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Topological Vertex, Instanton Counting, and Link Homology

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APPLICATIONS OF TOPOLOGICAL STRINGS

. . . . .

Twister amplitudes

Donaldson-Thomas

Seiberg-Witten Theory

Knot Theory

topological strings

Dijkgraaf-Vafa

Gromov-Witten Theory

Black Holes

Mirror Symmetry

Matrix Models

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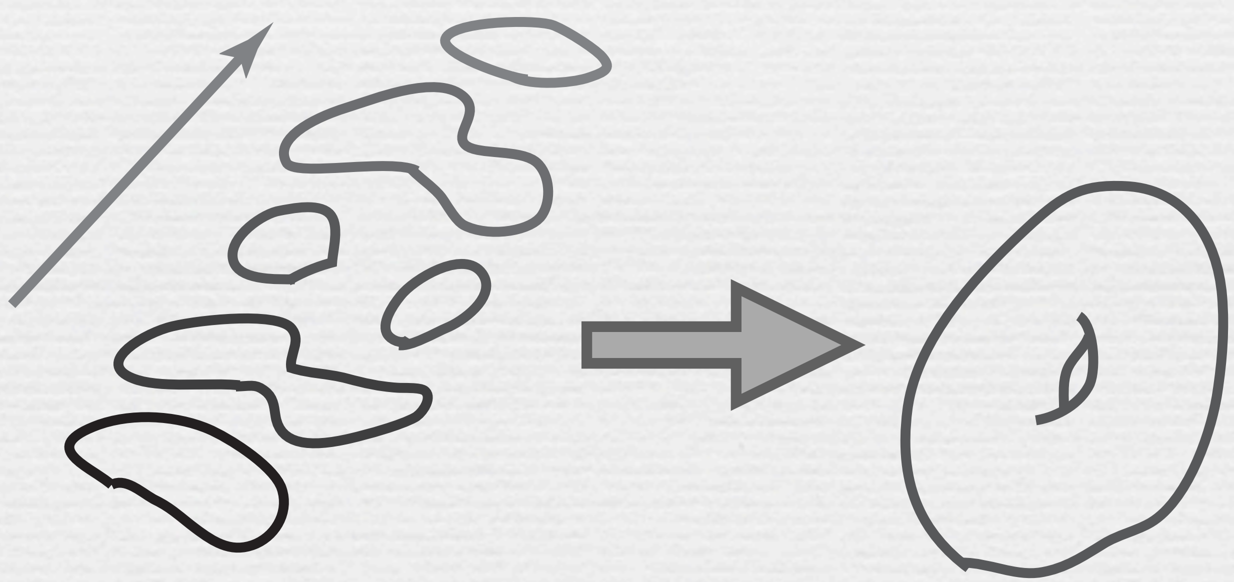
1. Gromov-Witten invariants & Topological Strings

- String Theory

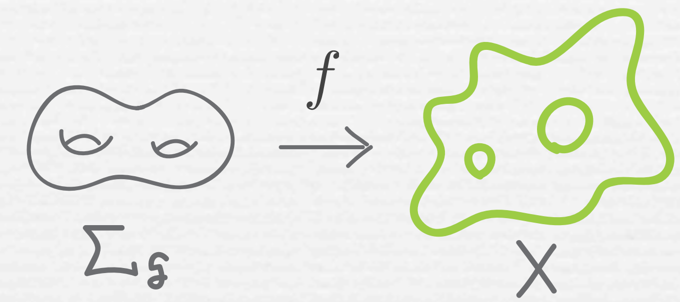
String theory is a quantum mechanics of 1-dimensional objects propagating the so-called target space X. We call the 2-dimensional surface which string sweep out "world-sheet".



Thus, a map from a worldsheet to the target space gives a configuration of the string.

In order to introduce the quantum theory quantity, we use the Feynmann path integral.

$$\sum_{\text{all } f} \exp(-S[f])$$



Let us consider a string model whose target space is a Calabi-Yau 3-fold and the action is

$$\begin{aligned} S^{\mathrm{A}} &= \int_{\Sigma_g} d^2 z \sqrt{g} G_{i\bar{j}} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^{\bar{j}} + i\epsilon^{\mu\nu} \partial_\mu X^i \partial_\nu X^{\bar{j}} + \cdots \\ &= \{\mathcal{Q}, V(X, \rho, \chi)\} + \int_{\Sigma_g} X^*(\omega) \end{aligned}$$

$$\Big\downarrow$$

SUSY localization $\qquad \partial_{\bar{z}} X^i = \partial_z X^{\bar{i}} = 0$

Thus A-model topological string theory counts **holomorphic** maps f from worldsheets to the Calabi-Yau.

$$F_g := \sum_{\text{hol. maps } f} e^{-\int_{\Sigma_g} f^*(\omega)}$$

- Topological Strings & Gromov-Witten invariants

Thus we can introduce the A-model topological string amplitudes and Gromov-Witten invariants for Calabi-Yau X

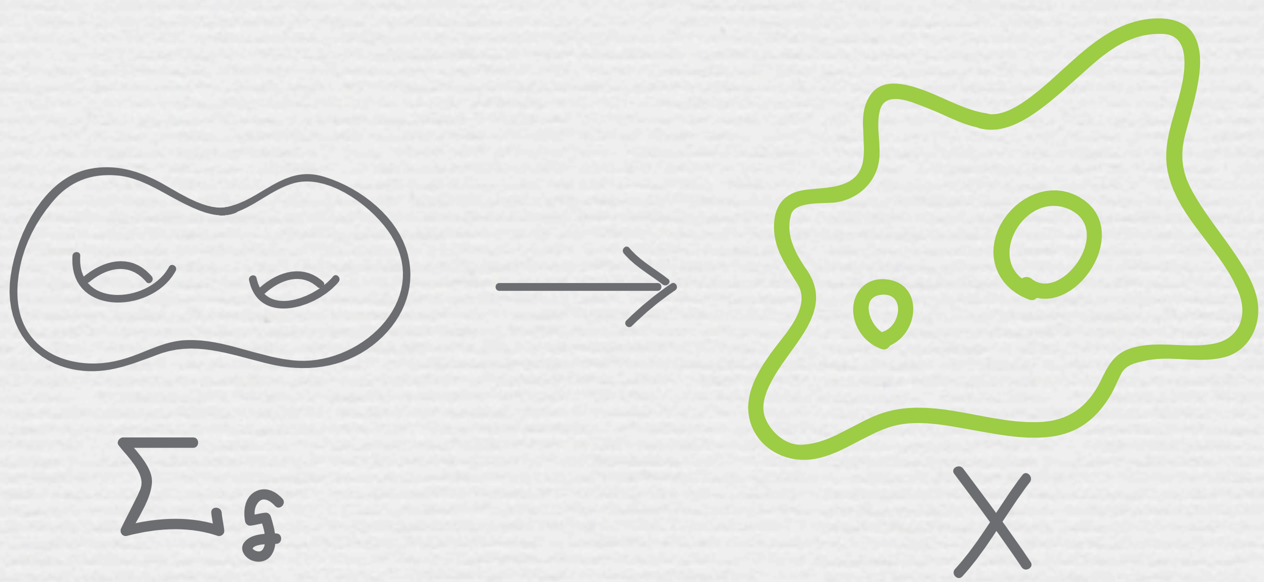
$$F_g = \sum_{\Sigma \in H_2(X,\mathbb{Z})} N_{g,\Sigma} e^{-t_\Sigma}$$

Gromov-Witten invariant

$f : \Sigma_g \to \Sigma \subset X$

$$Z = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t)$$

$\hbar$: topological string coupling constant



- Gromov-Witten invariant : mathematical(algebraic) definition

The definition of the invariant is given by the virtual fundamental class of the moduli space of the stable maps. (In this talk, we don't use this definition)

$$N^g_\beta = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1$$

$$= \deg[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}$$

$$\overline{\mathcal{M}}_g(X,\beta) = \{\text{stable maps } (f,\Sigma) \,|\, f_*([\Sigma]) = \beta \in H^2(X,\mathbb{Z})\}$$

- **localization**
- **mirror symmetry**

Example : $\mathcal{O}(-3) \to \mathbb{CP}^2$ [Chiang et.al. '99]
[Klemm-Zaslow '01]

$$F_0 = -\frac{t^3}{18} + 3Q - \frac{45Q^2}{8} + \frac{244Q^3}{9} - \frac{12333Q^4}{64} + \cdots$$

$$F_1 = -\frac{t}{12} + \frac{Q}{4} - \frac{3Q^2}{8} - \frac{23Q^3}{3} - \frac{3437Q^4}{16} + \cdots$$

$$F_2 = -\frac{2}{5720} + \frac{Q}{80} + \frac{3Q^3}{20} + \frac{514Q^4}{5} + \cdots$$

$$Q = e^{-t}$$

not integers but **rational** numbers !!

Thus topological string theory is constructed as a model of strings which propagate on Calabi-Yau 3-fold. In general, it is very hard to get the full partition function via straightforward computation. However, for the certain class of Calabi-Yau's, **we can compute the partition function exactly** using **string dualities** !!

→ **Example : topological vertex**

2. Gopakumar-Vafa invariants & M-theory

- **Modern "definition" of topological string theory [Gopakumar-Vafa '98]**

Topological strings on X

Type IIA superstrings (M-theory)

Topological strings count degeneracies of wrapped **M2-branes** (solitons)

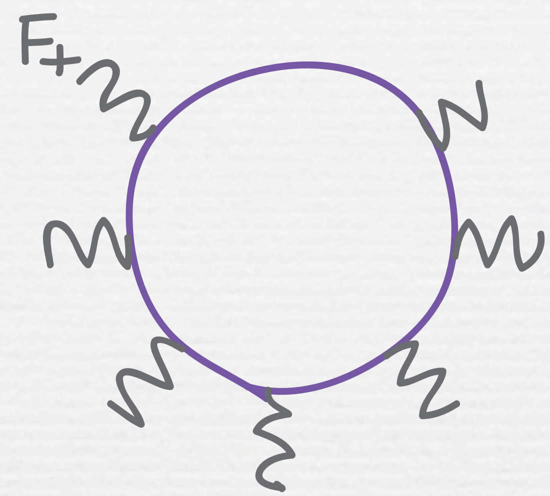
$$F_g = \sum_{\Sigma \in H_2(X,\mathbb{Z})} N_{g,\Sigma} e^{-t_\Sigma}$$

Counting the BPS(stable) state coming from branes wraps on cycles of CY

Labels (quantum numbers) of these particles are $\beta \in H^2(X,\mathbb{Z})$ and $SO(4)$ spin (j_L, j_R)

$N_\beta^{(j_L,j_R)}$: # of these particles

Diagrammatic computation implies that these wrapped branes gives the following contribution to the free energy



1-loop amplitude of BPS particles coupling to the background field

multiplicity

$$\mathcal{F} = \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{n \in Z} \sum_{j_L, j_R} \boxed{N_\Sigma^{(j_L,j_R)}} \log \det{}_{(j_L,j_R)} \left(\Delta + {m_{(\Sigma,n)}}^2 + 2 m_{(\Sigma,n)}\, \sigma_L\, F_+\right)$$

$$= \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} N_\Sigma^{j_L} \, (-1)^{-2\,j_L} \, e^{-k\,T_\Sigma} \, \frac{\sum_{l=-\,j_L}^{j_L} q^{-2kl}}{k\left(q^{k/2} - q^{-k/2}\right)^2}$$

$$F_+ = -\hbar$$

$$N_\Sigma^{j_L} = \sum_{j_R} (-1)^{-2\,j_R} (2\,j_R + 1)\, N_\Sigma^{(j_L,j_R)}$$

$$\Longrightarrow \quad \sum_{j_L} N_\Sigma^{j_L} [j_L] = \sum_{g=0}^{\infty} n_\Sigma^g [2(0) + (1/2)]^{\otimes g}$$

change of representation basis

Gopakumar-Vafa invariants

Proposal [Gopakumar-Vafa '98]

$$F(\hbar,t)=\sum_{g\geqslant 0}\sum_{\beta\in H^2(X,\mathbb{Z})}\sum_{d\geqslant 1} n^g_\beta \frac{1}{d}\left(2\sin\frac{d\hbar}{2}\right)^{2g-2} e^{-d\langle\beta,t\rangle}$$

$$=\sum_{g\geqslant 0}\sum_{\beta\in H^2(X,\mathbb{Z})} \hbar^{2g-2}N^g_\beta e^{-\langle\beta,t\rangle}$$

$$F_g(t)=\sum_\beta\left(\frac{|B_{2g}|n^0_\beta}{2g\,(2g-2)!}+\cdots-\frac{g-2}{12}n^{g-1}_\beta+n^g_\beta\right)\mathrm{Li}_{3-2g}(Q^\beta)$$

This expression solves some problems of the Gromov-Witten invariants

Example : genus zero

$$F_0(t)=\sum_\beta n^0_\beta\sum_{d=1}\frac{Q^{d\beta}}{d^3}$$

primitive curve $\beta\in H^2(X,\mathbb{Z})$

$\downarrow$

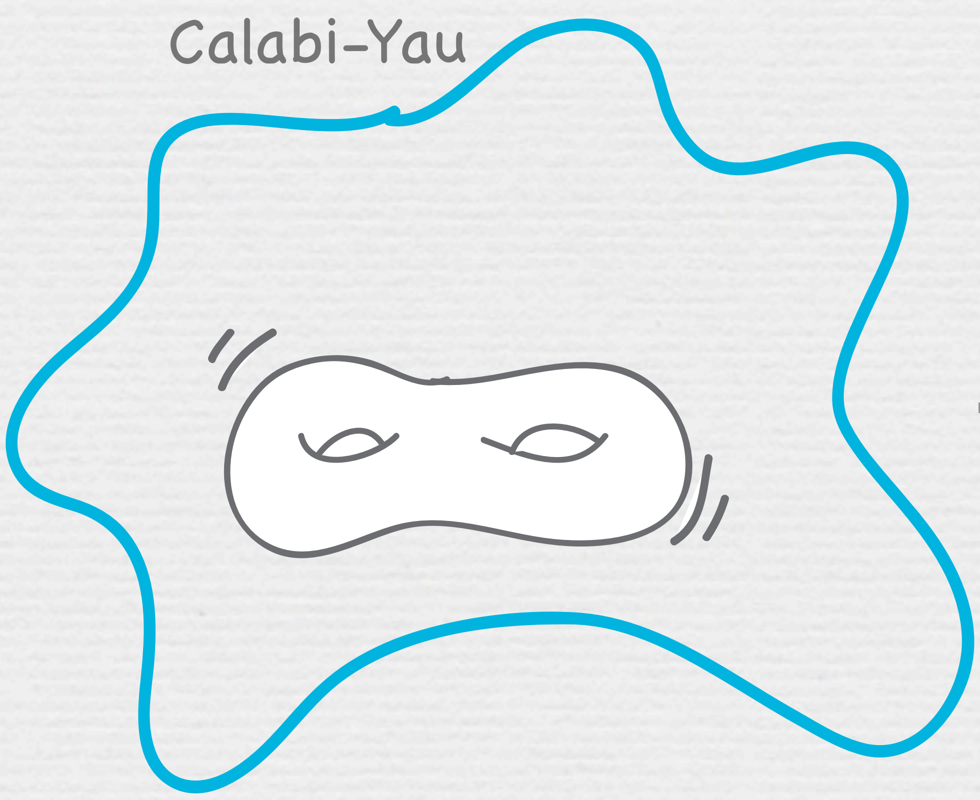
multicovering $d\beta$ with weight $1/d^3$

- Geometric Representation of Gopakumar-Vafa

Moduli space of wrapped D2-branes consists of

- U(1) gauge field living on branes
  $\parallel$
  flat connection on Σ_g $\longrightarrow$ $\mathrm{Jac}(\Sigma_g) = \boxed{\mathrm{T}^{2g}}$

- moduli space of geometric deformations of Σ_g inside the Calabi-Yau



$\longrightarrow$ $\boxed{\mathcal{M}_{g,d}}$

the moduli space of deformations

The degeneracies of bound states of wrapped M2-branes are extracted from the Hilbert space $H^*(\mathcal{M}_{g,d}) \times H^*(\mathrm{T}^{2g})$

$SU(2)_R$ $SU(2)_L$ $[(\mathbf{1/2}) \oplus \mathbf{2(0)}]^{\otimes g}$

These cohomologies have SU(2) actions (Lefshetz action)

The Hilbert space is graded with $SU(2)_R$ R-charge. We take trace over the charges with sign.

Let us consider curves inside the C-Y mfd in class $\Sigma = \sum_i d_i[\Sigma_i]$

Then the Gopakumar-Vafa invariants are given by

$$\boxed{n_d^g = (-1)^{\dim \mathcal{M}_{g,d}} \chi(\mathcal{M}_{g,d})}$$

Example : $\mathcal{O}(-3) \to \mathbb{CP}^2$ **(local $\mathbb{P}^2$) revisited**

branes wrap a degree d curve inside $\mathbb{P}^2$. Let us introduce the homogenerous coord. of $\mathbb{P}^2$: x, y, z. The curve is a zero-locus of the following polynomial

$$\sum_{i+j+k=d} a_{ijk}\, x^i y^j z^k = 0$$

$$a_{ijk} \in \mathbb{C}$$

moduli space of these curves is

$$\{\, a_{ijk} \,\}/ \text{ rescale by } \mathbb{C}^\times \longrightarrow \mathbb{CP}^{\frac{d(d+3)}{2}}$$

$${}_{d+2}\mathrm{C}_2 = \frac{d(d+3)}{2} + 1$$

genus-degree formula implies $\quad g = \dfrac{(d-1)(d-2)}{2}$

$$\boxed{n_d^{\frac{(d-1)(d-2)}{2}} = (-1)^{d(d+3)/2} \frac{(d+1)(d+2)}{2}}$$

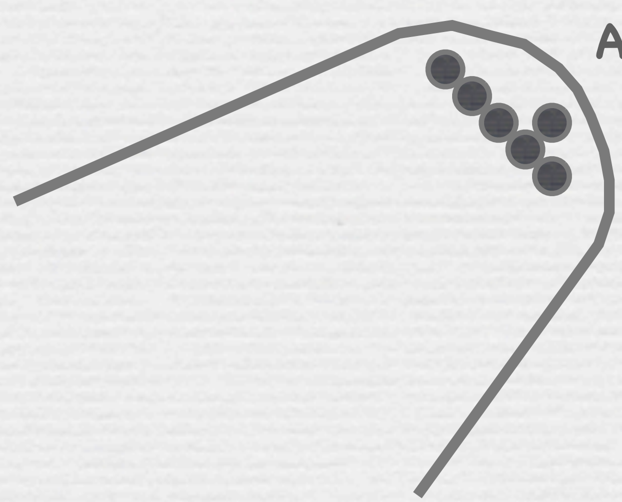
$$\longrightarrow \quad n_1^0 = 3 \quad n_2^0 = -6 \quad n_3^1 = -10$$

3. Geometric Transition & Gopakumar-Vafa invariants

- **Local Calabi-Yau manifolds & toric Calabi-Yau manifolds**

toric Calabi-Yau : **Local models** of Calabi-Yau manifolds
(describe the structure in neighborhood of **singularity**)

→ Geometric engineering
AdS/CFT
. . . . .

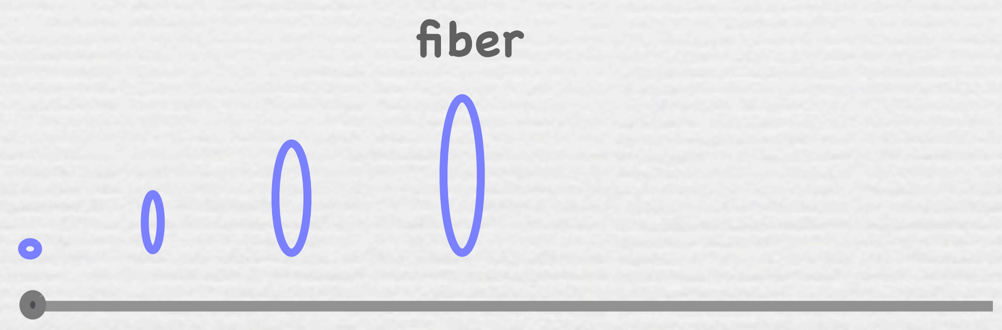


→ **ADE gauge symmetry**

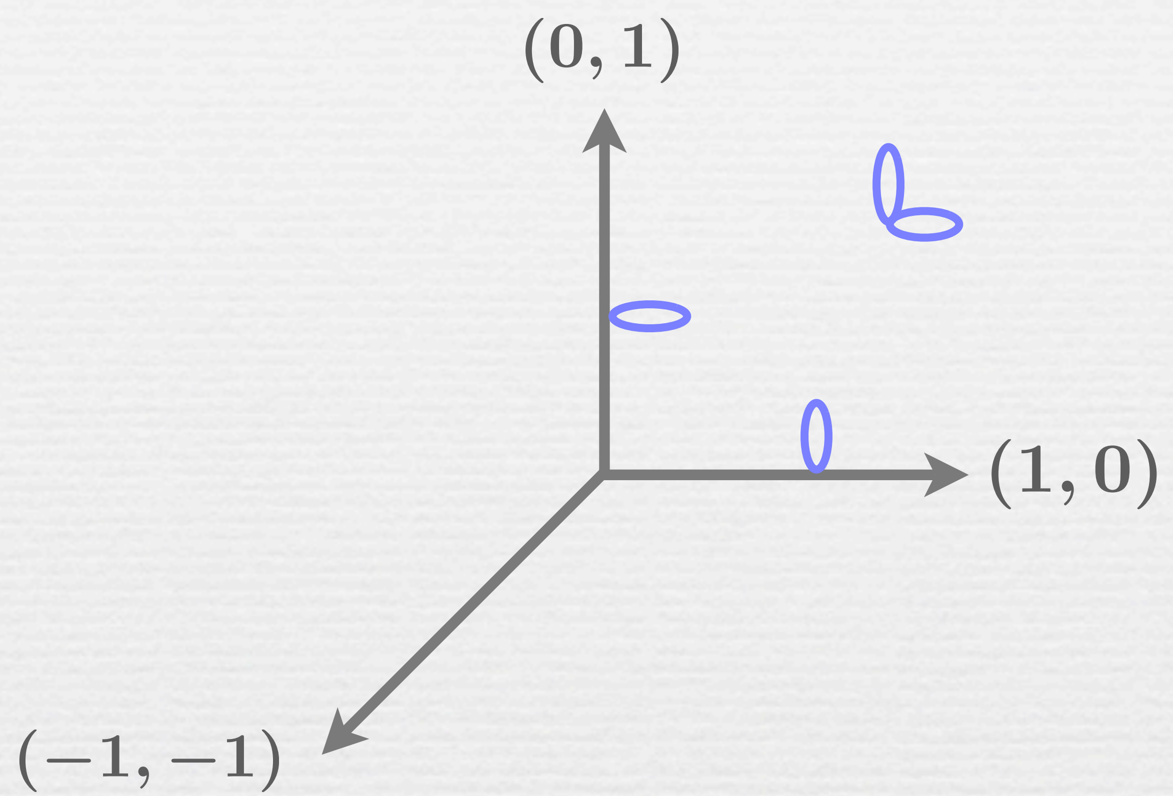
- Toric Calabi-Yau manifold

- $\mathbb{C}^1$

$$z = |z|e^{i\theta} \qquad T^1(\theta) \text{ fibration over } \mathbb{R}(|z|)$$



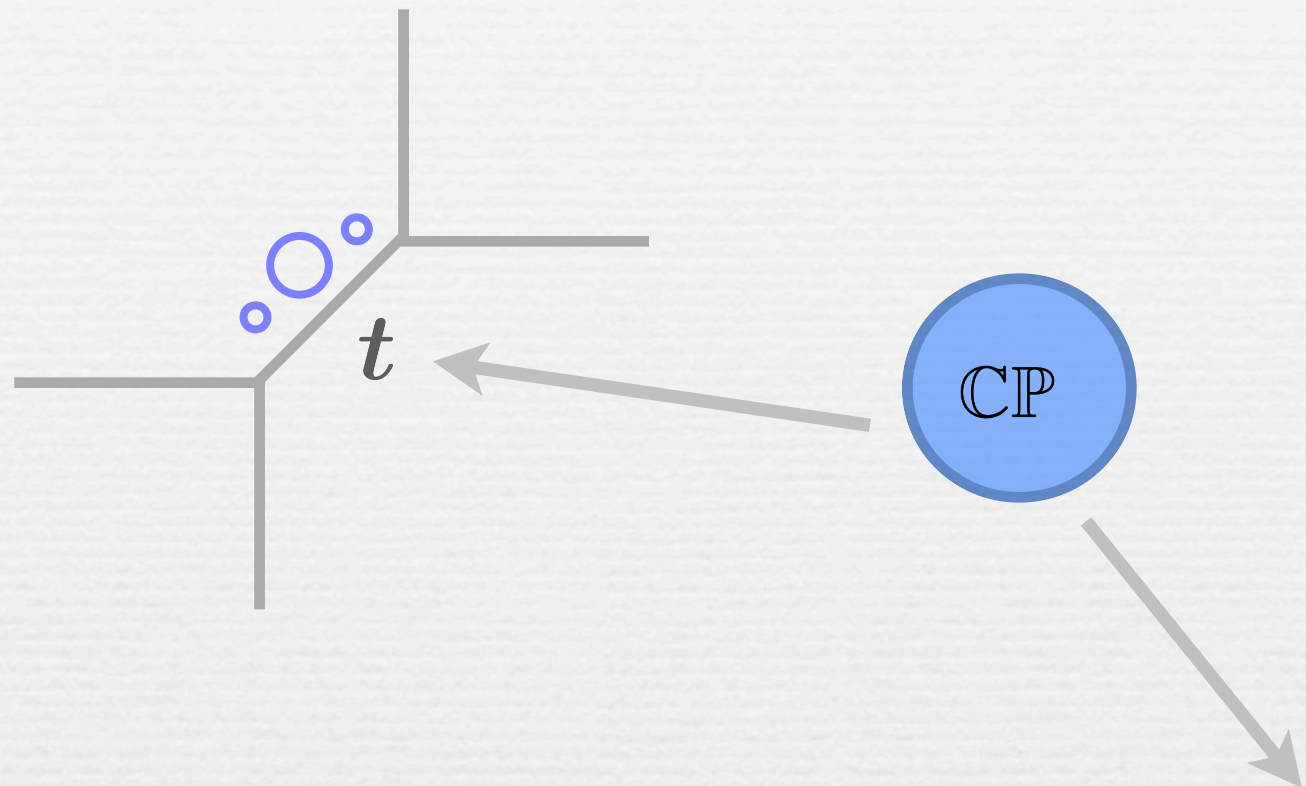
- $\mathbb{C}^3$



We are focusing on T^2-action

$$e^{\alpha r_\alpha + \beta r_\beta} : (z_1, z_2, z_3) \longrightarrow (e^{i\alpha} z_1, e^{-i\beta} z_2, e^{-i\alpha + i\beta} z_3)$$

- Resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$



$$\det\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = 0 \quad \longrightarrow \quad \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

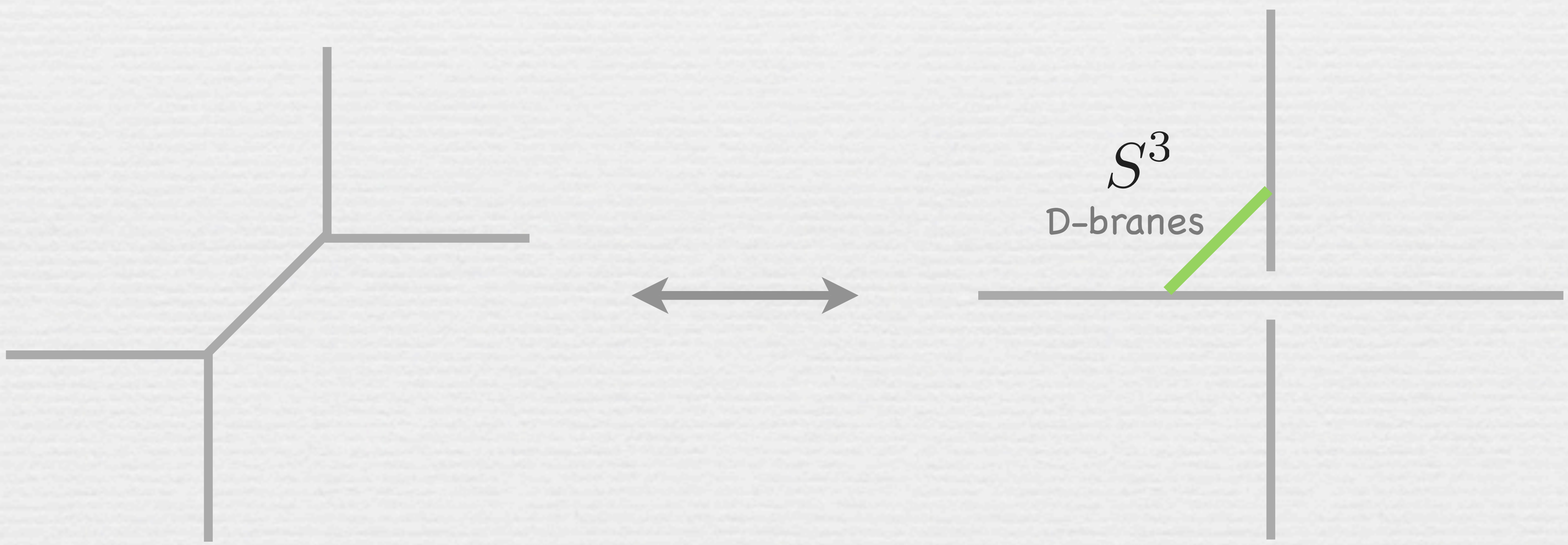
conifold (singular) resolved conifold

$A, B \in \mathbb{C}$

$$\mathcal{O}(n) \longrightarrow \mathbb{CP}^1 \qquad z_S = \frac{1}{z_N}$$

$$\{\phi\} \qquad \{z\} \qquad \phi_S = (z_N)^n \phi_N$$

- Geometric transition



closed A-model on
resolved conifold

open A-model on
deformed conifold

$\longrightarrow$ **Chern-Simons theory**
[Witten, '93]

- **Symplectic quotient**

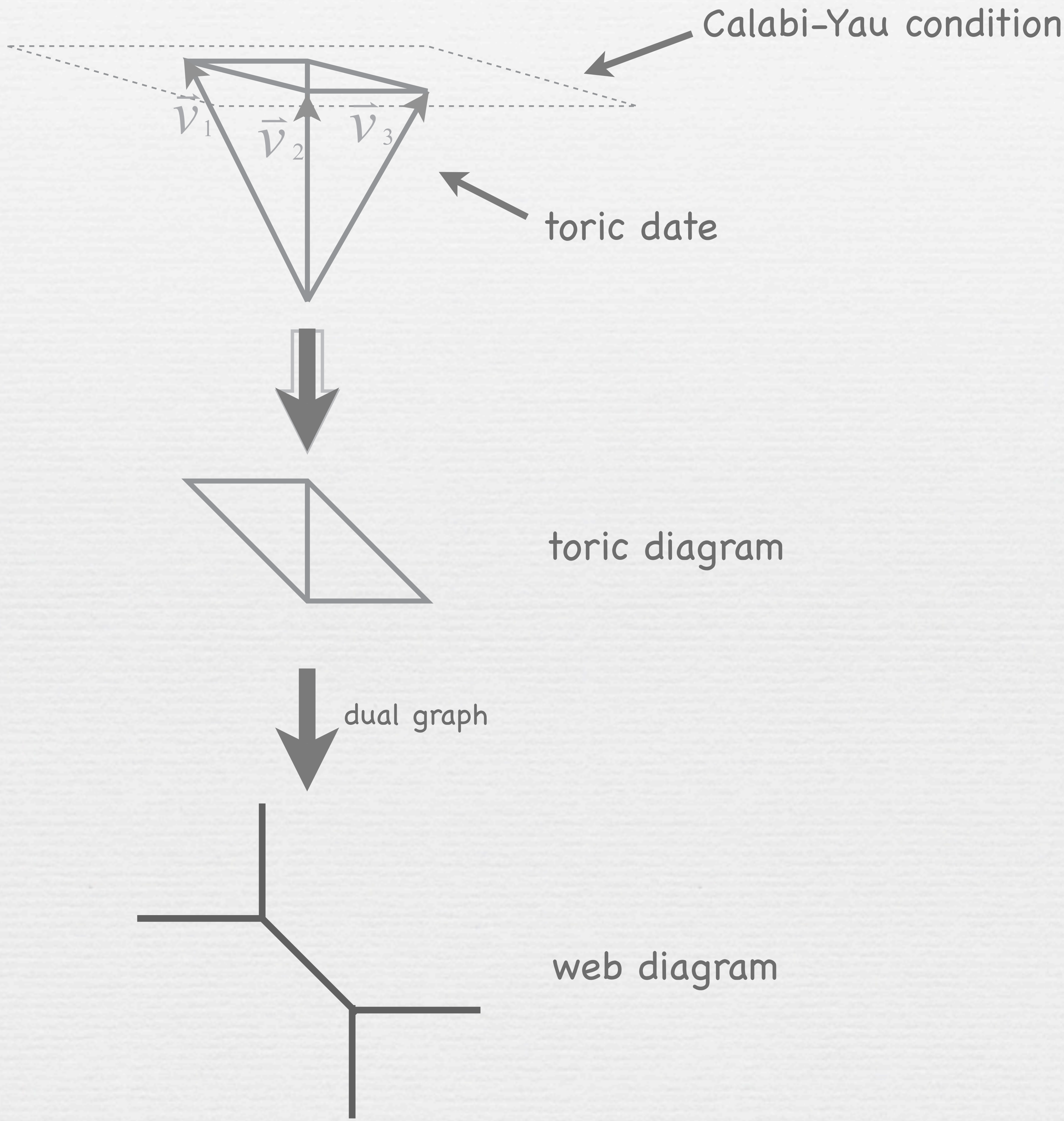
toric date $\vec{v}_i \in \mathbb{Z}^3$ $\longrightarrow$ $Q_i^a \in \mathbb{Z}$ s.t. $\sum_{i=1}^{N} Q_i^a \vec{v}_i = 0$

- moment map $\mu_a(z) = \sum_{j=1}^{N+3} Q_j^a |z_j|^2$

- $G = U(1)^N$ $z_j \to e^{i \sum_a Q_a^j \alpha_a} z_j$

$$\boxed{\begin{aligned} X &= \mathbb{C}^{N+3} // G \\ &= \cap_{a=1}^{N} \mu_a^{-1}(t_a)/G \end{aligned}}$$

$\sum_j Q_a^j = 0$: Calabi-Yau condition


Calabi-Yau condition
$\vec{v}_1$
$\vec{v}_2$
$\vec{v}_3$
toric date
toric diagram
dual graph
web diagram

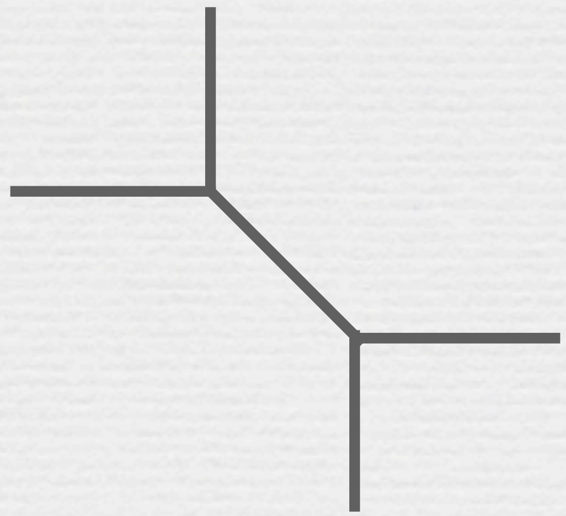
- SUMMARY

$$F_g = \sum_{\Sigma \in H_2(X,\mathbb{Z})} N_{g,\Sigma} e^{-t_\Sigma}$$

$$F(\hbar, t) = \sum_{g \geqslant 0} \sum_{\beta \in H^2(X,\mathbb{Z})} \sum_{d \geqslant 1} n^g_\beta \frac{1}{d} \left(2 \sin \frac{d\hbar}{2} \right)^{2g-2} e^{-d\langle \beta, \omega \rangle}$$

- $q = e^{-i\hbar}$ $\qquad F = \sum_{g \geq 0} \hbar^{2g-2} F_g(t)$
- $\beta = \sum_i d_i [\Sigma^i] \qquad \int_{[\Sigma^i]} \omega = t^i \Longrightarrow \langle \beta, \omega \rangle := \sum_i d_i t^i$

toric Calabi-Yau 3-fold



$\rightarrow$ **Chern-Simons theory**

knot theory

- large-N duality as gauge/gravity duality

'tHooft's idea

amplitudes of gauge theory → stringy genus expansion

$$F = \log Z^{\text{gauge}}$$
$$= \sum_{g,h} \hbar^{2g-2} \lambda^{2g-2+h} F_{g,h} \qquad \lambda = \hbar N$$
$$= \sum_{g} \hbar^{2g-2} F_g(\lambda) = \log Z^{\text{string}} \quad !?$$

Let us study SU(N) Chern-Simons theory as the gauge theory.

$$Z^{\text{CS}}(S^3) = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} 2 \sin^{N-j} \frac{j\pi}{k+N}$$

Let us introduce the following parameters

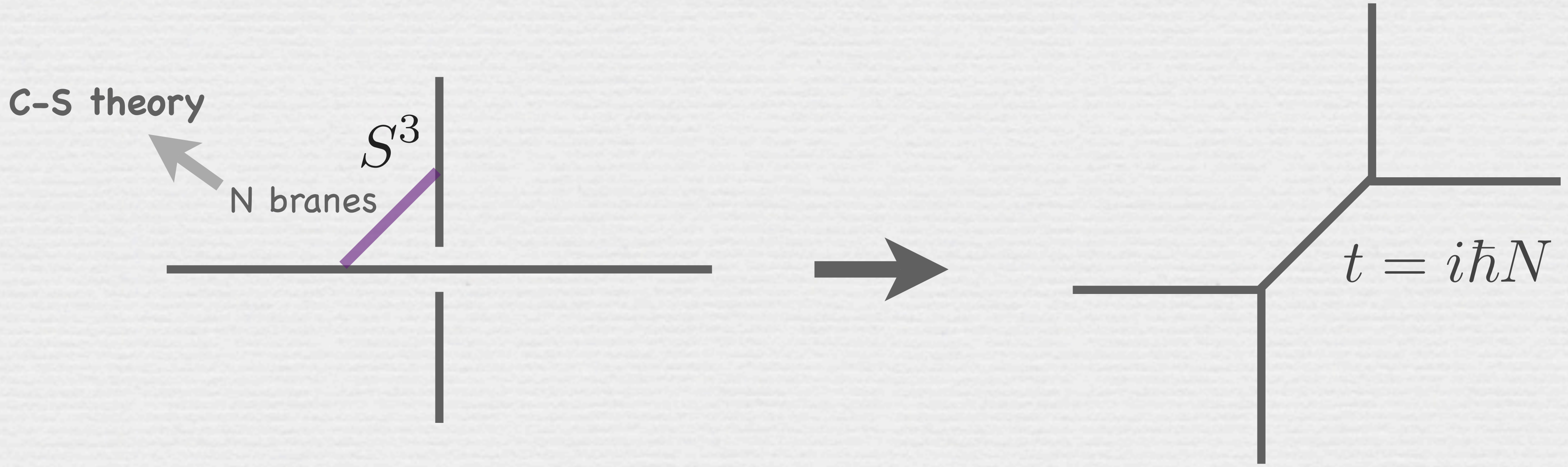
$$\hbar = \frac{2\pi i}{k+N} \qquad t = -\frac{2\pi N}{k+N}$$

$$F_{g>1}(t) = \frac{(-1)^g |B_{2g} B_{2g-2}|}{2g\,(2g-2)\,(2g-2)!} + \frac{|B_{2g}|}{2g\,(2g-2)!} \sum_{d\geq 1} \frac{e^{-dt}}{d^{3-2g}}$$

This recovers the G-V invariants for **resolved conifold** ([Faber-Pandharipande]) !!

$$\boxed{n^g_\beta = \delta_{g,0}\,\delta_{\beta,1}}$$

mechanism behind this phenomena : **geometric transition**



Lesson : Thus the Chern-Simons theory "computes" the G-V invariants !!

- G-V partition function for conifold

- $$F(\hbar,t)=\sum_{g\geqslant 0}\sum_{\beta\in H^2(X,\mathbb{Z})}\sum_{d\geqslant 1} n^g_\beta \frac{1}{d}\left(2\sin\frac{d\hbar}{2}\right)^{2g-2} e^{-d\langle\beta,t\rangle}$$
- $$n^g_\beta=\delta_{g,0}\,\delta_{\beta,1}$$

$\rightarrow$
$$F=\sum_d \frac{1}{d}\frac{Q^d}{(q^{d/2}-q^{-d/2})^2}$$
$$=\sum_n -\log(1-Qq^n)^n$$

$\rightarrow$
$$\boxed{\begin{aligned} Z &= \exp\big[-F(\hbar,t)\big] \\ &= \prod_{n=1}(1-Qq^n)^n \end{aligned}}$$

4. Topological Vertex Method

- Topological strings on toric Calabi-Yau manifolds via topological vertex

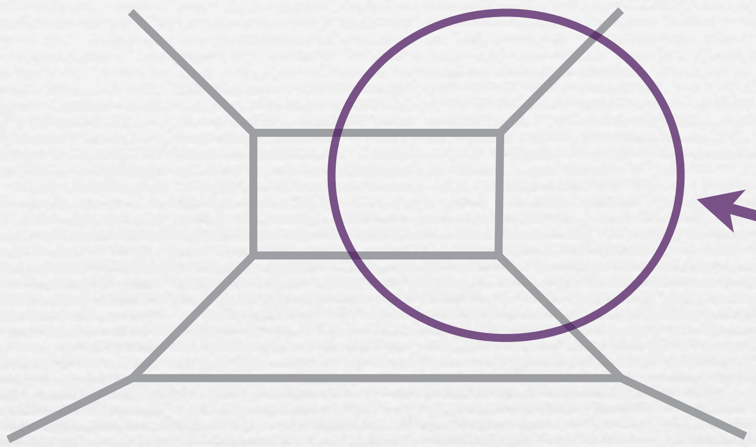
The topological string amplitude for a simplest toric Calabi-Yau manifold (conifold) is given by the Chern-Simons theory.

We can apply the geometric transition method to more complicated toric geometry.

The resulting rules for computation collect into a systematic formalism. This is the topological vertex.

- **Topological Vetex & toric Calabi-Yau manifolds**

How to compute topological string amplitudes for toric Calabi-Yau manifolds ?



Locally they look like a conifold

↓

Geometric transition enable us to calculate these amplitudes using Chern-Simons theory

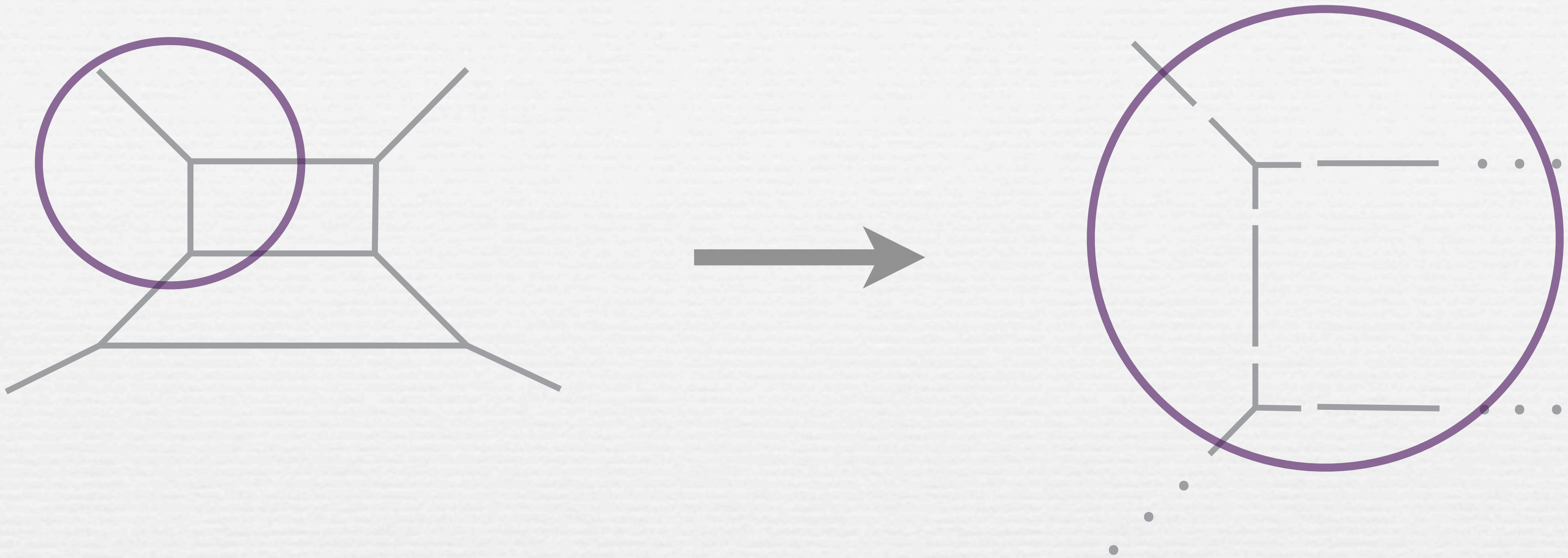
→ **Topological vertex [AMKV, '03]**

1. Decompose a toric web-diagram into vertices and propagators

2. Assign Young diagrams for each edges of these parts

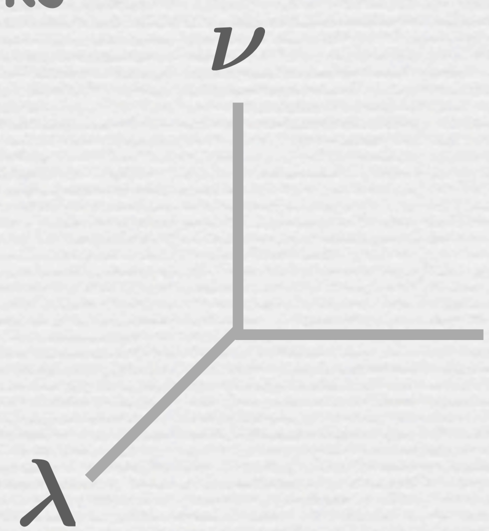
$$\mu = \{\mu_i \in \mathbb{Z}_{\geq 0} \,|\, \mu_1 \geq \mu_2 \geq \cdots\}$$

3. Glue them to get topological string partition function



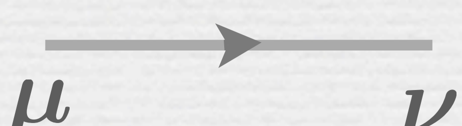
▲ Decomposition of toric web-diagram

Blocks

- 

μ : trivalent vertex . . . local $\mathbb{C}^3$ patch $\longrightarrow$ $C_{\lambda,\mu,\nu}$

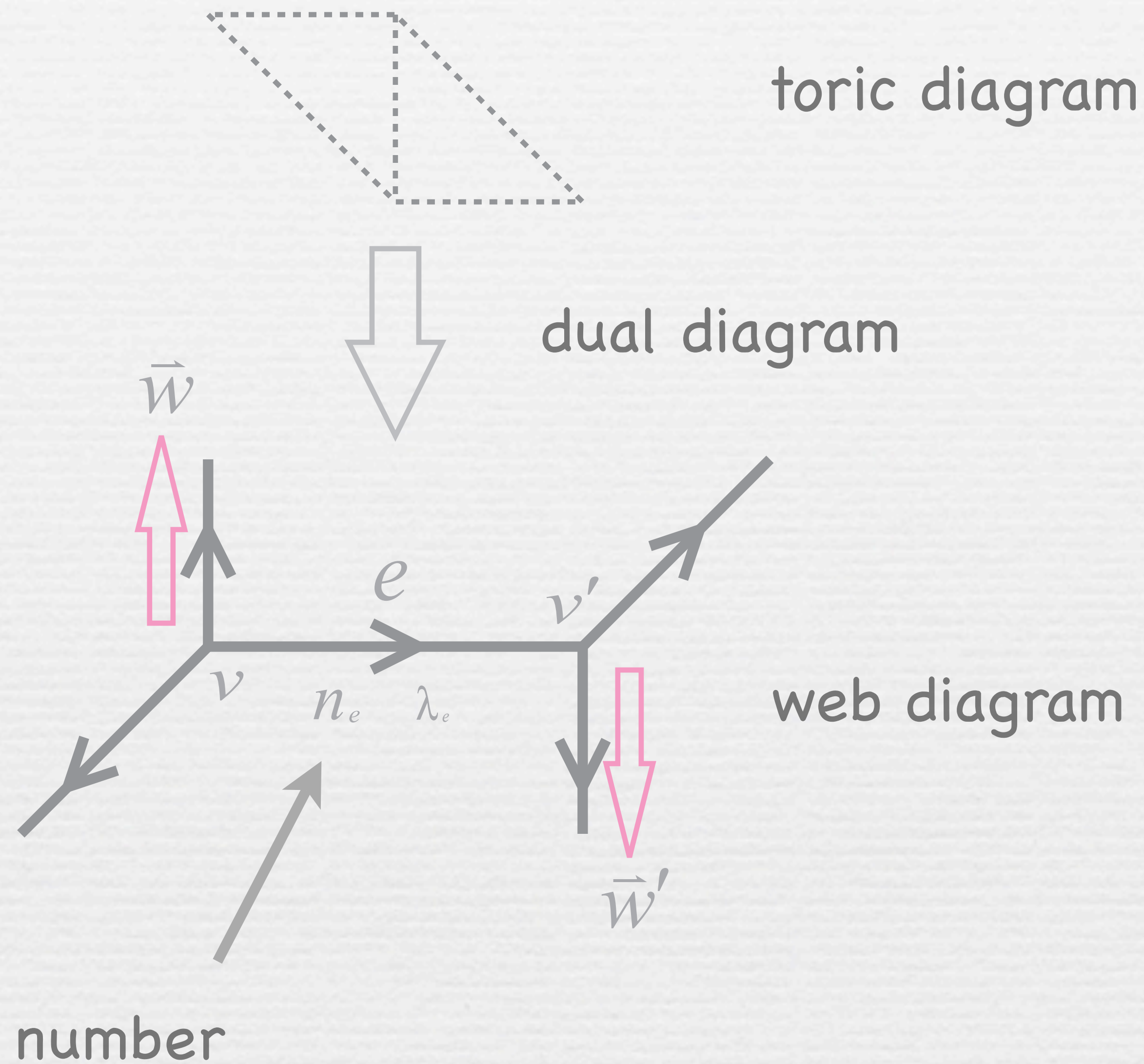
  vertex function

- 

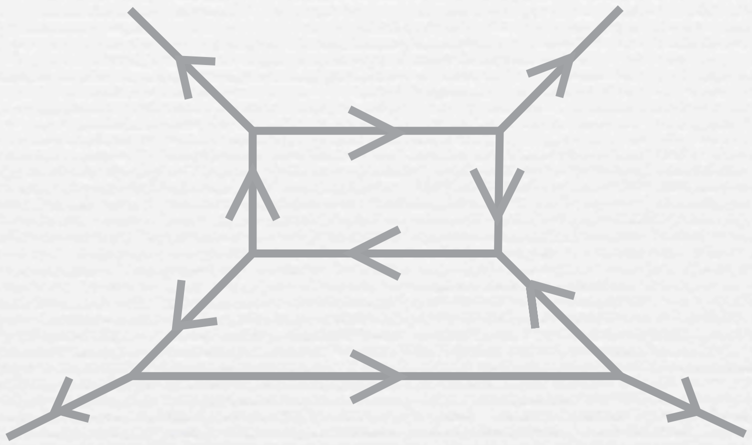
: edes . . . $\mathbb{CP}$ $\longrightarrow$ $f_{\mu}(q)^{n}(-1)^{|\mu|}e^{-t|\mu|}\delta\mu,\nu^{t}$

  framing factor & propagator

framing number

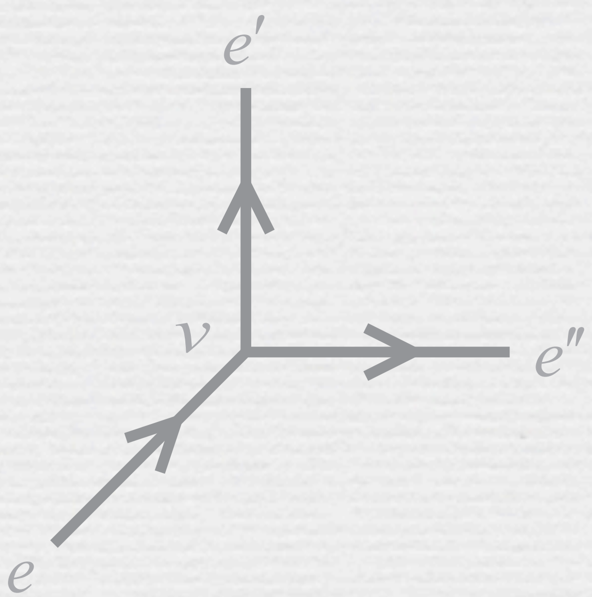


$$n = \det(\vec{w}, \vec{w}')$$



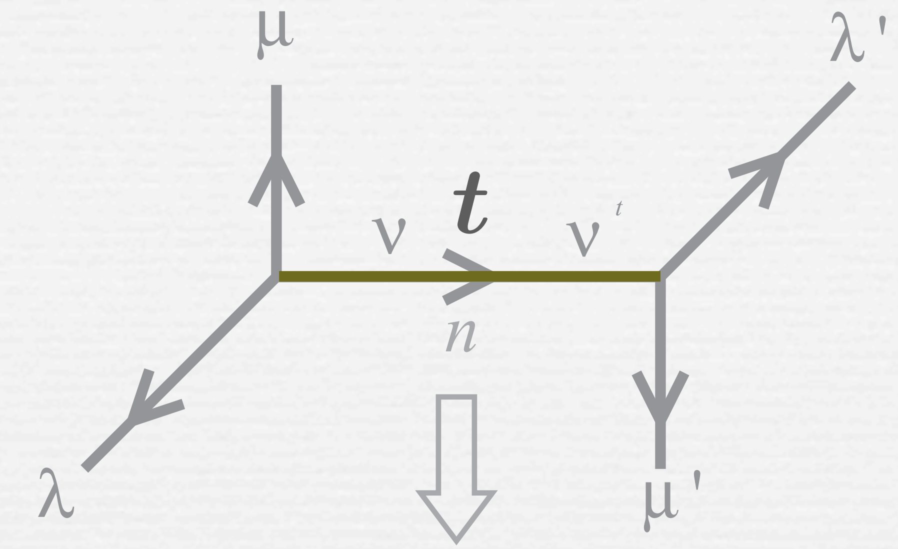
We assign Young diagrams for each edges of these parts

$$\boldsymbol{\mu} = \{\boldsymbol{\mu_i} \in \mathbb{Z}_{\geq \mathbf{0}} \,|\, \boldsymbol{\mu_1} \geq \boldsymbol{\mu_2} \geq \cdots\}$$

1. 

$$\longrightarrow \quad C_{\lambda_e^t \lambda_{e''} \lambda_{e'}}(q)$$

2.



- $\{f_\nu(q)\}^n$ framing factor
- $(-1)^{|\nu|}e^{-|\nu|t}$ propagator

Gluing along a leg is done by the following procedure

$$Z = \cdots \sum_{\nu} C_{\lambda\mu\nu}(-1)^{|\nu|}e^{-t|\nu|}(f_\nu)^n C_{\lambda'\mu'\nu^t} \cdots$$

vertex function propagator framing factor

$$C_{\lambda\mu\nu}(q) = q^{\kappa_\mu/2} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

- Conventions

$$C_{\lambda\mu\nu}(q) = q^{\kappa_\mu/2} s_{\nu^t}(q^{-\rho}) \sum_\eta s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho})$$

$$q^{-\rho} \longleftrightarrow x_i = q^{i-\frac{1}{2}}$$

$$i = 1, 2, 3, \cdots$$

$$q^{-\mu-\rho} \longleftrightarrow x_i = q^{-\mu_i+i-\frac{1}{2}}$$

$$|\mu| = \sum_i \mu_i \qquad ||\mu||^2 = \sum_i {\mu_i}^2$$

$$\begin{aligned}\kappa_\mu &= \sum_i \mu_i(\mu_i + 1 - 2i) \\ &= \sum_i {\mu_i}^2 - \sum_j {\mu_j^t}^2 = -\kappa_{\mu^t}\end{aligned}$$

- Schur functions

$$S_\lambda(x_1, x_2, \cdots, x_N) = \frac{\det({x_i}^{\lambda_j+N-j})}{\det({x_i}^{N-j})}$$

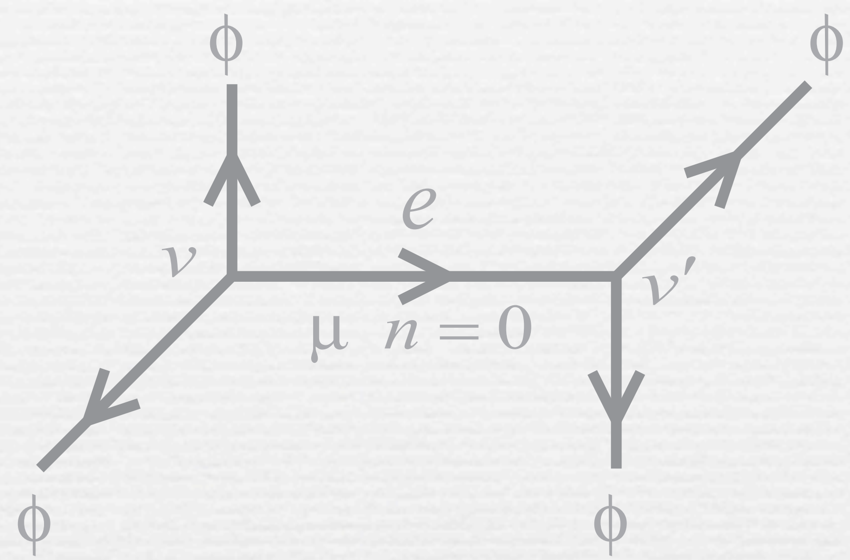
$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu}\, s_\lambda(x)$$

$$\Downarrow$$

skew Schur functions

$$s_{\lambda/\mu}(x) = \sum_\lambda c^\lambda_{\mu,\nu}\, s_\nu(x)$$

Ex: Conifold



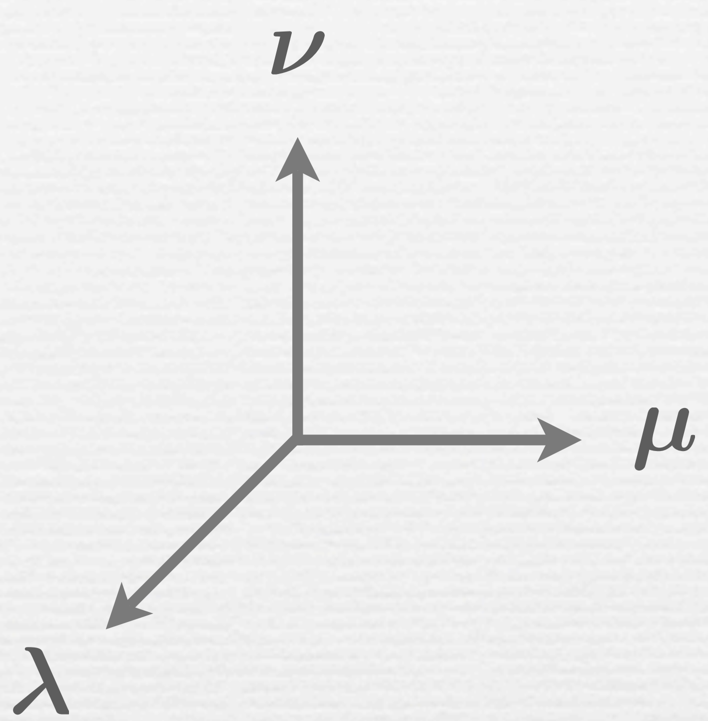
$$Q = e^{-t}$$
$$q = e^{-\hbar}$$

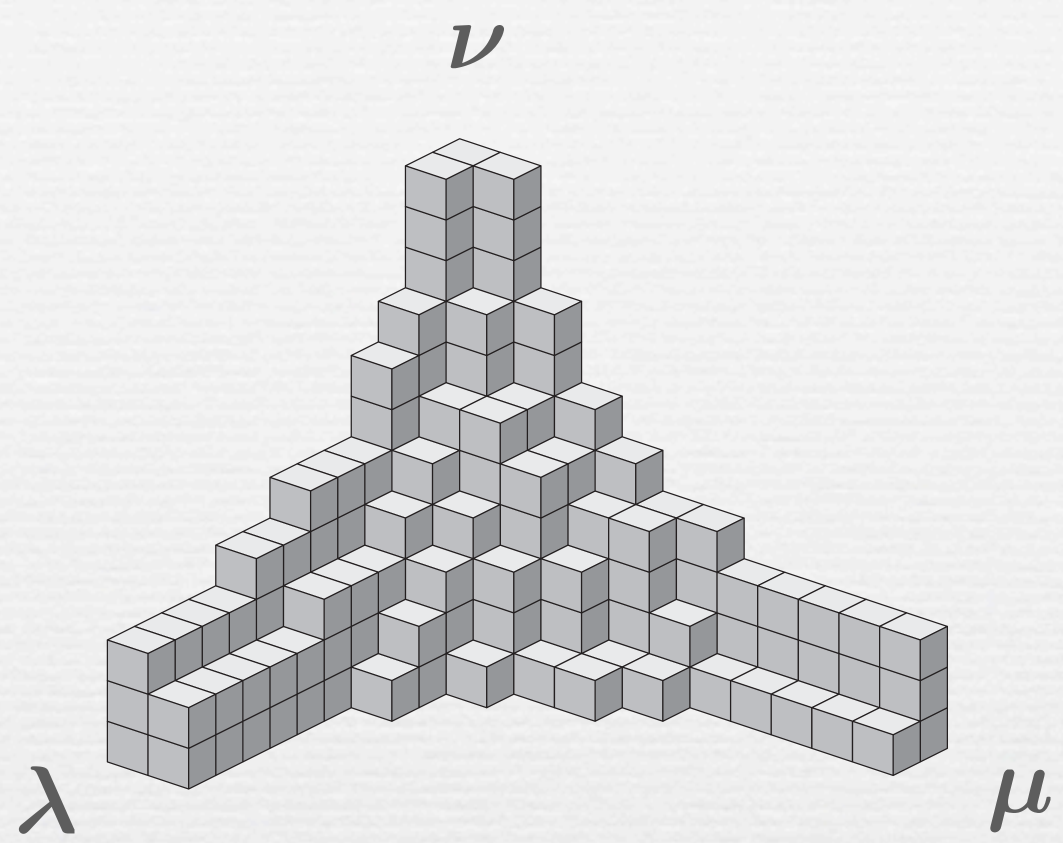
$$\begin{aligned} Z &= \sum_{\nu} C_{\phi\phi\nu}(-Q)^{|\nu|} C_{\phi\phi\nu^t} \\ &= \sum_{\nu} s_{\nu}(q^{-\rho})(-Q)^{|\nu|} s_{\nu^t}(q^{-\rho}) \\ &= \prod_{n=1}^{\infty} (1 - Qq^n)^n \end{aligned}$$

Formulae

$$\boxed{\begin{gathered} s_{\mu}(Qx) = Q^{|\mu|} s_{\mu}(x) \\ \sum_{\mu} s_{\mu^t}(x) s_{\mu}(y) = \prod_{i,j} (1 + x_i y_j) \end{gathered}}$$

- Duality to Crystal melting [Okounkov-Leshetikin-Vafa, '04]





$\mathbb{C}^{\mathbf{3}}$ -patch

melting crystal corner

Plane Partitons

grand-canonical ensemble melting crystals

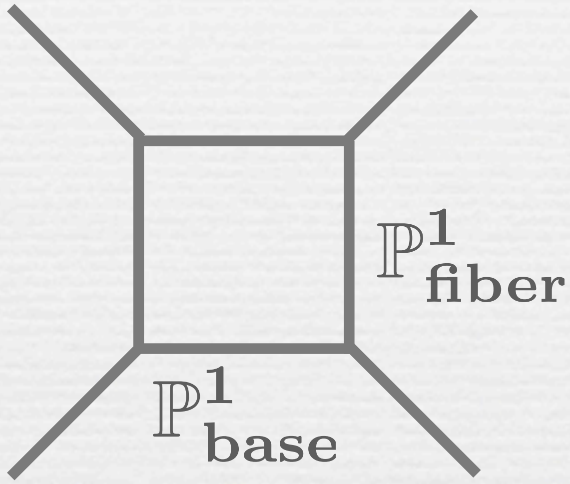
topological vertex !

$$Z_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} = \sum_{\text{crystals}} e^{-\hbar\#(\text{boxes})} \longrightarrow C_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}$$

$$\frac{1}{k_B T} = \hbar$$

- **Geometric Engineering** [Iqbal-KashaniPoor, '04],[Eguchi-Kanno, '04]
[Konishi-Sakai, '04],[Zhou, '04]

Local Hirzhebruch : $K \to \mathbb{P}^1 \times \mathbb{P}^1$ ⟶ **SU(2) gauge group**



$$Z = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} {Q_F}^{|\mu_1|+|\mu_3|} \, {Q_B}^{|\mu_2|+|\mu_4|} \, q^{-\kappa_{\mu_1}/2 + \kappa_{\mu_2}/2 - \kappa_{\mu_3}/2 - \kappa_{\mu_4}/2}$$

$$\times \, C_{\phi\, \mu_1\, {\mu_4}^t} \, C_{\phi\, {\mu_2}^t\, {\mu_1}^t} \, C_{\mu_2\, \phi\, \mu_3} \, C_{\phi\, \mu_4\, {\mu_3}^t}$$

$$= \sum_{\mu_2,\mu_4} {Q_B}^{|\mu_2|+|\mu_4|} \, q^{+\kappa_{\mu_2}/2 - \kappa_{\mu_4}/2} \, K_{\mu_4\, \mu_2}(Q_F) \, K_{{\mu_2}^t\, {\mu_4}^t}(Q_F)$$

$$K_{\mu\nu} = \sum_{\lambda} {Q_F}^{|\lambda|}\, q^{-\kappa_\lambda/2}\, C_{\phi\lambda\,\mu^t}\, C_{\nu^t\,\lambda^t\,\phi}$$

$$= s_{\mu^t}(q^{-\rho})\, s_\nu(q^{-\rho}) \sum_{\lambda} {Q_F}^{|\lambda|}\, s_\lambda(q^{-\mu-\rho})\, s_\lambda(q^{-\nu^t-\rho})$$

$$= q^{\|\mu\|^2/2+||\nu^t||^2/2}\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_\nu(q) \prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{-\mu_i-{\nu^t}_j+i+j-1}}$$

$$s_\mu(q^{-\rho}) = q^{||\mu^t||^2/2} \prod_{s\in\mu} \left(1-q^{h_\mu(s)}\right)^{-1} = q^{||\mu^t||^2/2}\, \tilde{Z}_\mu(q)$$

Let us extract the instanton part from the above partition function

$$Z(Q_F, Q_B) = Z^{\text{pert.}}(Q_F) Z^{\text{inst.}}(Q_F, Q_B)$$

$$Z^{\text{pert.}}(Q_F) \equiv K_{\phi\phi}\,(Q_F)^2 = \left[\prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{i+j-1}}\right]^2$$

Then we obtain

$$Z^{\text{inst.}}(Q_F, Q_B) \sum_{\mu,\nu} Q_B{}^{|\mu|+|\nu|}\, q^{||\mu||^2+||\nu^t||^2}\, \tilde{Z}_\mu(q)\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_\nu(q)\, \tilde{Z}_{\nu^t}(q) \left[\prod_{i,j=1}^{\infty} \frac{1 - Q_F\, q^{+i+j-1}}{1 - Q_F\, q^{-\mu_i - \nu^t{}_j + i + j - 1}} \right]^2$$

Identification with gauge theory parameters

$$Q_B = (\beta\Lambda)^4 \qquad Q_F = e^{-4\beta a} \qquad q = e^{-2\beta\hbar}$$

Under this identification, this topological string partition function is precisely the Nekrasov partition function of SU(2) gauge theory on $\mathbb{R}^4 \times S^1{}_\beta$

$$Z^{\text{inst.}} = \sum_{k=0}^{\infty} 4^k (e^{-\beta a} \beta\Lambda)^{4k} \times \sum_{|\mu|+|\nu|=k} W_\mu^2(q) W_{\nu^t}^2(q) \prod_{n\in\mathbb{Z}} (1 - e^{-2\beta(2a+n\hbar)})^{-2C_n(\mu,\nu^t)}$$

$$W_\mu(q) = s_\mu(q^\rho)$$

5. Instanton Countig

- Seiberg-Witten theory

A a section of SU(N) principal bundle $P(SU(N), M^4)$

$$\lambda, \psi \in C^\infty(M^4, \mathrm{ad}_{\mathbb{C}} P \otimes S)$$

$$\Phi \in C^\infty(M^4, \mathrm{ad}_{\mathbb{C}} P)$$

$\mathcal{N} = 2$ vector multiplet

Low energy effective action of $\mathcal{N} = 2$ gauge theory is solved by Seiberg-Witten prepotential

$$\mathcal{F}(\vec{a}, \Lambda)$$

Seiberg-Witten solved this theory by introducing an auxiliary elliptic curve to describe the prepotential.

What is the origin of the prepotential from the perspective of instanton computation ??

- Nekrasov formulae [Nekrasov, '02]

Nekrasov gave the generating function of Seiberg-Witten prepotential via instanton caluculus

$$\boldsymbol{Z^{\mathrm{Nek.}}(a,\Lambda,\hbar) = \exp\sum_{g=0}^{\infty}\hbar^{2g-2}\mathcal{F}_g(a,\Lambda)} \quad , \quad \boldsymbol{\mathcal{F}_0(a,\Lambda) = \mathcal{F}^{\mathrm{SW}}(a,\Lambda)}$$

In general, Nekrasov formula has two parameters $\hbar \longrightarrow \boldsymbol{\epsilon_1, \epsilon_2}$

It is given by a character on the instanton moduli space

$\mathcal{M}(N,k)$ moduli space of ADS connection (instanton of SU(N) gauge theory) on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with

$$\int_{\mathbb{R}^4} c_2 = k$$

$\mathcal{A}(N,k)$ holomorphic functions on $\mathcal{M}(N,k)$

representation space of

$$SU(N) \times U(2) \supset T^{N-1} \times T^2$$

$$\mathrm{diag}(e^{a_1}, \cdots, e^{a_N}, e^{\epsilon_1}, e^{\epsilon_2})$$

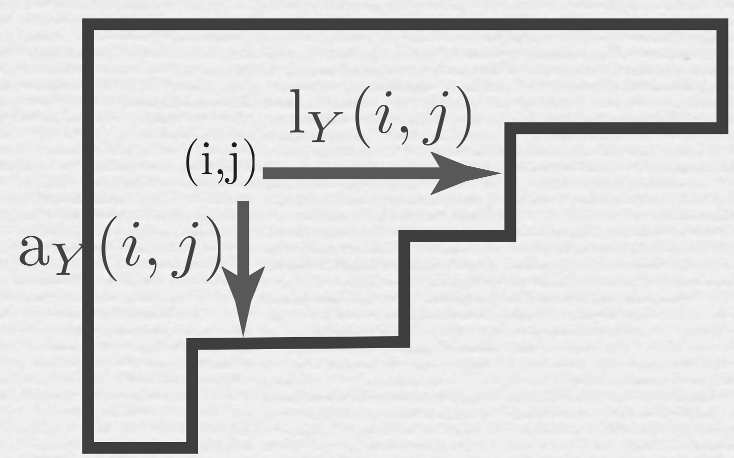
$$Z(\vec{a}, \Lambda, \epsilon_1, \epsilon_2) = \sum_k \Lambda^{2kN} \mathrm{ch}\, \mathcal{A}(N, k)$$

$$Z^{\mathrm{inst}}(\vec{a}, \Lambda, \epsilon_1, \epsilon_2) = \sum_{\vec{Y}} \frac{\Lambda^{2N_c|\vec{Y}|}}{\prod_{\alpha,\beta=1}^{N_c} n_{\alpha,\beta}^{\vec{Y}}(\vec{a}, \epsilon_1, \epsilon_2)}$$

$$\begin{aligned} \mathrm{n}_{\alpha,\beta}^{\vec{Y}}(\vec{a}, \epsilon_1, \epsilon_2) = & \prod_{(i,j)\in Y_\alpha} (-l_{Y_\beta}(i,j)\epsilon_1 + (a_{Y_\alpha}(i,j)+1)\epsilon_2 + a_\alpha - a_\beta) \\ & \times \prod_{(i,j)\in Y_\beta} ((l_{Y_\alpha}(i,j)+1)\epsilon_1 - a_{Y_\beta}(i,j)\epsilon_2 + a_\alpha - a_\beta) \end{aligned}$$

$$\mathrm{a}_Y(i,j) = Y_i - j$$

$$\mathrm{l}_Y(i,j) = {Y^t}_j - i$$



Recall that the partition function is precisely the topological string partition function of SU(2)-geometry for $\epsilon_1 = -\epsilon_2 = \hbar$

$$Z^{\mathrm{Nek.}}(\vec{a}, \Lambda, \epsilon_1 = -\epsilon_2 = \hbar) = Z^{\mathrm{A-model}}(t_i, \hbar)$$

geometric engineering

- **AGT conjecture** [Alday-Gaiotto-Tachikawa, '09]

Nekrasov partition function of instanton counting	$\Longleftrightarrow$	correlators (conformal blocks) of 2-dim CFT

$\uparrow$

$\mathcal{N}=2$ superconformal $\mathrm{SU}(2)$ quiver gauge theory

[Gaiotto, '09] [Marshakov-Mironov-Morozov, '09]

Nekrasov partition function of SU(2) pure SYM	$\Longleftrightarrow$	Shapovalov form of Virasoro algebra

- SU(3) SYM gives the Shapovalov form of $\mathcal{W}_3$-algebra [M.T, '09]

Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

highest weight representation

weight : eigenvalue of L_0

vector $|\Delta\rangle$: $L_{n>0}|\Delta\rangle = 0 \quad L_0|\Delta\rangle = \Delta|\Delta\rangle$

$\longrightarrow V_\Delta$

action of Virasoro generators

weight	**basis**
Δ	$\lvert\Delta\rangle$
$\Delta + 1$	$L_{-1}\lvert\Delta\rangle$
$\Delta + 2$	$L_{-2}\lvert\Delta\rangle \qquad {L_{-1}}^2\lvert\Delta\rangle$
	$\cdots$
$\Delta + n$	$L_{-Y}\lvert\Delta\rangle = L_{-Y_1}L_{-Y_2}\cdots\lvert\Delta\rangle \quad \lvert Y\rvert = n$

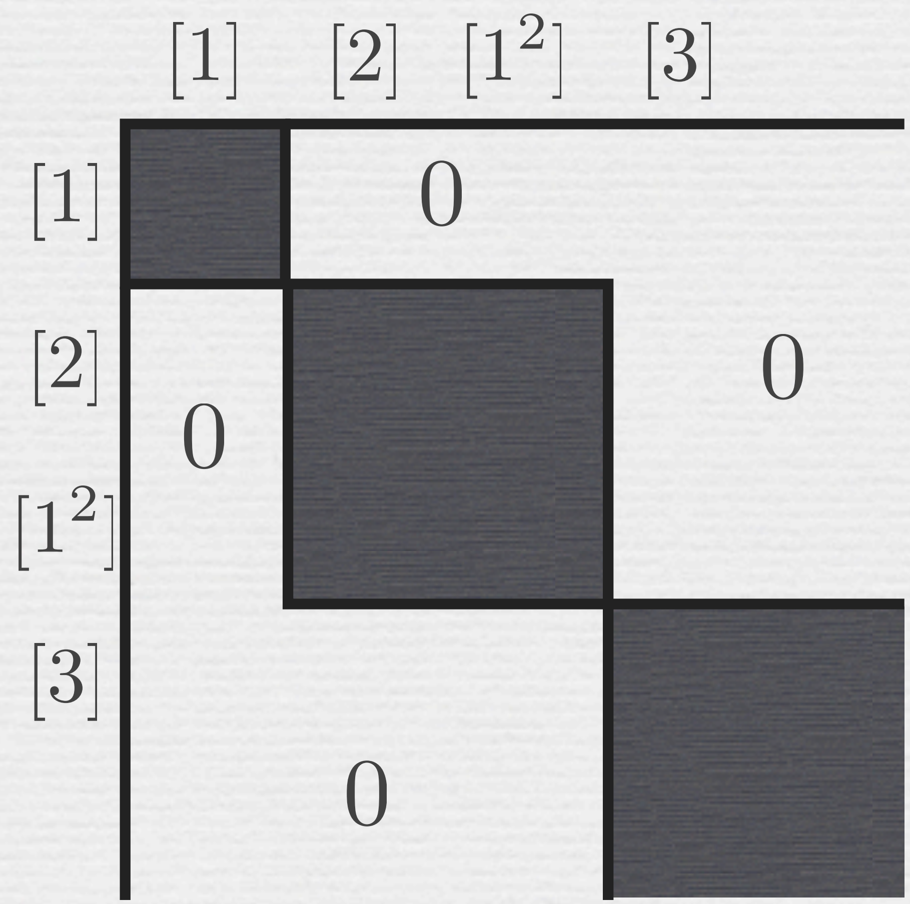
$$\langle\Delta|L_0 = \Delta\langle\Delta| \qquad \langle\Delta|L_{0<0} = 0 \quad \longrightarrow \quad {V_\Delta}^* \qquad \langle\Delta|\cdot|\Delta\rangle = 1$$

pairing (bilinear form) of V_Δ and ${V_\Delta}^*$: Shapovalov form

Shapovalov matrix

$$Q_\Delta(Y_1\,;\,Y_2) = \Big(\,\langle\Delta|L_{Y_2}L_{-Y_1}|\Delta\rangle\,\Big)$$

$\longrightarrow$ block diagonal w.r.t. $|Y_1| = |Y_2| = n$



$$e_E = \frac{\epsilon_E}{\sqrt{-\epsilon_1\epsilon_2}}, \quad E = 1, 2$$

$$e := e_1 + e_2$$

$$Z^{\text{inst}}(a, \Lambda, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} \Lambda^{4k} Z_k(a, \epsilon_1, \epsilon_2)$$
$$= \sum_{k=0}^{\infty} \frac{\Lambda^{4k}}{(-\epsilon_1\epsilon_2)^{2k}} Z_k\left(\frac{a}{\sqrt{-\epsilon_1\epsilon_2}}, e_1, e_2\right)$$

- **non-conformal AGT conjecture**
  [Gaiotto, '09] [Marshakov-Mironov-Morozov, '09]

$$Z^{\text{inst}}_{n}(a,\ e_1,\ e_2) = Q^{-1}_{\Delta}([1^n];[1^n])$$

$$c = 1 - 6e^2$$

$$\Delta = a^2 - \frac{e^2}{4}$$

- Refinement

Recall that the partition function is precisely the topological string partition function of SU(N)-geometry for $\epsilon_1 = -\epsilon_2 = \hbar$

$$Z^{\mathrm{Nek.}}(\vec{a}, \Lambda, \epsilon_1 = -\epsilon_2 = \hbar) = Z^{\mathrm{A-model}}(t_i, \hbar)$$

So it is very natural to expect that there exist the extension of topological string which recover the Nekrasov's partition function for $\epsilon_1 \neq -\epsilon_2$

- **Refined Topological Vertex**

It is very hard to find a guiding star for refining the topological vertex formalism

- **Awata-Kanno's idea**

The Macdonald functions are a multi-parameter extension of the Schur functions.

$$s_\mu(q^{-\rho}) = q^{\frac{\|\mu^t\|^2}{2}} \tilde{Z}_\mu(q)$$

$$\downarrow$$

$$P_{\nu^t}(t^{-\rho}; q, t) = t^{\frac{1}{2}||\nu||^2} \tilde{Z}_\nu(t, q), \quad \tilde{Z}_\mu(t, q) = \prod_{(i,j)\in\nu} (1 - t^{\nu_j^t - i + 1} q^{\nu_i - j})^{-1}$$

Macdonald function

$$\tilde{Z}_\mu(t, q) \xrightarrow{t \to q} \tilde{Z}_\mu(q)$$

So we may obtain a refined vertex by replacing the Schur functions with the specialized Macdonald functions !

[Awata-Kanno, '05, '08]

$$C_{\lambda\mu}{}^{\nu}(t,q) = f_\nu(t,q)^{-1} P_{\mu^t}(t^\rho;q,t)$$
$$\times \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} (-1)^{|\lambda|+|\eta|} \iota P_{\lambda^t/\eta^t}(t^{\mu^t} q^\rho; t, q) P_{\nu/\eta}(t^\rho q^\nu; q, t)$$

skew Macdonald function

The framing factor is [M.T. '07]

$$f_\nu(t,q) = (-1)^{|\nu|} t^{-\frac{||\nu^t||^2}{2}} q^{\frac{||\nu||^2}{2}}$$

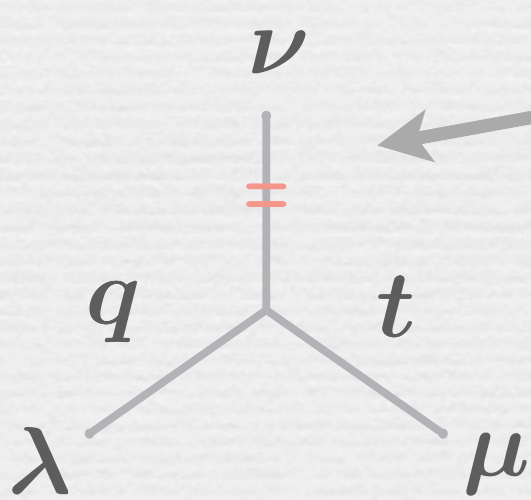
For the geometric engineering Calabi-Yau's, the amplitudes computed using the vertex give the Nekrasov's partition functions [Awata-Kanno, '08].

[Iqbal-Kozcaz-Vafa]

$$C_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa_\mu}{2}} P_{\nu^t}(t^{-\rho};q,t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

IKV's refined vertex breaks the cyclic symmetry for three legs of the vertex !

$$C_{\lambda,\mu,\nu}(t,q) \neq C_{\mu,\nu,\lambda}(t,q) \neq C_{\nu,\lambda,\mu}(t,q)$$



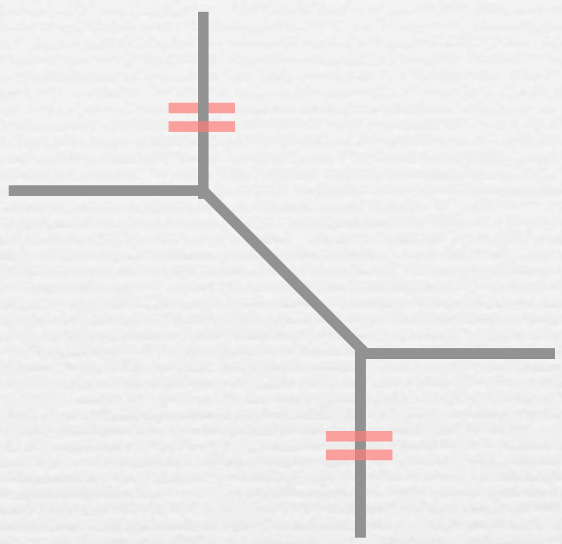
preferred direction

additional rule

take all the direction parallel (same $\mathbb{CP}$)

→ There exists the ambiguity about a choice of preferred direction when one construct the amplitudes using the refined vertex !

Flop invariance [M.T, '08]



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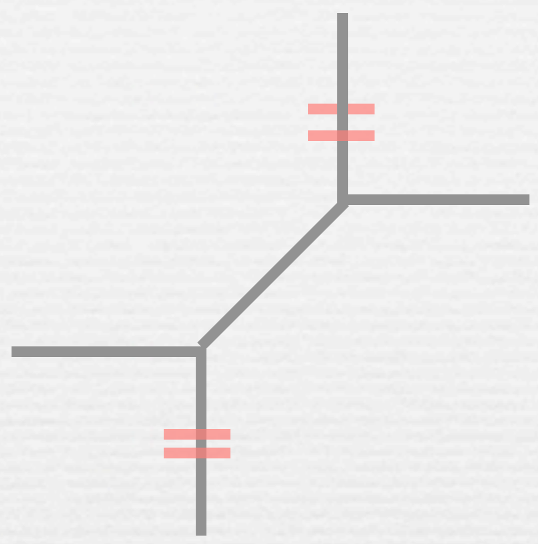
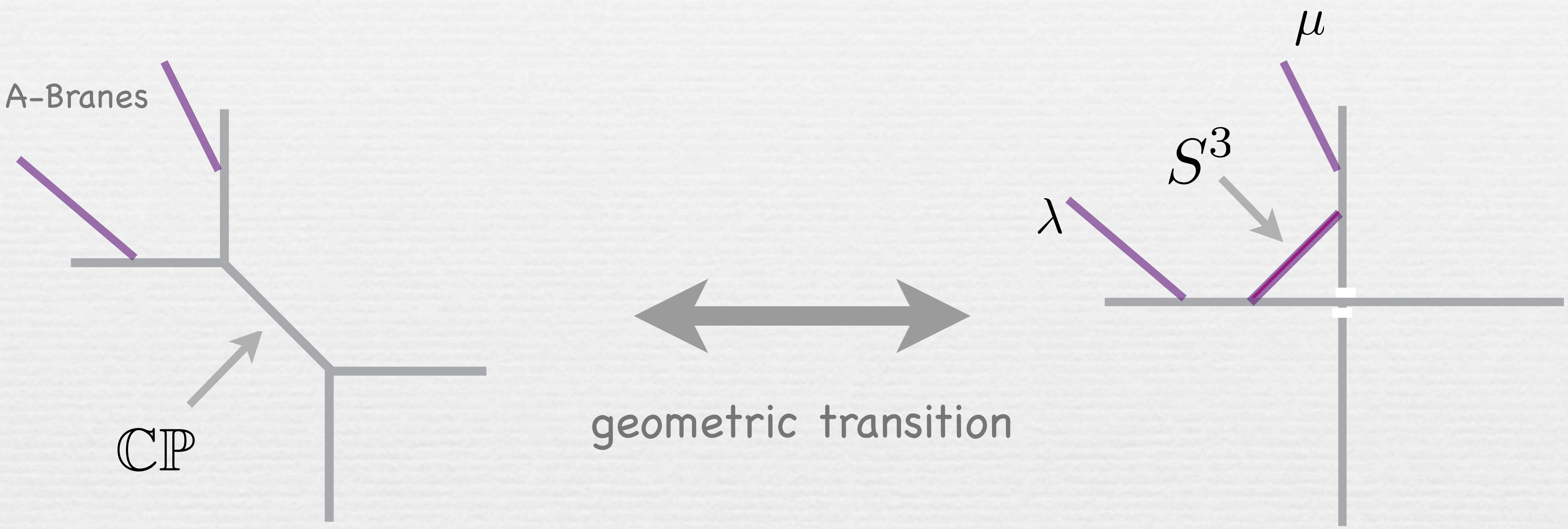


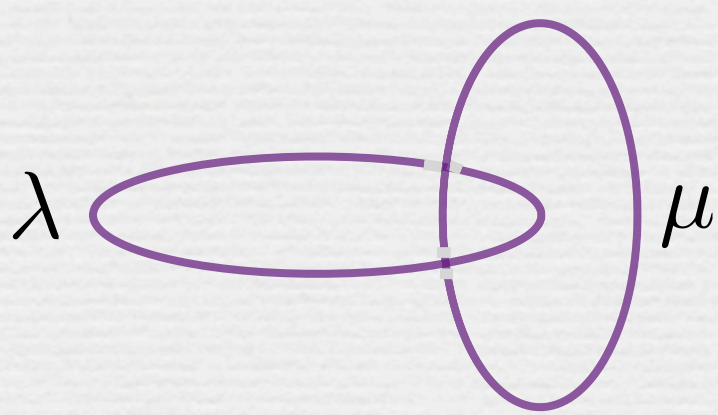
$$t \longrightarrow -t$$

6. Link Invariant & Topological Strings

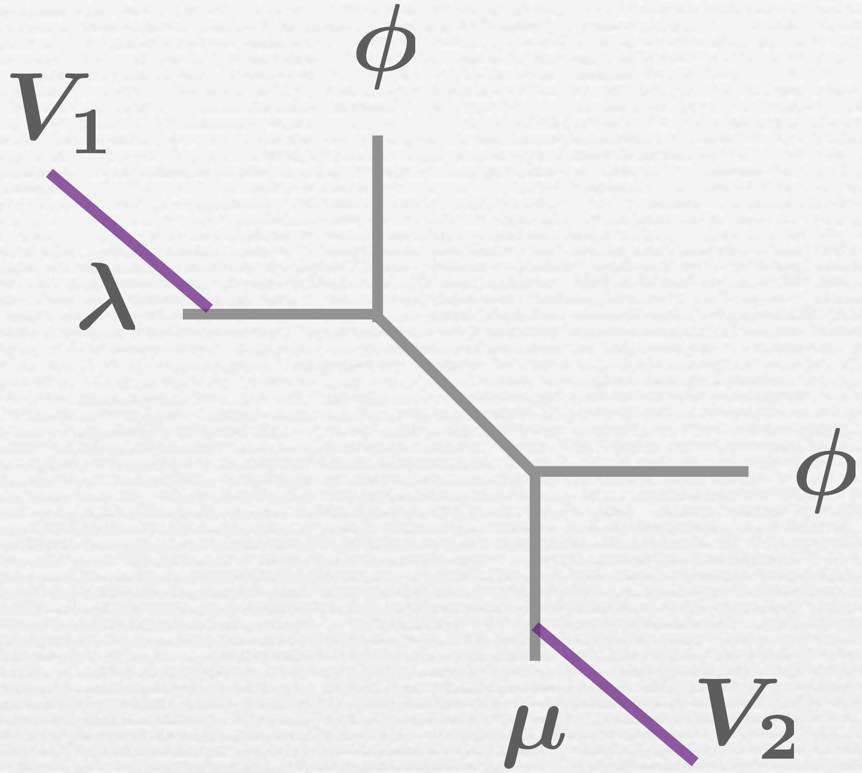
- **Chern-Simons invariants in topological strings [Ooguri-Vafa]**



Boundaries of worldsheets make Wilson loop in S^3



Hopf link



$$Z = \sum_{\lambda,\mu} Z_{\lambda,\mu}(q;Q)\,\mathrm{Tr}_\lambda V_1\,\mathrm{Tr}_\lambda V_2$$

$$\downarrow$$

$$Z_{\lambda,\mu}(q;Q) = \sum_{\nu} C_{\lambda,\nu,\phi}(q)(-Q)^{|\nu|}C_{\nu^t,\mu,\phi}(q)$$

$$\downarrow \quad Q = q^N$$

$$\boxed{W_{\lambda\mu} = q^{\kappa_\mu/2} s_\lambda(q^{-\rho}) s_\mu(q^{-\rho-\lambda}, q^{-N+\rho}) \prod_{(i,j)\in\lambda} (1 - q^{-N+i-j})}$$

Hopf link invariant !!

- Homological link invariants

Polynomial invariants of knots and links

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^j \mathbf{q}^i \dim \mathcal{H}^{sl(N),R_1,\cdots,R_k}_{i,j}(L)$$

Euler characteristic

Conjectural cohomology

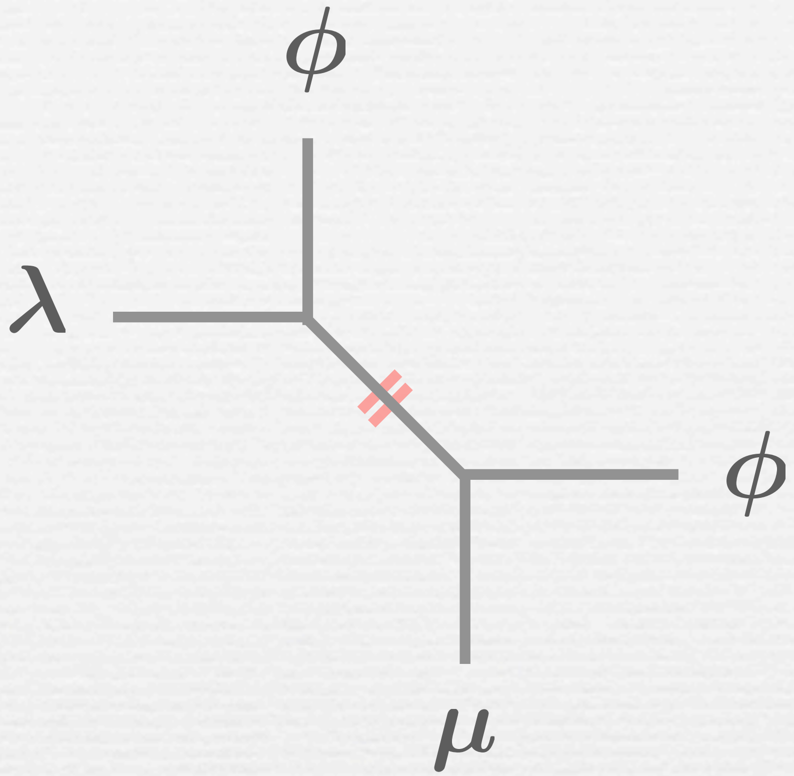
Homological invariant

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q},\mathbf{t}) = \sum_{i,j\in\mathbb{Z}} \mathbf{q}^i \mathbf{t}^j \dim \mathcal{H}^{sl(N),R_1,\cdots,R_k}_{i,j}(L)$$

Poincare characteristic

Mathematical theory of homological link invariant is formulated for some representations.

Gukov-Iqbal-Kozcaz-Vafa embedded it into refined topological strings expecting that it gives some insights into their formulation.



$$Z_{\lambda\mu}(t,q,Q) = \sum_{\nu} (-Q)^{|\nu|} C_{\phi\mu\nu}(t,q) C_{\lambda\phi\nu^t}(q,t)$$

Recall the refined topological vertex

$$C_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa_\mu}{2}} P_{\nu^t}(t^{-\rho};q,t) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$$

From the partition function we get the superpolynomial

Gukov-Iqbal-Kozcaz-Vafa

$$\bar{\mathcal{P}}_{\lambda.\mu}(\mathbf{q},\mathbf{t},\mathbf{a}) = \sum_{\nu} (-Q)^{|\nu|} t^{\frac{1}{2}||\nu||^2} q^{\frac{1}{2}||\nu^t||^2} \tilde{Z}_\nu(q,t) \tilde{Z}_{\nu^t}(t,q) s_\lambda(t^{-\rho}q^{-\nu^t}) s_\mu(t^{-\rho}q^{-\nu^t})$$

- Superpolynomial proposal of Gukov-Iqbal-Kozcaz-Vafa

Superpolynomial [Gukov-Iqbal-Kozcaz-Vafa, '07]

$$\bar{\mathcal{P}}_{\lambda.\mu}(\mathbf{q},\mathbf{t},\mathbf{a}) = \sum_{\nu}(-Q)^{|\nu|}t^{\frac{1}{2}||\nu||^2}q^{\frac{1}{2}||\nu^t||^2}\tilde{Z}_\nu(q,t)\tilde{Z}_{\nu^t}(t,q)s_\lambda(t^{-\rho}q^{-\nu^t})s_\mu(t^{-\rho}q^{-\nu^t}) \times \prod_{i,j=1}(1-Qt^{1-1/2}q^{j-1/2})^{-1}(-1)^{|\lambda|+|\mu|}\left(Q^{-1}\sqrt{\frac{q}{t}}\right)^{\frac{|\lambda|+|\mu|}{2}}\left(\frac{q}{t}\right)^{|\lambda||\mu|}$$

New parameters

$$\sqrt{t}=\mathbf{q}, \quad \sqrt{q}=-\mathbf{t}\mathbf{q}, \quad Q=-\mathbf{t}/\mathbf{a}^2.$$

$\mathbf{a}=\mathbf{q}^N$ (Large-N duality)

Homological link invariants for Hopf link

$$\bar{\mathcal{P}}^{sl(N)}_{R_1,\cdots,R_k}(\mathbf{q},\mathbf{t})$$

They must give homological invariants

- **Example**

Hopf link

$$\bar{\mathcal{P}}_{\square,\square}(\mathbf{q},\mathbf{t},\mathbf{a}) = \frac{1}{\mathbf{a}^2}\left(\frac{1-\mathbf{q}^2+\mathbf{q}^4\mathbf{t}^2}{(1-\mathbf{q}^2)^2} - \mathbf{a}^2\frac{1+\mathbf{q}^2\mathbf{t}^2-\mathbf{q}^2+\mathbf{q}^4\mathbf{t}^2}{(1-\mathbf{q}^2)^2} + \mathbf{a}^4\frac{\mathbf{q}^2\mathbf{t}^2}{(1-\mathbf{q}^2)^2}\right) \qquad (*)$$

$(*)$ gives the Kovanov-Rozansky invariant for Hopf link

$$\boxed{\bar{\mathcal{P}}_{\square,\square}(\mathbf{q},\mathbf{t},\mathbf{a}=\mathbf{q}^N) = \mathbf{q}^{-2N}KhR(2_1^2)}$$

Polynomial !

$$KhR_{N=3}(2_1^2) = 1+\mathbf{q}^2+\mathbf{q}^4+\mathbf{q}^4\mathbf{t}^2+2\mathbf{q}^6\mathbf{t}^2+2\mathbf{q}^8\mathbf{t}^2+\mathbf{q}^{10}\mathbf{t}^2$$

$$KhR_{N=4}(2_1^2) = 1+\mathbf{q}^2+\mathbf{q}^4+\mathbf{q}^4\mathbf{t}^2+\mathbf{q}^6+2\mathbf{q}^6\mathbf{t}^2+3\mathbf{q}^8\mathbf{t}^2+3\mathbf{q}^{10}\mathbf{t}^2 +2\mathbf{q}^{12}\mathbf{t}^2+\mathbf{q}^{14}\mathbf{t}^2$$

[M.T, ‘08] Slicing invariance hypothesis → simple expression !!

Unfortunately amplitudes for generic representations do not satisfy this property.
However slicing invariance holds for some simple rep.s.!

[Awata-Kanno, ‘09]

$$\frac{Z_{[1^r],[1^s]}(Q;q,t)}{Z_{\phi,\phi}(Q;q,t)} = (-1)^s t^{-\frac{s(s-1)}{2}} e_r(t^\rho) e_s\left(\sqrt{\frac{q}{t}} Q q^{[1^r] t^\rho , t^{-rho]}}\right) \prod_{i=1} r \left(1 - Q q^{\frac{1}{2}} t^{-i+\frac{1}{2}}\right)$$

Summary

- We reviewed topological vertex method of A-model caluculation

- We saw the relation between topological vertex and instanton counting

- We apply the refined vertex for homological link invariants