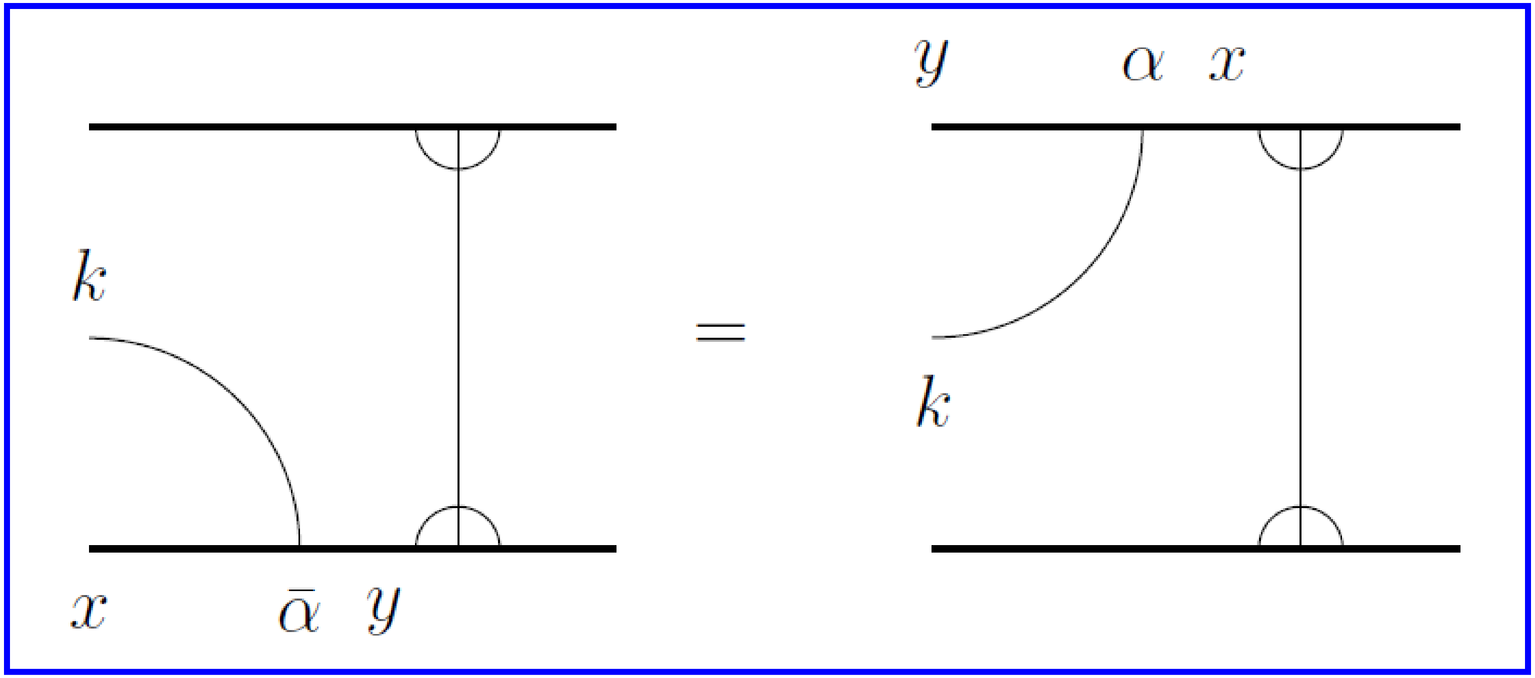
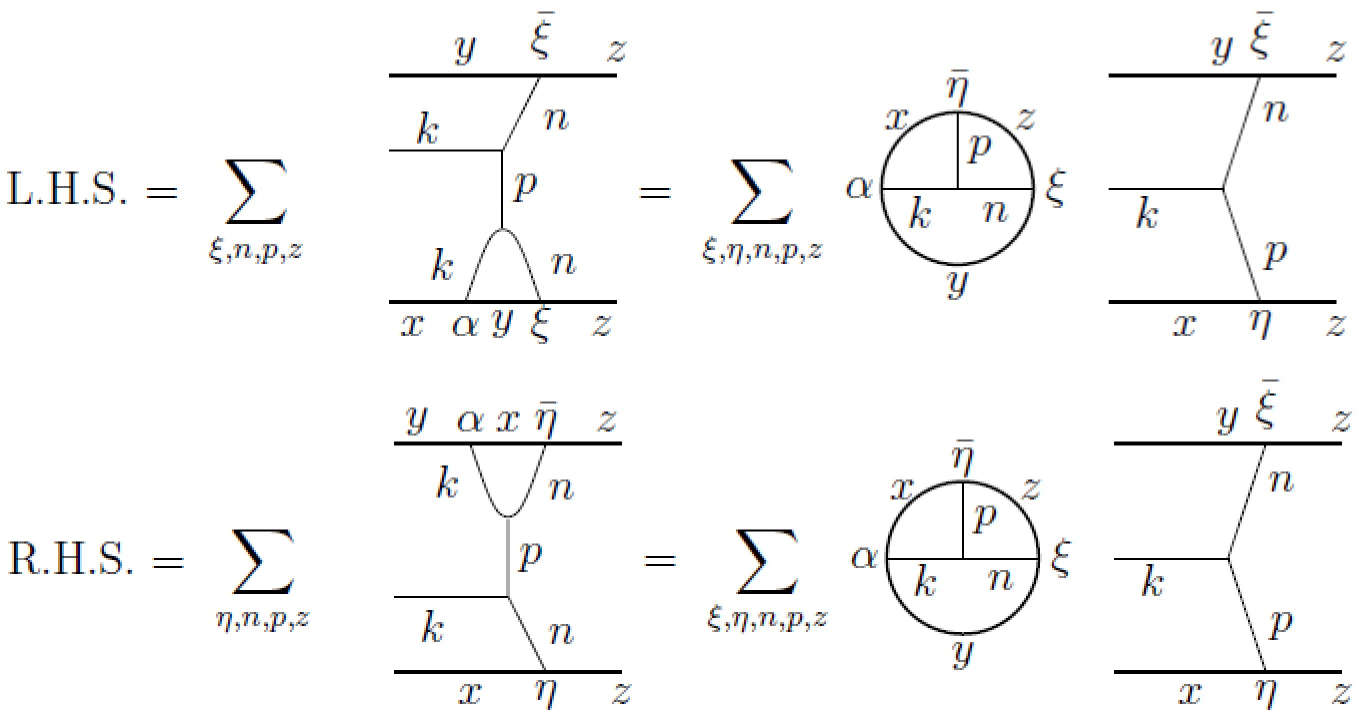


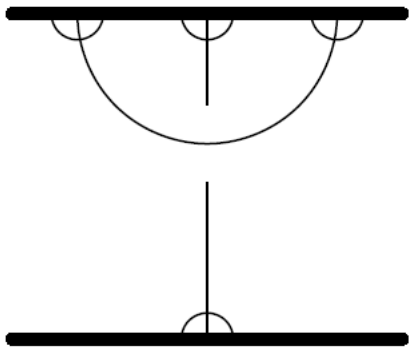
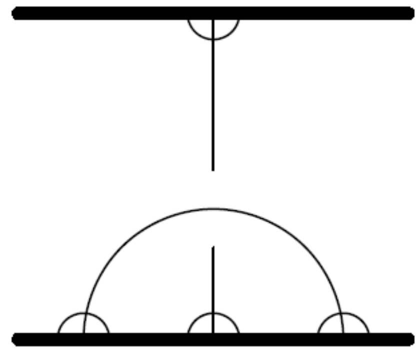
Figure 19:



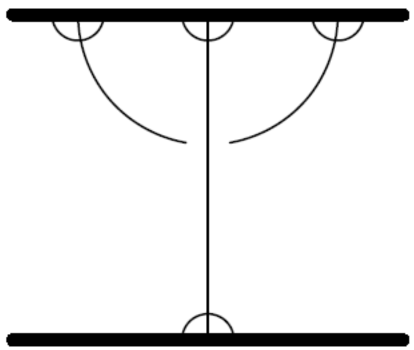
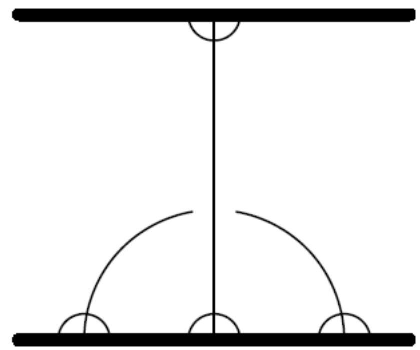
k
x
ᾱ
y
=
y
α
x
k



L.H.S. =
ξ,n,p,z
y ξ̄ z
k n
p
k n
x α y ξ z
=
ξ,η,n,p,z
η̄
x z
α p ξ
k n
y
y ξ̄ z
n
k
p
x η z
R.H.S. =
η,n,p,z
y α x η̄ z
k n
p
k n
x η z
=
ξ,η,n,p,z
η̄
x z
α p ξ
k n
y
y ξ̄ z
n
k
p
x η z

$\Psi_+ \equiv$  $=$ 

Chiral
left projector

$\Psi_- \equiv$  $=$ 

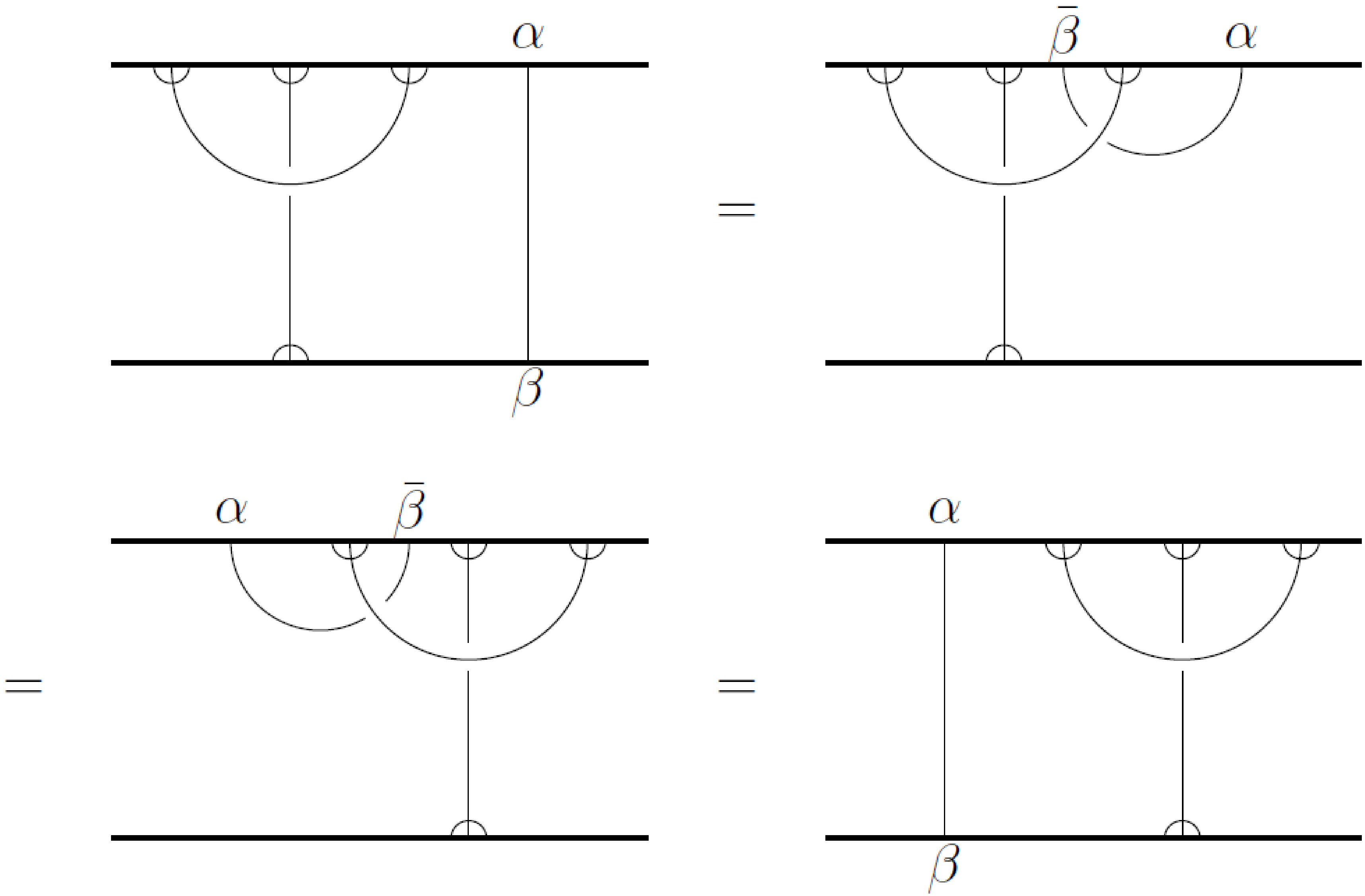
Chiral
right projector

Ambichiral projector

$$\Psi_\pm \equiv \Psi_+ * \Psi_- = \Psi_- * \Psi_+$$

Ψ_+, Ψ_-, $\Psi_\pm$ are central projections in $(\mathcal{A}, *)$

Chiral projectors are central



Gaps and minimal central projections

$$\mathrm{gap}(K) \overset{\text{def}}{=} \min\{n>0 \mid \mathrm{EssPath}^{(n)}_{a,a}K \neq 0,\ {}^{\forall}a \in \mathrm{Vert}\,K\}$$

$$0\text{-}\mathrm{gap}(K) \overset{\text{def}}{=} \min\{n>0 \mid \mathrm{EssPath}^{(n)}_{0,0}K \neq 0\}$$

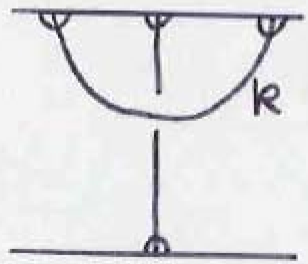
"0" represents the distinguished vertex (of K).

K	$\mathrm{gap}(K)$	$0\text{-}\mathrm{gap}(K)$
A_n	∞	∞
D_{2n+1}	∞	$4n-2$
D_{2n}	$4n-4$	$4n-4$
E_6	6	6
E_7	16	8
E_8	10	10

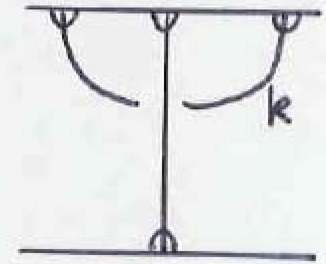
Chiral generators

K	gap(K)	0-gap(K)
A_n	∞	∞
D_{2n+1}	∞	$4n-2$
D_{2n}	$4n-4$	$4n-4$
E_6	6	6
E_7	16	8
E_8	10	10

Prop. $p_k^+ \equiv$



, $p_k^- \equiv$



are minimal central projections in $(\mathcal{A}, *)$ if $k < \frac{\mathrm{gap}(K)}{2}$.

Prop $p_k^+ \perp p_l^+$, $p_k^- \perp p_l^-$ if $k \neq l$, $k+l < \mathrm{gap}(K)$.

Prop $p_k^+ \perp p_l^-$ if $k \neq l$, $k+l < 0\text{-}\mathrm{gap}(K)$.

Strategy ： How to find all K-K irreducible connections

- Extend Kauffman–Lins’ recoupling theory
- Define double triangle algebra (DTA) with convolution product （⇒ finite dim C*–algebra)
    - minimal central projections
    - irreducible ＊-representations of K-K DTA
    - irreducible K-K connections
- （minimal central projections of DTA, ・product ）＝ a system (FRA) of K-K irreducible connections

A system (FRA) of on connections

Direct sum of connections

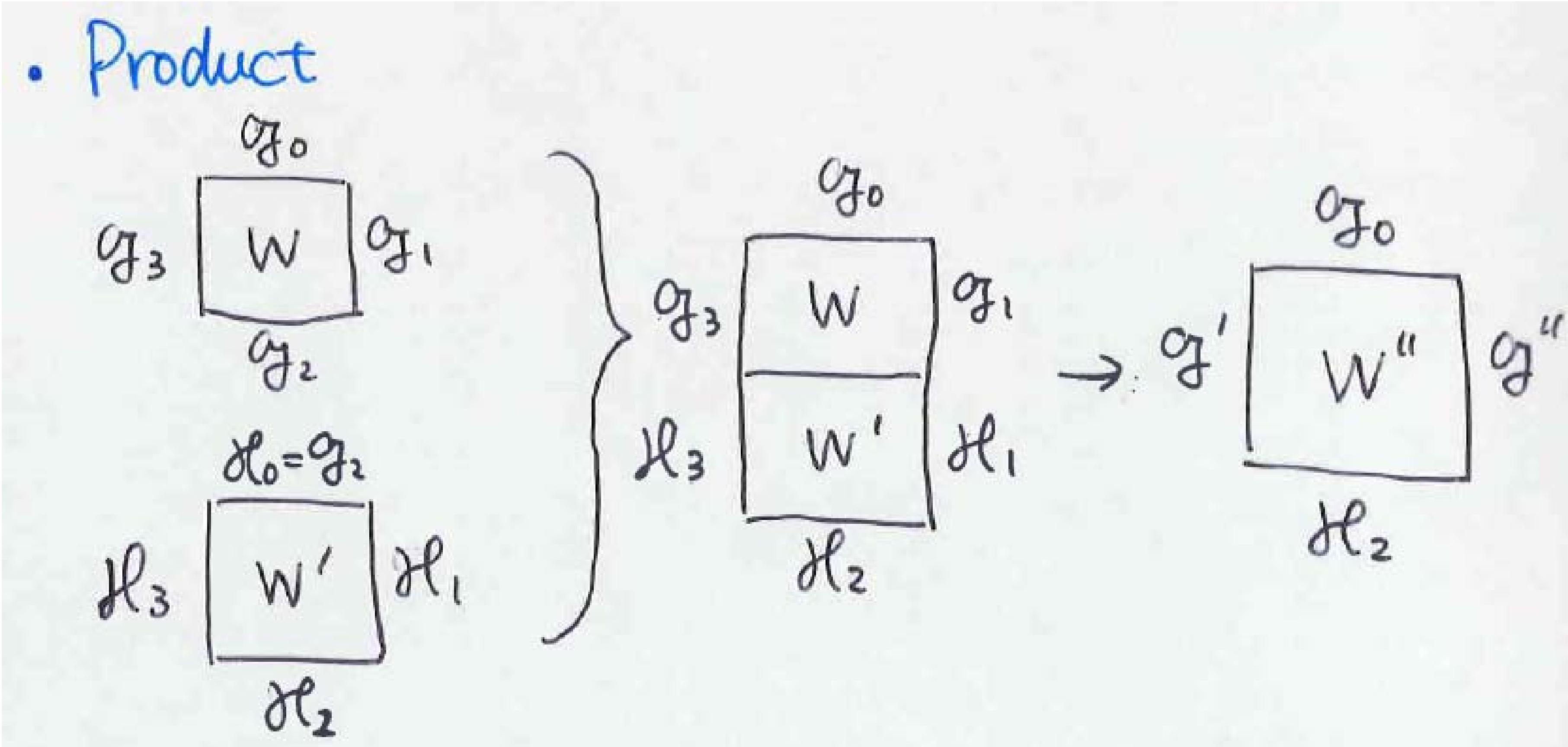
- Direct sum

$$
\begin{array}{ccc} & \mathcal{G}_0 & \\ \mathcal{G}_3 & \boxed{W_1} & \mathcal{G}_1 \\ & \mathcal{G}_2 & \end{array}
\oplus
\begin{array}{ccc} & \mathcal{G}_0 & \\ \mathcal{G}_3' & \boxed{W_2} & \mathcal{G}_1' \\ & \mathcal{G}_2 & \end{array}
\equiv
\begin{array}{ccc} & \mathcal{G}_0 & \\ \mathcal{G}_3 \sqcup \mathcal{G}_3' & \boxed{W} & \mathcal{G}_1 \sqcup \mathcal{G}_1' \\ & \mathcal{G}_2 & \end{array}
$$

$$W = W_1 \oplus W_2 .$$

$$
W\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 \downarrow & \xrightarrow{\qquad} & \downarrow \xi_1 \\ & \xrightarrow[\xi_2]{\qquad} & \end{array} \right)
=
\begin{cases}
W_1\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 \downarrow & \xrightarrow{\qquad} & \downarrow \xi_1 \\ & \xrightarrow[\xi_2]{\qquad} & \end{array} \right) & \text{if } \xi_1 \in \mathcal{G}_1 ,\ \xi_3 \in \mathcal{G}_3 \\
W_2\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 \downarrow & \xrightarrow{\qquad} & \downarrow \xi_1 \\ & \xrightarrow[\xi_2]{\qquad} & \end{array} \right) & \text{if } \xi_1 \in \mathcal{G}_1' ,\ \xi_3 \in \mathcal{G}_3' \\
0 & \text{otherwise}
\end{cases}
$$

Product of connections



Irreducibility of connections

- Reducibility, Irreducibility.

$$W \cong W_1 \oplus W_2 \quad {}^{\exists} W_1, W_2$$

↳ up to vertical gauge choice

~~total~~

Remark Vertical gauge choice = Total gauge choice if two horizontal graphs $\mathcal{G}_0, \mathcal{G}_2$ are trees.

A system (FRA) of connections

Def K : connected finite bipartite graph

$_K W_K$: a set of equiv. classes of K-K bi-unitary connections ↳ up to vertical gauge.

$_K W_K$ is called a system of K-K connections if it is closed under direct sum, product, conjugation, irreducible decomposition.

↳ renormalization

Frobenius reciprocity of connection system

Prop (Frobenius reciprocity)

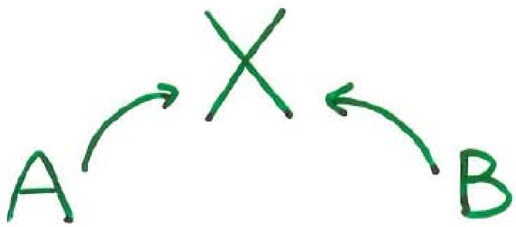
K, L, M : connected finite bipartite graphs

${}_K\alpha_L,\ {}_L\beta_M,\ {}_K\gamma_M$: connections

If γ appears m times in $\alpha \cdot \beta$,

then $\begin{cases} \alpha & \text{" in } \gamma \cdot \bar{\beta} \\ \beta & \text{" in } \bar{\alpha} \cdot \gamma \end{cases}$.

Bimodule（両側加群）



X : Hilbert space

A, B : 作用素環

群の表現	Bimodule X, Y
$\oplus$	$\oplus$　　$X \oplus Y$
$\otimes$	$\otimes_A$ 相対テンソル積　$X \otimes_A Y$
contragredient rep.	conjugate bimod.　$\overline{X}$
Frobenius reciprocity	Frobenius reciprocity.
dimension	$(\text{Jones index})^{1/2}$
既約（分解）	既約（分解）

Examples of connection system

Examples

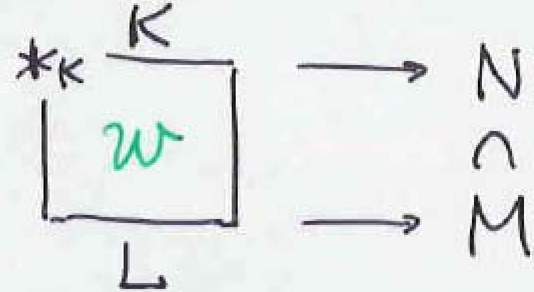
(1) trivial, all K-K connections

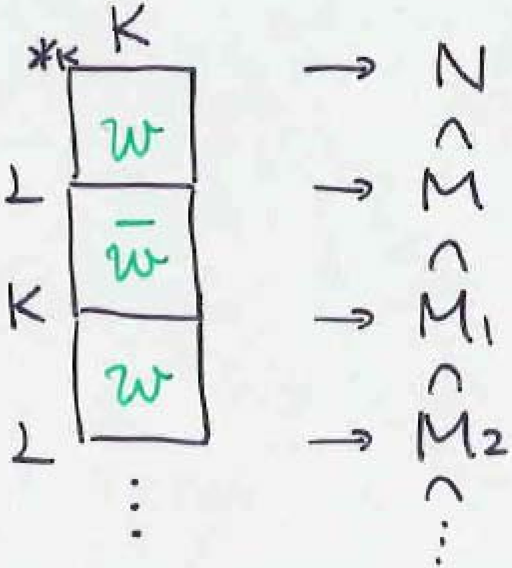
(2) NCM, K : pr.gr. / L : dual pr.gr } → Galois functor → [K-K, K-L, L-K, L-L: 4 kinds of connections]

(3) w : K-L connection, singly generated system

$w \cdot \bar{w} \cdot w \cdots$ irred. decomp. ↑ (to the 4 kinds of connections)

Given K-L connection ${}_K w_L$





${}_K(w \cdot \bar{w})^m_K$ → irred. K-K connections

${}_K(w \cdot \bar{w})^m w_L$ → irred. K-L 〃

${}_L(\bar{w} \cdot w)^m \bar{w}_K$ → irred. L-K 〃

${}_L(\bar{w} \cdot w)^m_L$ → irred. L-L 〃

We get a system of 4 kinds of connections which is closed under product, conjugation, irreducible decomposition.

Strategy : How to find all K-K irreducible connections

- Extend Kauffman–Lins' recoupling theory
- Define double triangle algebra (DTA) with convolution product (⇒ finite dim C*–algebra)
  - minimal central projections
  - irreducible ∗-representations of K-K DTA
  - irreducible K-K connections
- (minimal central projections of DTA, ・product) ＝ a system (FRA) of K-K irreducible connections

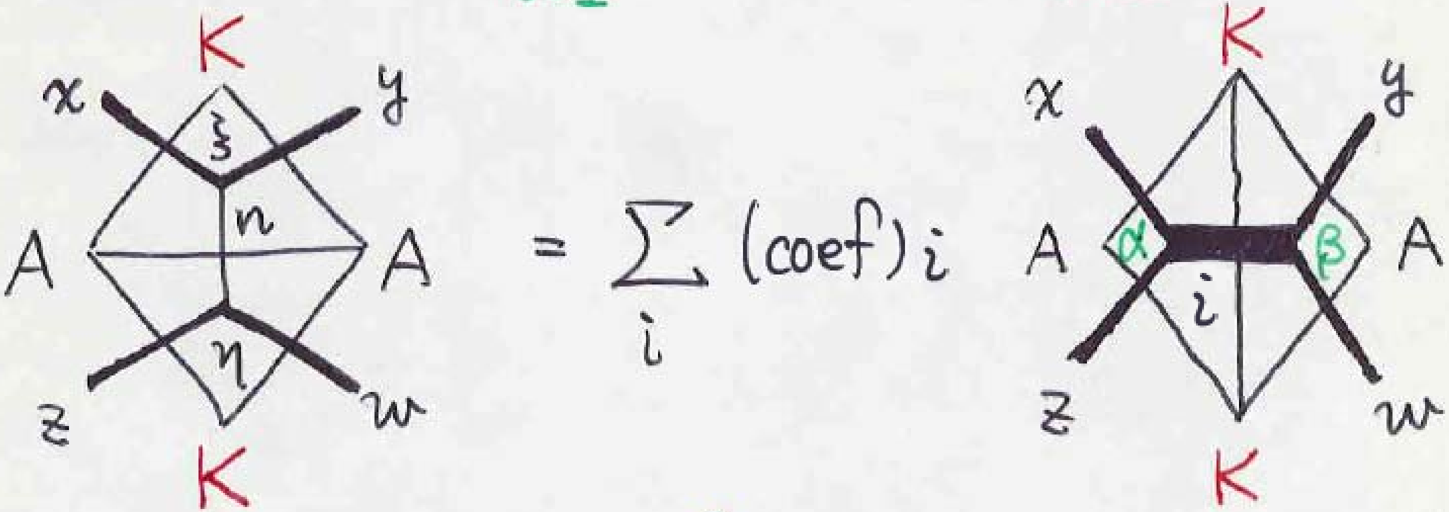
Correspondence between connections and *−representations of DTA

Extension of recoupling

K : one of the Dynkin diagram A_n. D_n. $E_{6.7.8}$

$(\mathcal{A}, *)$: DTA. finite dim C^*-alg

$$(\mathcal{A}, *) \triangleq \bigoplus_{i \in I} H_i \otimes \overline{H_i} \cong \bigoplus_{i \in I} \mathrm{End}(H_i)$$



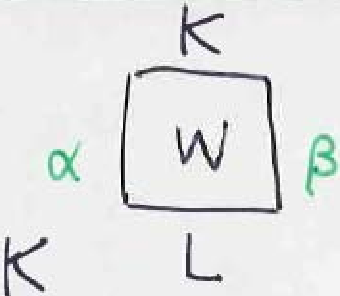
extension of recoupling

Correspondence between connections and *-representations of DTA

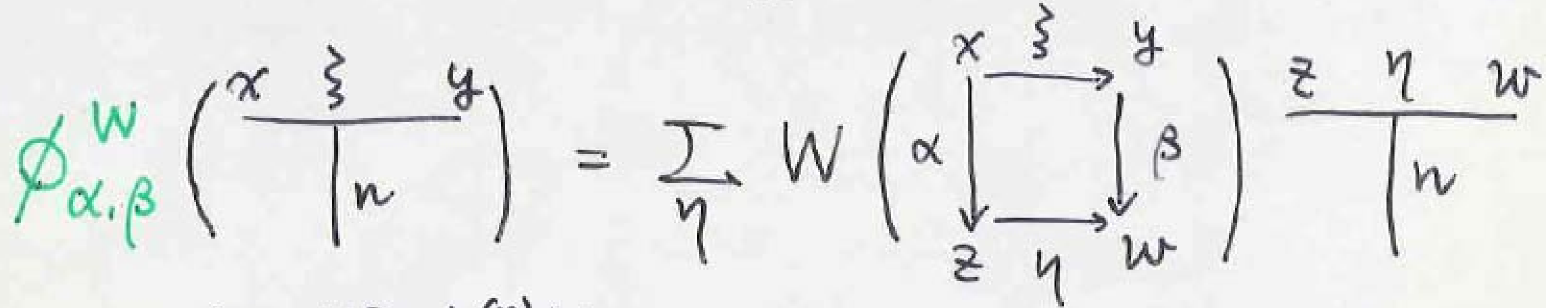
Theorem

There is a one-to-one correspondence between unitary equivalence classes of irreducible matricial *-representations of the K-K DTA and equivalence classes of irreducible K-K biunitary connections.

(proof) Given connection W



Define a map $\phi^{W}_{\alpha,\beta}$ on HPath K $= \bigoplus_n \mathrm{HPath}^{(n)} K$ by.



$\xi \in \mathrm{HPath}^{(n)} K$

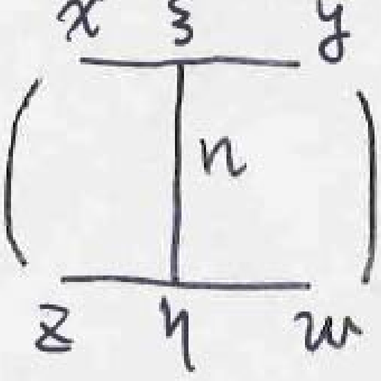
Properties

- $\phi^{W}_{\alpha\beta}(c_k(\xi)) = c_k(\phi^{W}_{\alpha,\beta}(\xi))$
- $\sum_\beta \phi^{W}_{\alpha,\beta}(\xi) \circ \phi^{W}_{\beta,\gamma}(\eta) = \phi^{W}_{\alpha,\gamma}(\xi \circ \eta)$

c_k: k-th annihilation operator

$\circ$: concatenation

Define $\Phi^W : (\mathcal{A}, *) \to Mat_{\Lambda \times \Lambda}(\mathbb{C})$



$$\left[\Phi^W \left(\text{(diagram above)} \right) \right]_{\alpha,\beta} \overset{\text{def}}{=} W \left(\begin{array}{ccc} x & \xrightarrow{\xi} & y \\ \alpha \downarrow & & \downarrow \beta \\ z & \xrightarrow[\eta]{} & w \end{array} \right)$$

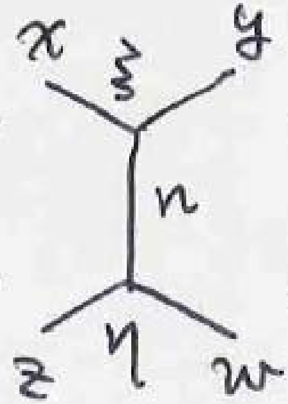
$$= \langle \phi^W_{\alpha,\beta}(\xi), \eta \rangle \quad \text{(hom)}$$

$$\Rightarrow \sum_\beta \Phi^W_{\alpha\beta}(a_1)\, \Phi^W_{\beta,\gamma}(a_2) = \Phi^W_{\alpha,\gamma}(a_1 * a_2)$$

- Renormalization rule $\Rightarrow$ ($*$-preserving)

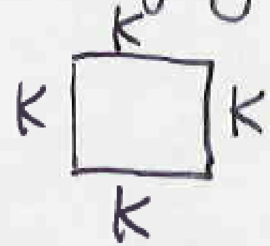
Conversely, given $*$-rep Φ of $(\mathcal{A}, *)$

$$W^{\Phi}\left(\begin{array}{ccc} x & \xrightarrow{\ \xi\ } & y \\ \alpha\downarrow & & \downarrow\beta \\ z & \xrightarrow[\ \eta\ ]{} & w \end{array}\right) \overset{\text{def}}{=} \left[\Phi\left(\quad \right)\right]_{\alpha,\beta}$$



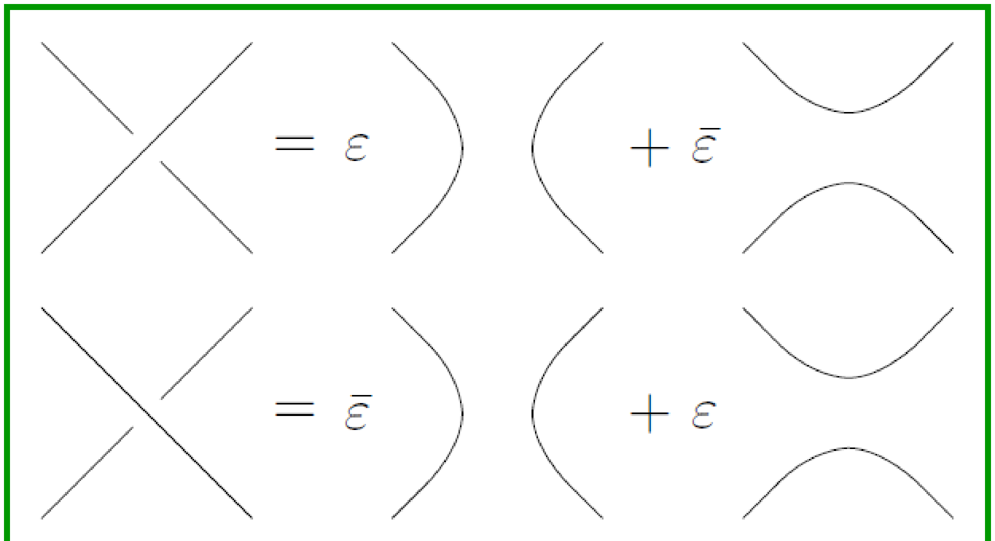
$$\left.\begin{array}{l}\text{homomorphism} \\ *\text{-preserving}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\text{bi-unitarity.} \\ \text{renormalization}\end{array}\right.$$

Corollary. The minimal central projections $p_1^{\pm}$ of the K-K DTA correspond to the two mutually complex conjugate bi-unitary connections on



, $(K \in \{A, D, E\})$

In particular $p_1^{+} = p_1^{-}$ when $K = A_n$



Corollary The fusion rule algebra of K-K bi-unitary connections is isomorphic to the center $\mathcal{Z}$ of the K-K DTA $(\mathcal{A}, *)$ with $\cdot$ product, $(\mathcal{Z}, \cdot)$.

Strategy ： How to find all K-K irreducible connections

- Extend Kauffman–Lins' recoupling theory
- Define double triangle algebra (DTA) with convolution product （⇒ finite dim C*–algebra)
    - minimal central projections
    - irreducible ＊-representations of K-K DTA
    - irreducible K-K connections
- （minimal central projections of DTA, ・product ）
  ＝ a system (FRA) of K-K irreducible connections

Classification of irreducible bi-unitary connections on the Dynkin diagrams

Classification of irreducible A-K connections

A : Dynkin diagram A_n, $K \in \{A_n, D_n, E_{6,7,8}\}$

Prop.

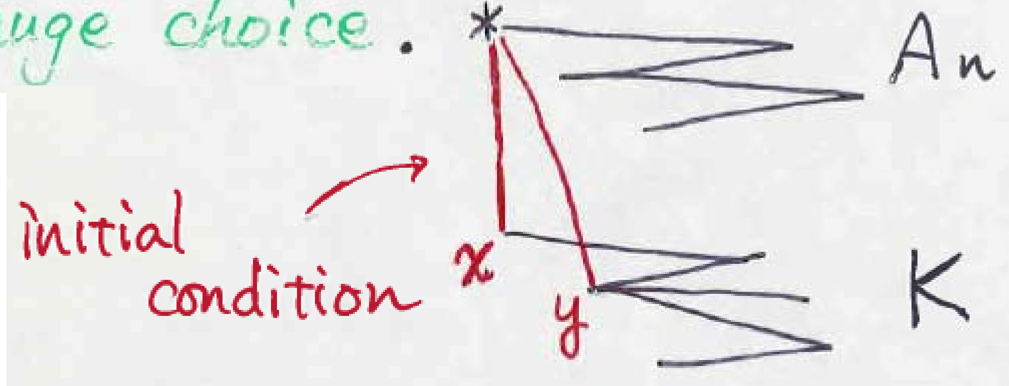
$$\mathcal{G}_3 \;\begin{array}{c} A \\ \boxed{\;W\;} \\ K \end{array}\; \mathcal{G}_1$$

: bi-unitary connection W

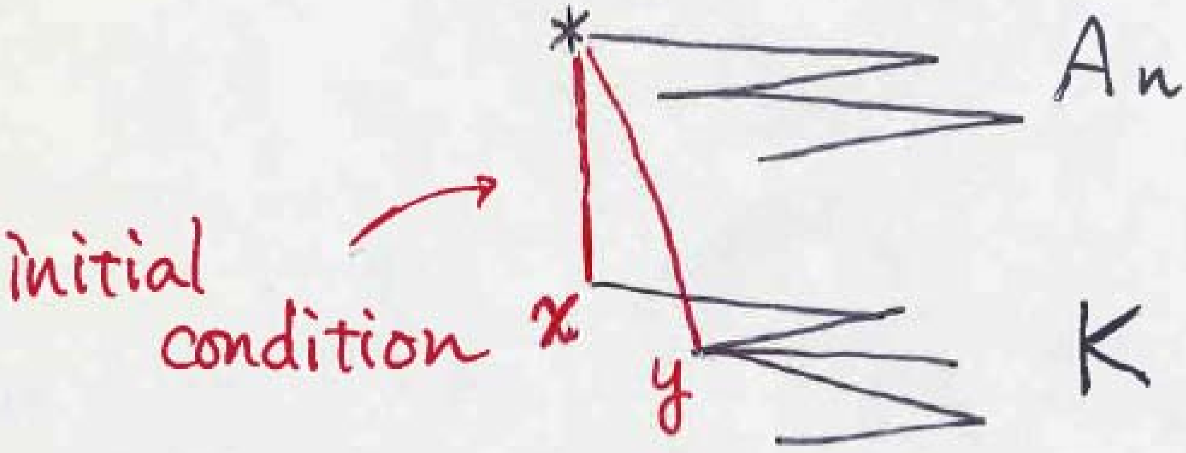
The vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are uniquely determined by the initial condition, i.e.,

the condition of edges connected to the distinguished vertex of A.

Moreover such a connection is unique up to vertical gauge choice.



Moreover such a connection is unique up to vertical gauge choice.



(proof) (1) dimension estimate
or (2) estimate of global index.

Global index for a system of connections

For a system of (singly generated) K-K bi-unitary connections $\{{}_K w_K\}$, we define global index for the system by

$$\sum_{{}_K w_K} [{}_K w_K]$$

Here $[{}_K w_K]$ represents the index of the bimodule corresponding to a connection ${}_K w_K$ and the summation runs over all irreducible K-K connections in the system.

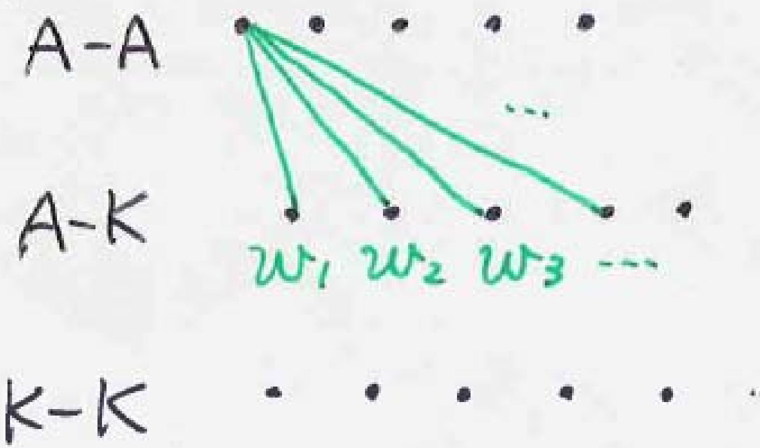
Theorem $K \in \{A_n, D_n, E_{6,7,8}\}$

The number of all equiv. classes of irreducible A-K and K-K connections are finite.

Moreover, they have the same global index as that of the system of all irreducible A-A bi-unitary connections.

$$\left(|{}_A Z_A| = |{}_A Z_K| = |{}_K Z_K| \right)$$

(proof)



$${}_A w_K = \bigoplus_i {}_A w^i{}_K$$

direct sum of all irred. A-K connections.

$$\begin{cases} {}_A w_K \cdot {}_K \overline{w}_A \to \text{all A-A} \\ {}_K \tilde{w}_A \cdot {}_A w_K \to \text{all K-K} \end{cases}$$

singly generated ⇒ global indices coincide.

Theorem $K \in \{A_n, D_n, E_{6,7,8}\}$

There is a one-to-one correspondence between vertices of the graph K and equivalence classes of irreducible A-K connections.

Classification of irreducible K–K connections

Coset decomposition

$K \in \{D_n, E_{6,7,8}\}$

Prop. (coset decomposition)

${}_K\mathcal{Z}_K$: the system of all K-K connections

$\cup$

$\mathcal{B}$: a fusion rule subalgebra

Then $\mathcal{Z}$ decomposes into left and right cosets of $\mathcal{B}$. (disjoint)

$$ {}_K\mathcal{Z}_K = \bigcup_{x \in X} x \cdot \mathcal{B} = \bigcup_{y \in Y} \mathcal{B} \cdot y $$

(proof) By Frobenius reciprocity.

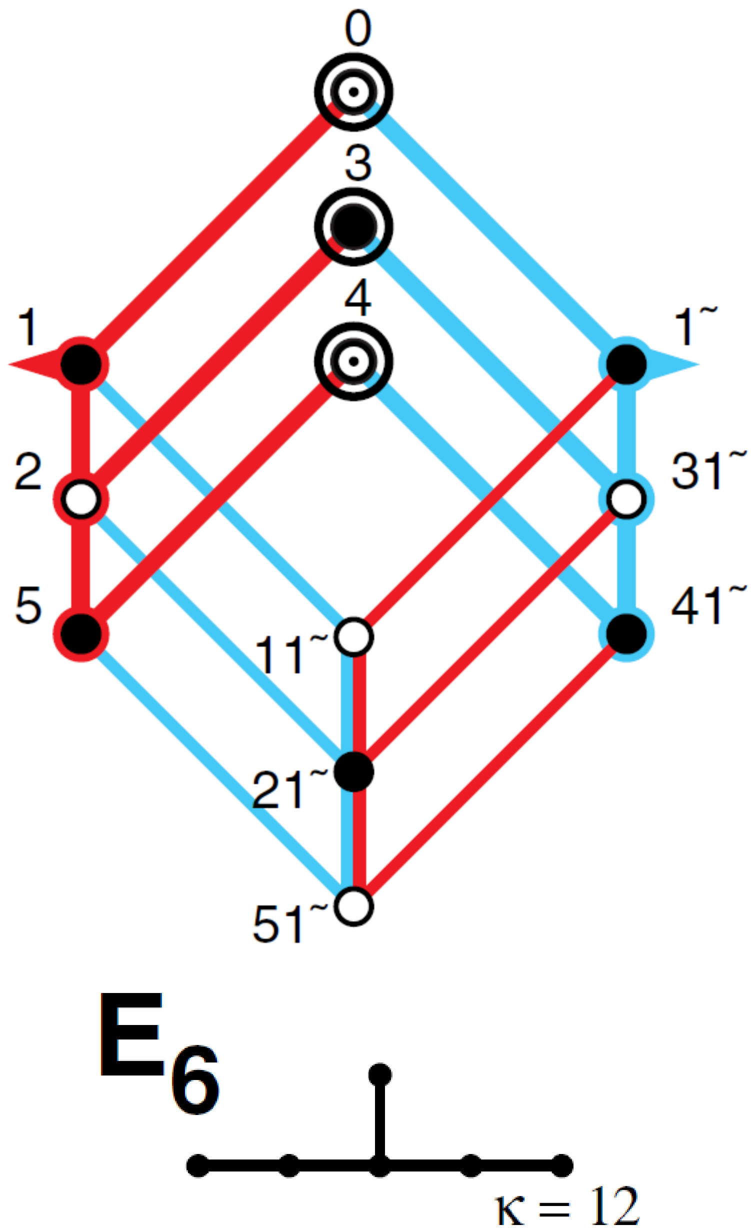
Classification of irreducible K–K connections

- minimal central projections $p_K^{\pm}$ (gap(K))
- global index
- coset decomposition.
- comparison of indices of connections
- decomposition rule ⇒ fusion rule graph

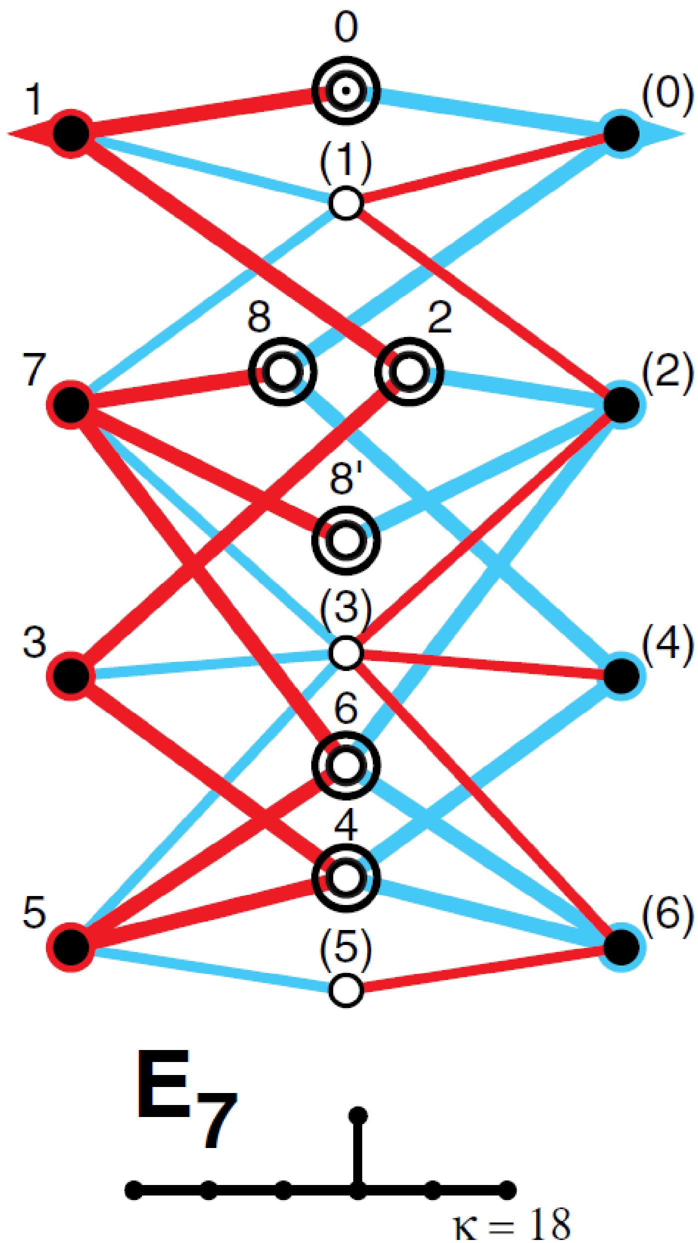
etc.

⇒ We can classify all irred. K-K connections and we get a system of all K-K connections ${}_K\Sigma_K$.

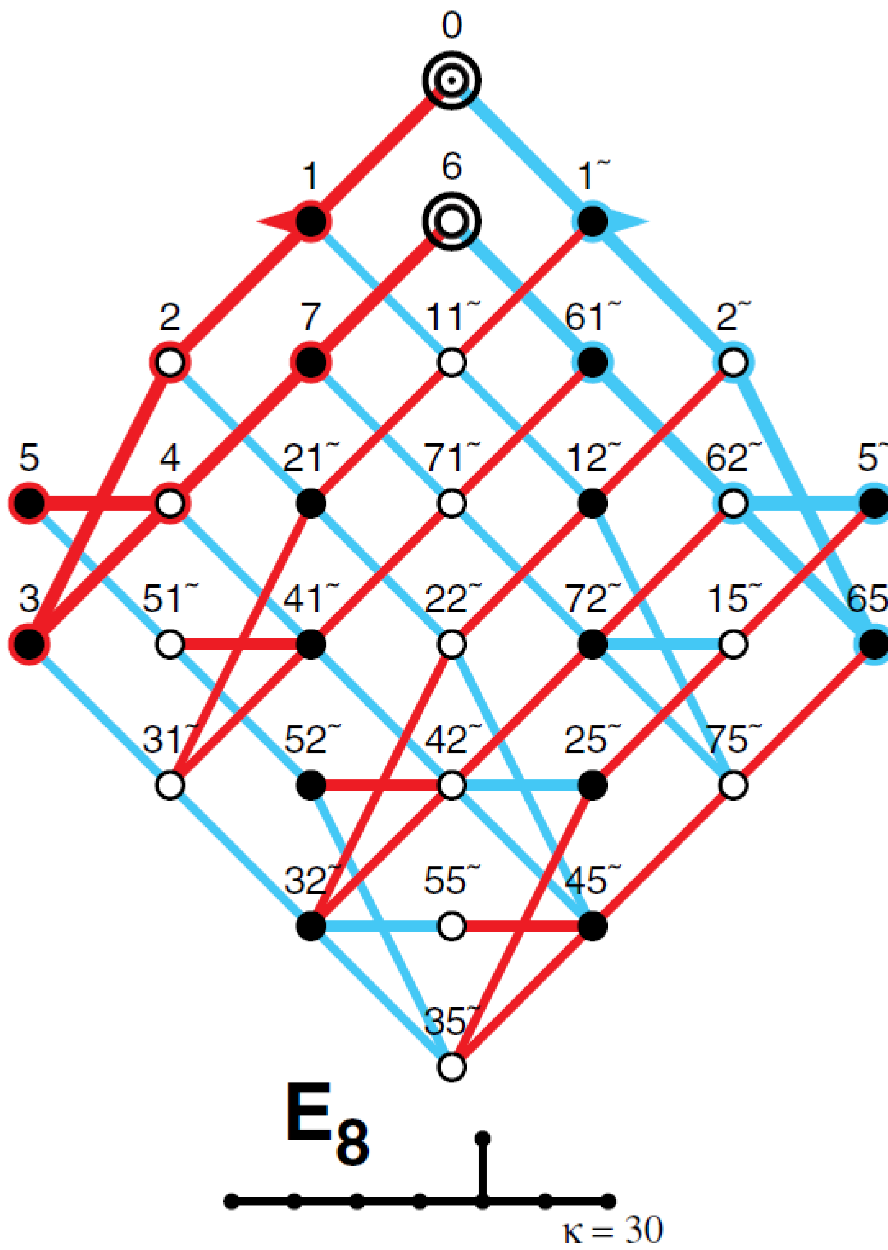
QUANTUM SYMMETRY FOR COXETER GRAPHS



QUANTUM SYMMETRY FOR COXETER GRAPHS



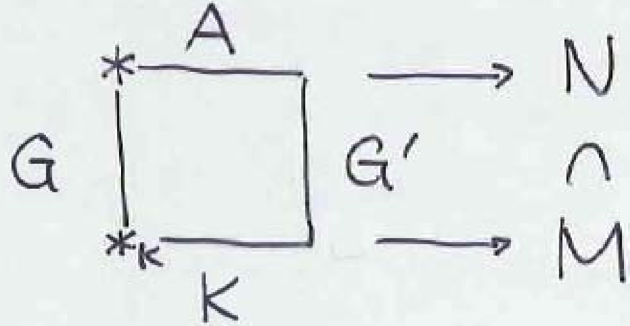
QUANTUM SYMMETRY FOR COXETER GRAPHS



Classification of quantum subgroups of quantum SU(2)

Goodman–de la Harpe–Jones subfactors

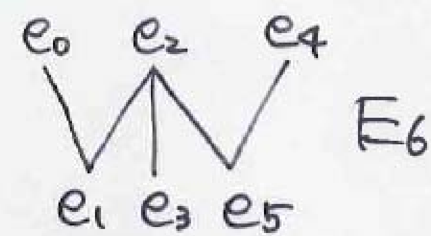
- GHJ subfactors

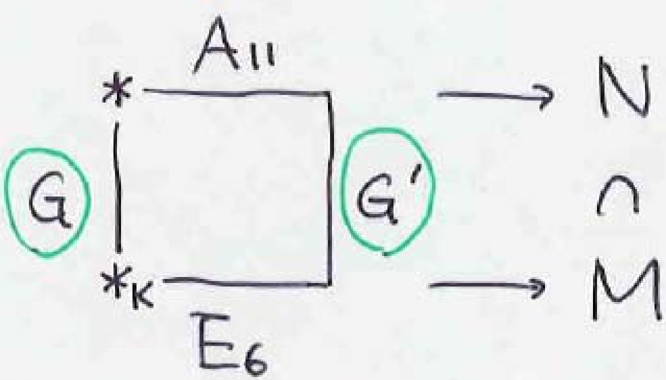


GHJ$(K, *_K)$
↑ choice of vertices of K

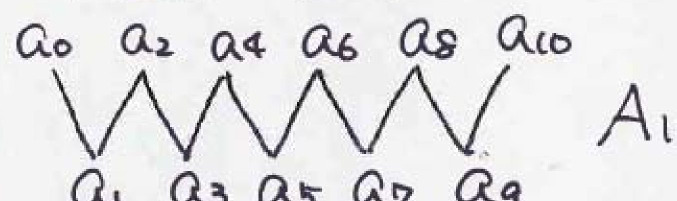
$K \in \{A, D, E\}$

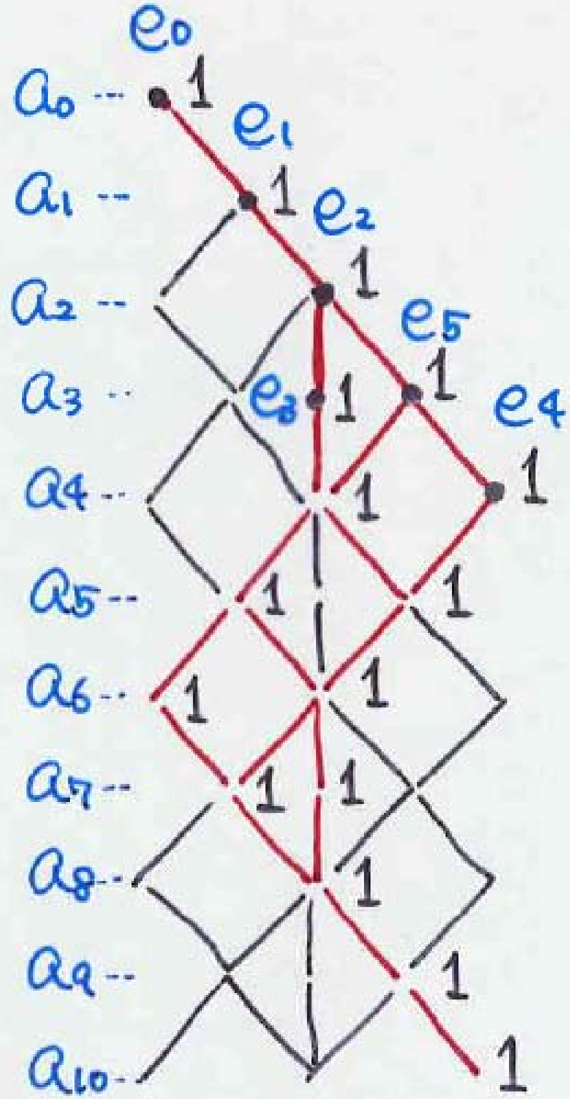
Example GHJ$(E_6, * = e_0)$



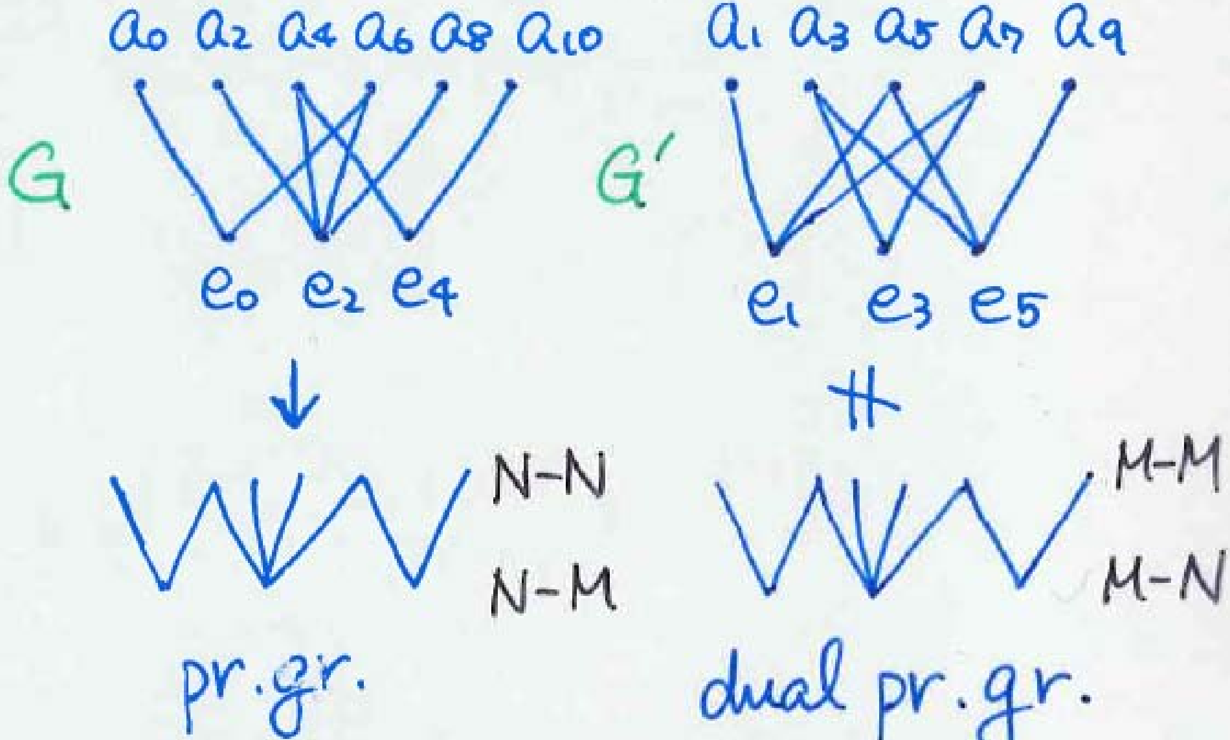


index $= 3 + \sqrt{3}$





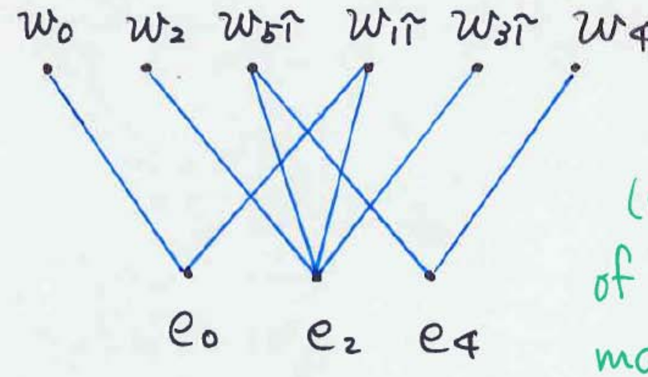
Vertical graphs G and G' can be easily obtained by looking at the essential paths.



- The dual principal graphs.

Example GHJ(E_6, $*=e_0$)

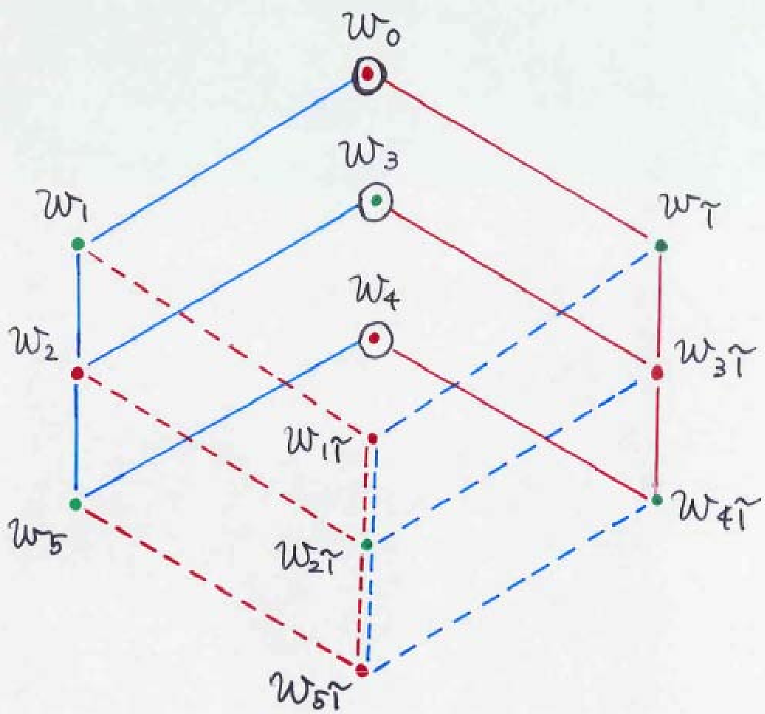
index $= 3+\sqrt{3}$



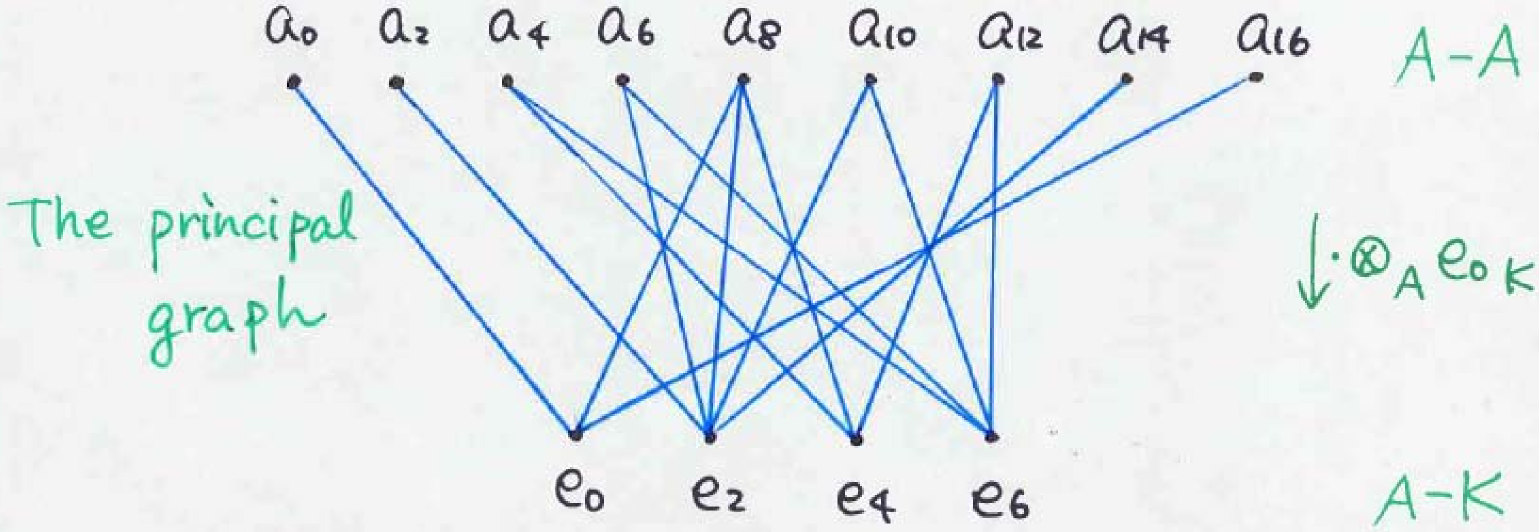
# of edges w_i — e_k

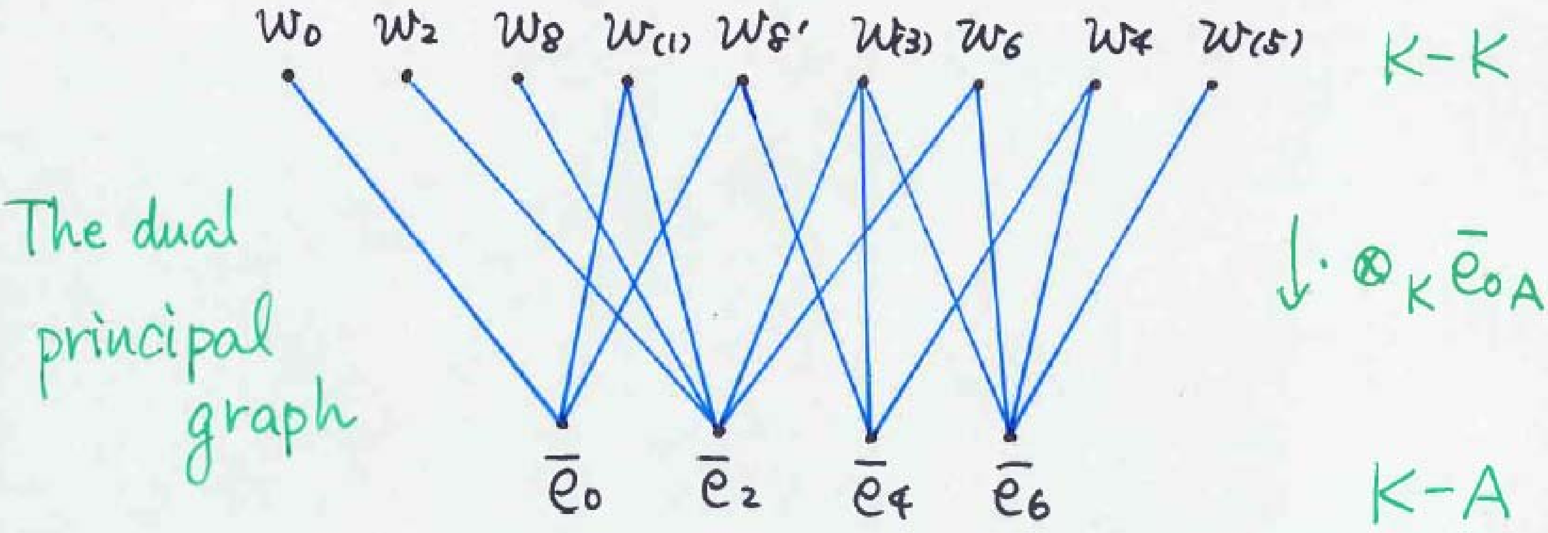
$\|$

(e_0, e_k)-th entry of the incidence matrix.

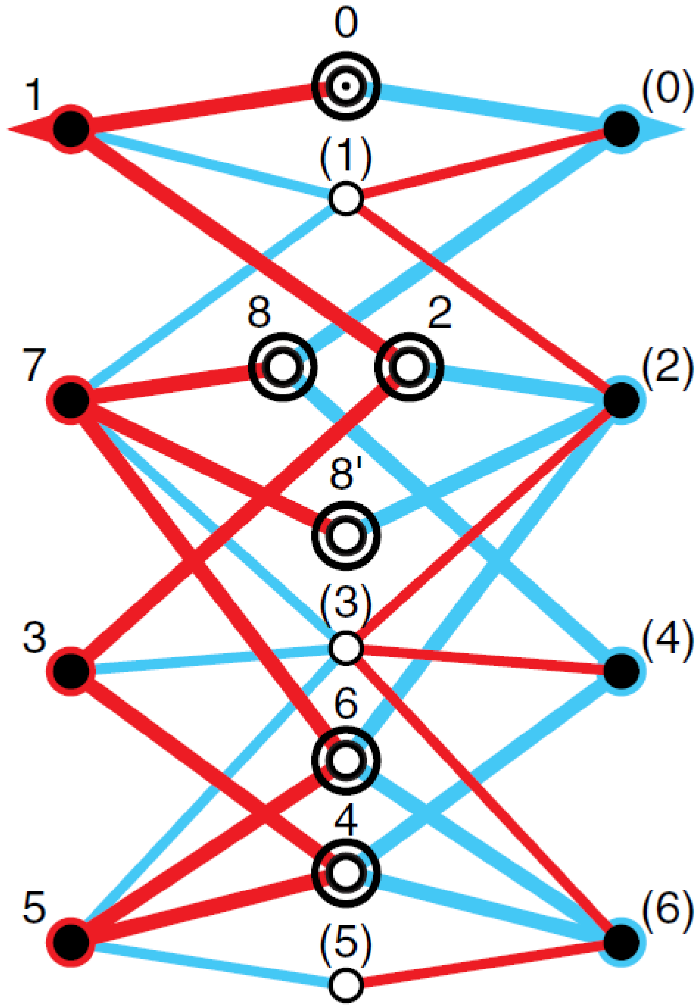


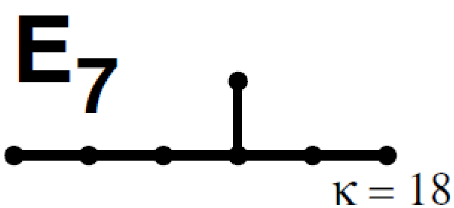
Example $GHJ(E_7, *_K = e_0)$ index $\fallingdotseq 7.759$



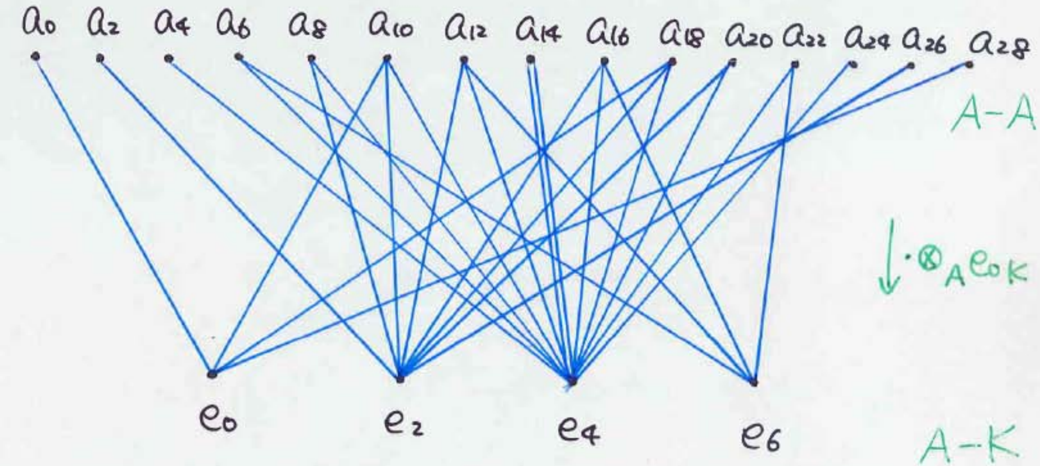


QUANTUM SYMMETRY FOR COXETER GRAPHS

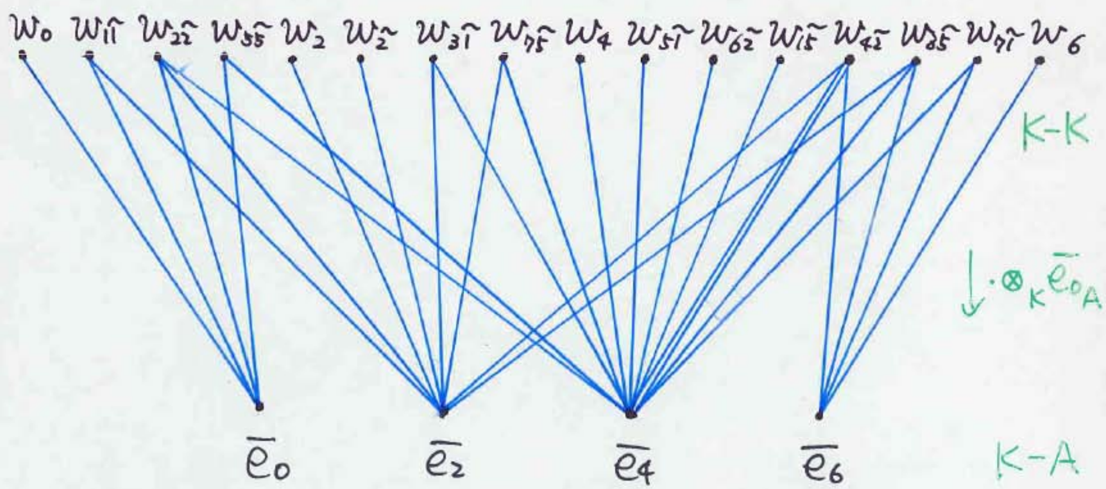




Example GHJ(E_8, $*_K = e_0$) index ≒ 19.48

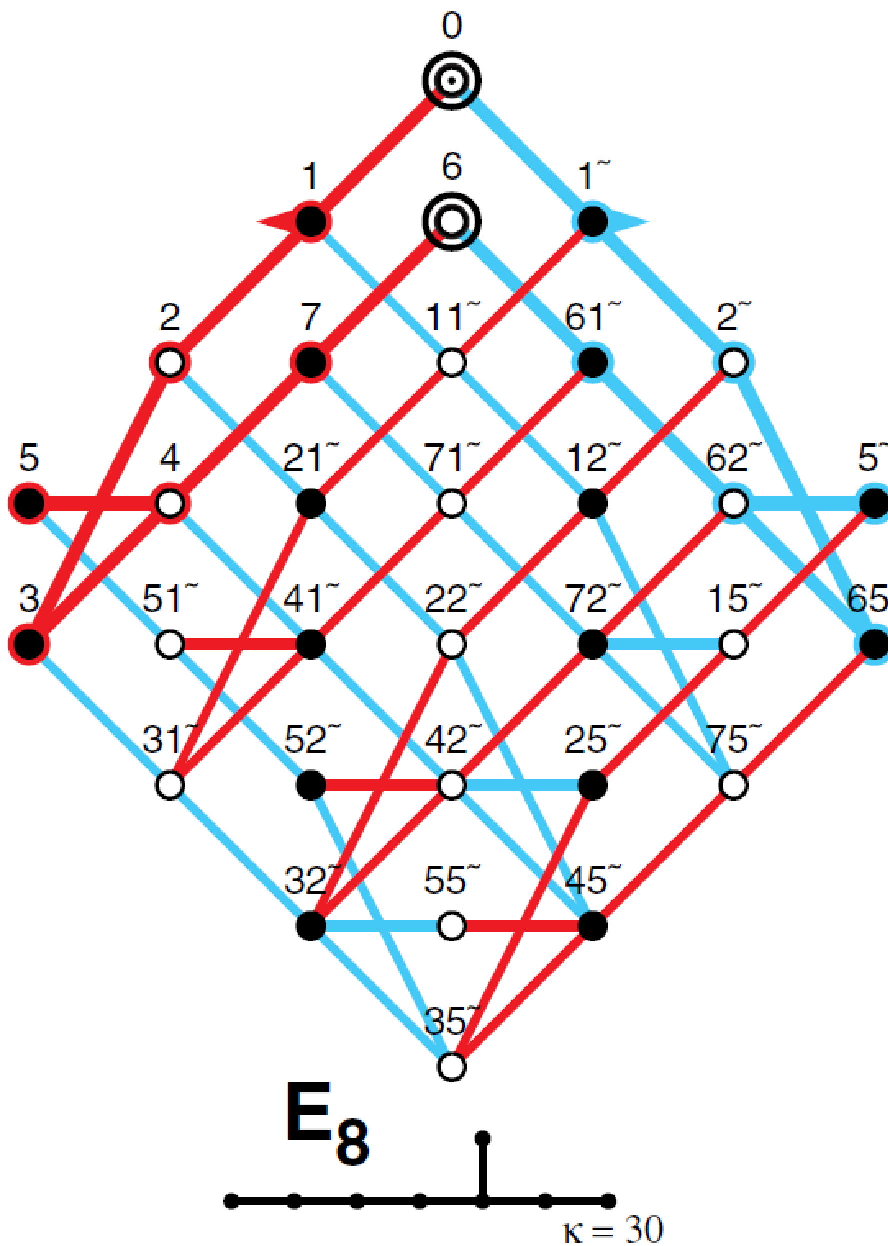


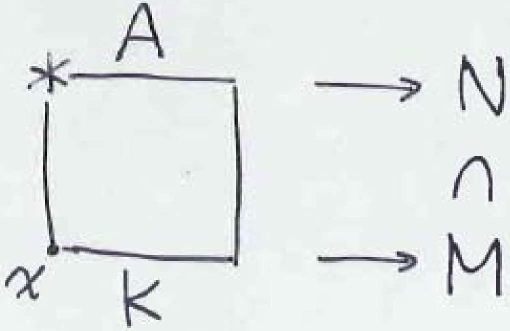
The principal graph



The dual principal graph

QUANTUM SYMMETRY FOR COXETER GRAPHS





$GHJ(K, *_K = x)$

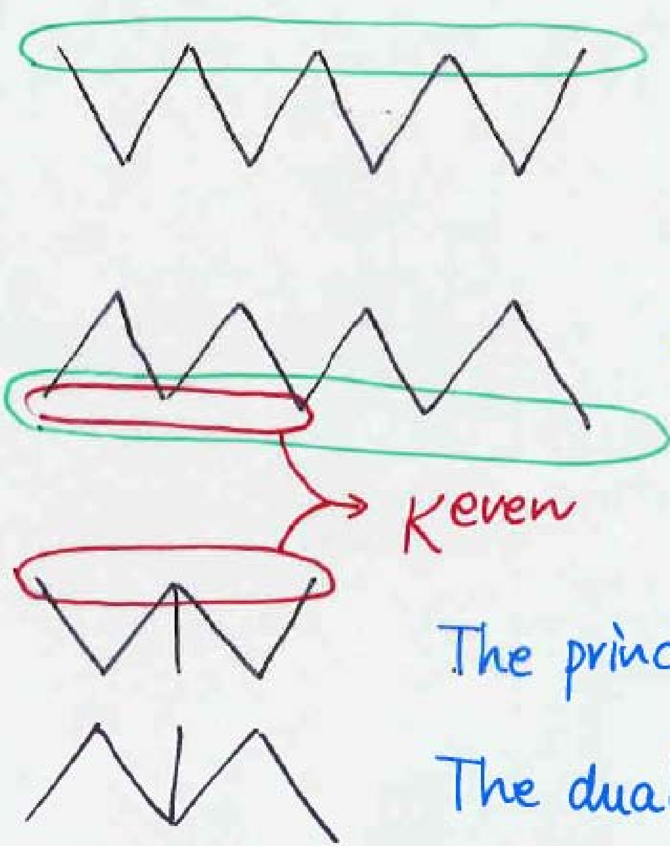
Take $*_K = x$ so that the index $\neq 2$.

N-N

N-M

M-N

M-M



$\cong A_n^{even}$

principal graph

dual pr. gr.

$\cong {}_K\Sigma_K^{even}$

The principal graph K

The dual pr. gr. K

Classification of quantum subgroups of quantum SU(2)

Corollary We have the following subequivalent paragroups.

These are strictly subequivalent. (not equivalent)

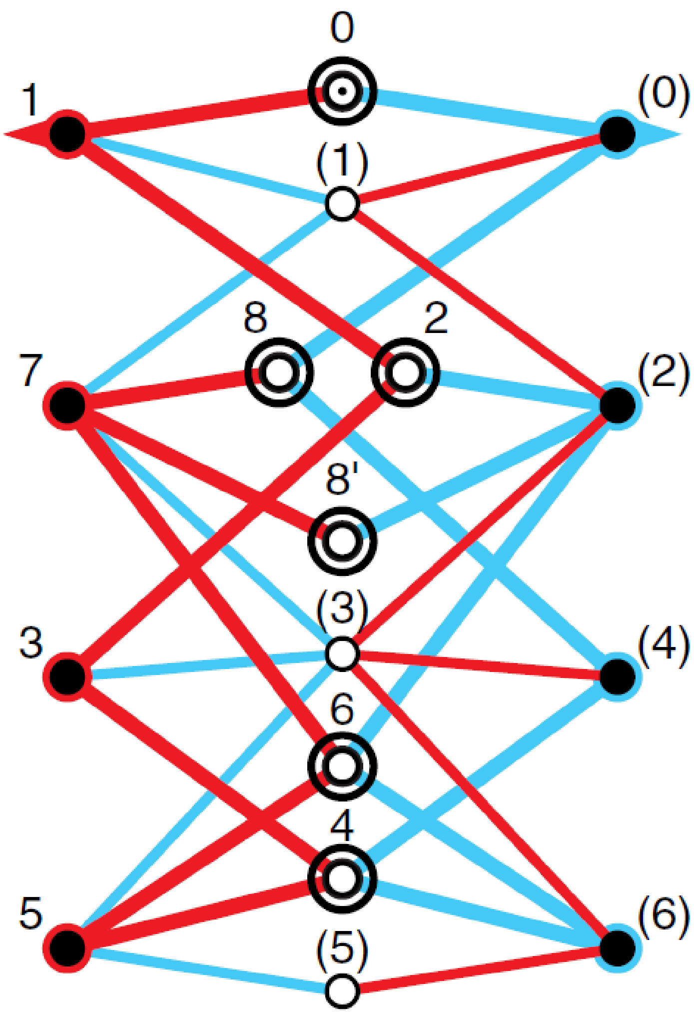
$A_{4n-3} \gnsim D_{2n}$

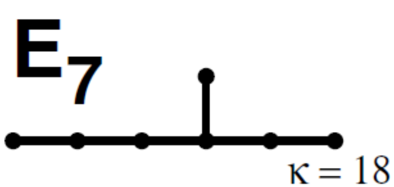
$A_{11} \gnsim$ ~~D_7~~ E_6

$A_{17} \gnsim D_{10}$, ~~E_7~~

$A_{29} \gnsim D_{16}, E_8$

From Ocneanu Graph to Modular invariants





$$
E_7 \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Ocneanu Graph の描き方（1）

- Ocneanuのオリジナルの方法
- Connection (bimodule) の system (fusion rule algebra, 略してFRA) による方法

- SU(2) の場合（modular invariant が不要）
- SU(N) (N≧3) の場合は，SU(2)より複雑で SU(2)とは異なる新しいアイディアが必要．
（modular invariant の分類を使用）

Ocneanu Graph の描き方（2）

- Boeckenhauer -Evans-Kawahigashi による方法
- Sector のFRAによる.
- Conformal Field Theory (CFT), WZW model, confomal inclusion
- Braiding, α-induction
- Modular invariant
- SU(N) (N≧３) で実行可能

Braiding, α-induction

a unitary operator $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$

Definition 2.2. *We say that a system* Δ *of endomorphisms is* **braided** *if for any pair* $\lambda, \mu \in \Delta$ *there is a unitary operator* $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$ *subject to initial conditions*

$$\varepsilon(\mathrm{id}_A, \mu) = \varepsilon(\lambda, \mathrm{id}_A) = \mathbf{1},$$

and whenever $t \in \mathrm{Hom}(\lambda, \mu\nu)$ *we have the braiding fusion equations (BFE's)*

$$\begin{aligned} \rho(t)\,\varepsilon(\lambda, \rho) &= \varepsilon(\mu, \rho)\,\mu(\varepsilon(\nu, \rho))\,t, \\ t\,\varepsilon(\rho, \lambda) &= \mu(\varepsilon(\rho, \nu))\,\varepsilon(\rho, \mu)\,\rho(t), \\ \rho(t)^*\,\varepsilon(\mu, \rho)\,\mu(\varepsilon(\nu, \rho)) &= \varepsilon(\lambda, \rho)\,t^*, \\ t^*\,\mu(\varepsilon(\rho, \nu))\,\varepsilon(\rho, \mu) &= \varepsilon(\rho, \lambda)\,\rho(t)^*, \end{aligned}$$

for any $\lambda, \mu, \nu \in \Delta$.

Braiding, α-induction

$$\varepsilon(\mathrm{id}_A, \mu) = \varepsilon(\lambda, \mathrm{id}_A) = \mathbf{1},$$

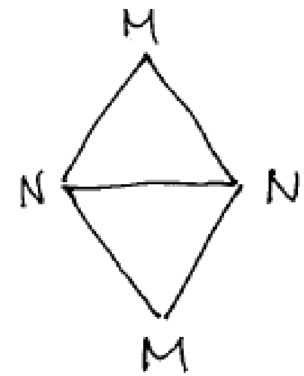
$$\begin{aligned}
\rho(t)\,\varepsilon(\lambda,\rho) &= \varepsilon(\mu,\rho)\,\mu(\varepsilon(\nu,\rho))\,t\,,\\
t\,\varepsilon(\rho,\lambda) &= \mu(\varepsilon(\rho,\nu))\,\varepsilon(\rho,\mu)\,\rho(t),\\
\rho(t)^*\,\varepsilon(\mu,\rho)\,\mu(\varepsilon(\nu,\rho)) &= \varepsilon(\lambda,\rho)\,t^*,\\
t^*\,\mu(\varepsilon(\rho,\nu))\,\varepsilon(\rho,\mu) &= \varepsilon(\rho,\lambda)\,\rho(t)^*,
\end{aligned}$$

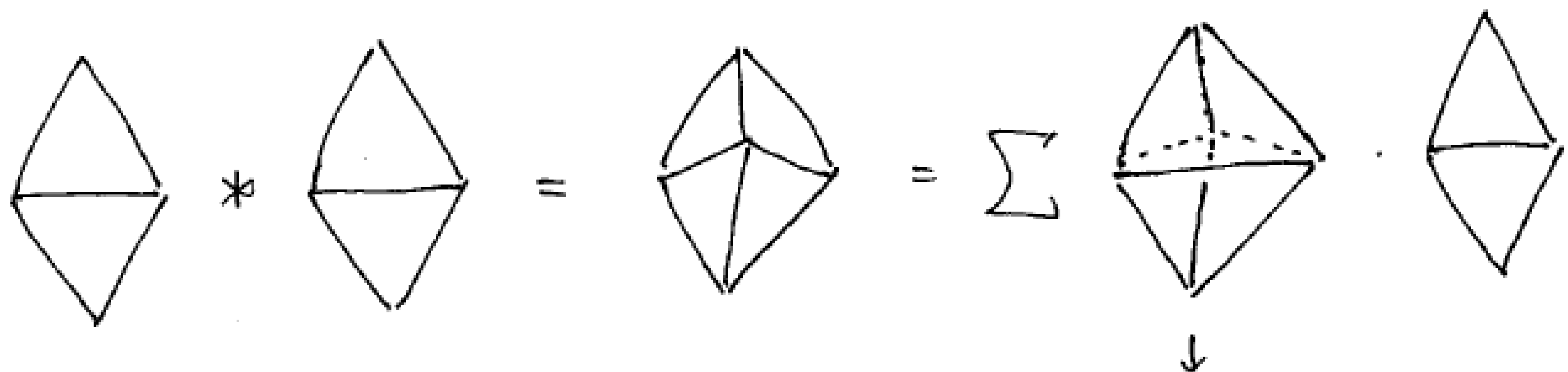
$$\varepsilon^-(\lambda,\mu) = (\varepsilon^+(\mu,\lambda))^*,\ \varepsilon^+(\mu,\lambda) \equiv \varepsilon(\mu,\lambda)$$

$$\alpha_\lambda^\pm = \bar{\iota}^{-1} \circ \mathrm{Ad}(\varepsilon^\pm(\lambda,\theta)) \circ \lambda \circ \bar{\iota}.$$

Problem   N-N, N-M, M-N given ⇒ find M-M

Double triangle algebra $\mathcal{A}$





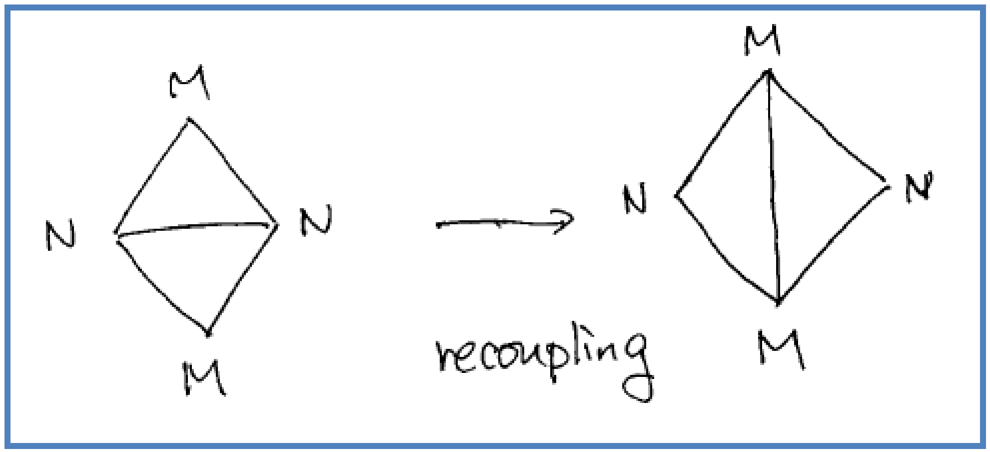
convolution product

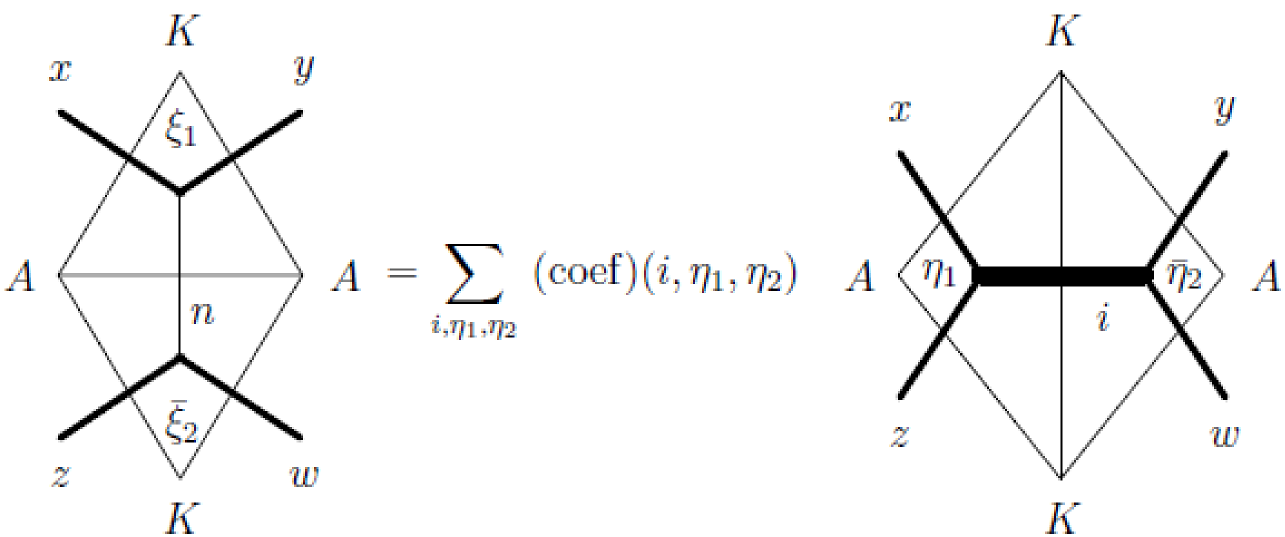
coef $\in \mathbb{C}$

minimal central projection of $(\mathcal{A}, *)$

$\updownarrow$

irred. M-M sector

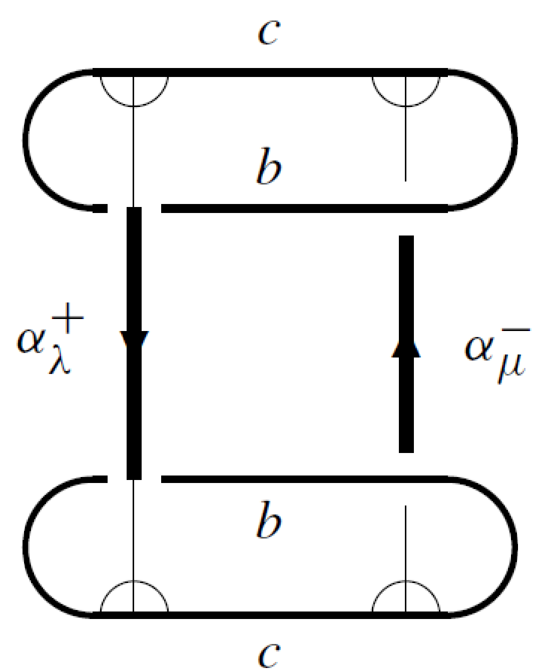




Modular invariants

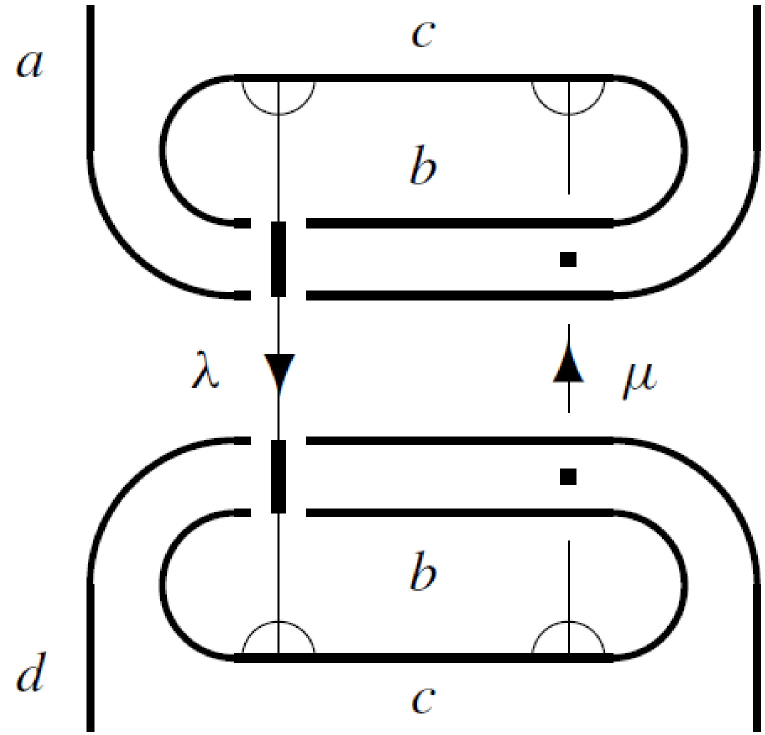
we define a matrix Z with entries $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle, \lambda, \mu \in {}_N\mathcal{X}_N.$

$$Z_{\lambda,\mu} = \sum_{b,c} \frac{d_b d_c}{w d_\lambda d_\mu}$$



Graphical representation of $Z_{\lambda,\mu}$

$$\sum_{a,b,c,d} \frac{d_b d_c}{w^2}$$



A vertical projector $q_{\lambda,\mu}$

Theorem 6.8. *Under Assumption 5.9, the vertical projector $q_{\lambda,\mu}$ is either zero or a minimal central projection in $(\mathcal{Z}_h, *_v)$. We have mutual orthogonality $q_{\lambda,\mu} *_v q_{\lambda',\mu'} = \delta_{\lambda,\lambda'}\delta_{\mu,\mu'} q_{\lambda,\mu}$ and the vertical projectors sum up to the multiplicative identity of $(\mathcal{Z}_h, *_v)$: $\sum_{\lambda,\mu \in {}_N\mathcal{X}_N} q_{\lambda,\mu} = e_0$. Moreover, $q_{\lambda,\mu} = 0$ whenever $Z_{\lambda,\mu} = 0$ and otherwise the simple summand $q_{\lambda,\mu} *_v \mathcal{Z}_h$ is a full $Z_{\lambda,\mu} \times Z_{\lambda,\mu}$ matrix algebra, where $Z_{\lambda,\mu}$ is the (λ, μ)-entry of the modular invariant mass matrix of Definition 5.5.*

$$A_8 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad D_6 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad D_7 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Modular invariant matrices of type A_8, D_6, D_7

$$E_6 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad E_7 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Modular invariant matrices of type E_6 and E_7

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\end{array}\right)
$$

Modular invariant matrix of type E_8

Ocneanu Graph の描き方(3)

- Coquereaux, Schieber, Trincheroらの方法
- 純代数的な方法(FRA の相対テンソル積, weak Hopf algebra (quantum groupoid), 行列計算)
- Ocneanu Graphに代数的な構造の意味づけがなされる.

- SU(N) (N≧3) の場合にも(SU(2)より複雑だが)実行可能.
- (SU(2)の場合) Modular invariant, toric matrixなども結果的に構成される.

Classification of quantum subgroups of quantum SU(N) （Ocneanu）

- First row of modular invariant ⇒ Gap
- Gap には level によらない uniform bound がある.
- Exceptional case は level に比例して Gap が大きくなる.

⇓

- SU(N) のexceptional graph (E series) には level の比較的小さいところにしかない.
- （cf. Ganon の計算 level < 10000 ）