# Classical Kummer Surfaces and Mordell-Weil Lattices

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**Abstract**

Suggested by the classical theory, we study the Kummer surface of a genus two Jacobian variety as an elliptic surface with special type of singular fibres. We determine the Mordell-Weil lattice together with the explicit generators in the general case.

## 1 Introduction

Let $C$ be a curve of genus two and let $J = J(C)$ be its Jacobian variety. The Kummer surface $S = \mathrm{Km}(J)$ is a smooth K3 surface obtained from the quotient surface $J/\iota_J$ of $J$ by the inversion $\iota_J$, by resolving the 16 singular points corresponding to the points of order 2 on $J$. We assume for simplicity that the base field $k$ is algebraically closed and $\mathrm{char}(k) \neq 2, 3$.

It is known in the classical theory of Kummer surfaces ([2]) that there is a beautiful symmetry called the $16_6$-configuration. There are two sets of 16 disjoint (-2)-curves on $S = \mathrm{Km}(J)$ which can be labeled as $\{A_{ij}\}$ and $\{Z_{ij}\}$ $(i, j \in I = \{1, 2, 3, 4\})$ in such a way that $A_{ij}$ and $Z_{kl}$ intersect if and only if $i = k$ or $j = l$ but not both; for instance, $Z_{11}$ meets the 6 curves $A_{12}, A_{13}, A_{14}, A_{21}, A_{31}, A_{41}$ and only these curves. Geometrically the 16 curves $\{A_{ij}\}$ are the exceptional curves corresponding to the 2-torsion points on $J$, while $\{Z_{ij}\}$ arise from embedding $C$ into $J$ as symmetric theta divisors (cf. [5], [7]).

Using the curves in the $16_6$-configuration, we can define various elliptic fibrations on $S$, since on a K3 surface it is equivalent to giving a divisor consisting of (-2)-curves which has the same type as in Kodaira's list of singular fibres ([3], [7]). In this note, we focus on an especially neat elliptic fibration $f : S \to \mathbf{P}^1$ which has two disjoint singular fibres of type $I_0^*$:

$$\Phi_1 = 2Z_{11} + A_{12} + A_{13} + A_{21} + A_{31},$$

$$\Phi_2 = 2Z_{44} + A_{24} + A_{34} + A_{42} + A_{43}.$$

For general $C$, we have six more singular fibres of type $I_2$. On the other hand, the curves $Z_{12}, Z_{13}, Z_{21}, Z_{24}, Z_{31}, Z_{34}, Z_{42}, Z_{43}$ are sections of this elliptic surface, since each of them intersects $\Phi_1$ with intersection number 1. Choose one of them as the zero-section. Then, using the height formula ([9]), we can see that three of them form the 2-torsion sections and that the remaining four are sections of height 1.

In this paper, we study more closely the elliptic surface in question (called an *elliptic Kummer surface* for short) and some related ones, in terms of explicit equations. (The geometric theory of $16_6$-configuration is used only for the motivation.) First we look at the twisted rational elliptic surface which has six $I_2$ fibres (or some confluent $I_4$ fibres) (§3). We determine the generators of the Mordell-Weil lattice (MWL) by using the height formula (§4). Then we turn to the study of the elliptic Kummer surface (§5). By using the correspondence on the curve $C$ and its image in the Kummer surface $S = \mathrm{Km}(J)$, we obtain some "new" elements in the Néron-Severi group $\mathrm{NS}(S)$, which give rise to some nontrivial rational points in the Mordell-Weil lattice (§6). Also we clarify the relation of some automorphisms of $C$ and the confluence of singular fibres (producing type $I_4$). In §7, we find a rational point of height 1 which gives an explicit generator of the MWL modulo torsion in the general case. As a byproduct, we obtain elliptic K3 surfaces with twelve $I_2$ fibres with positive rank, depending on 3 moduli.

# 2 The defining equation

Let us take the equation of a genus two curve $C$ as follows:

$$y^2 = f_5(x) = x(x^4 + c_1x^3 + c_2x^2 + c_3x + c_4) = x\prod_{i=1}^{4}(x - d_i) \tag{2.1}$$

where we always assume that $\{d_1, d_2, d_3, d_4, 0, \infty\}$ are mutually distinct 6 points of the $x$-line, normalized so that

$$c_4 = \prod_{i=1}^{4} d_i = 1. \tag{2.2}$$

As the Jacobian variety $J$ of a genus $g$ curve $C$ is birationally equivalent to the $g$-th symmetric product of $C$ in general ([11]), the function field of $J$, $k(J)$, is generated (in case $g = 2$) by the symmetric functions

$$x_1 + x_2, x_1x_2, y_1 + y_2, y_1y_2, x_1y_2 + x_2y_1$$

of two independent generic points $(x_1, y_1)$ and $(x_2, y_2)$ of $C$ over $k$. As the inversion $\iota_J$ is induced by the hyperelliptic involution $(x, y) \mapsto (x, -y)$, we see that the function field $k(S) = k(J/\iota_J)$ is generated by

$$X = x_1 + x_2,\ t = x_1x_2,\ Y = y_1y_2. \tag{2.3}$$

Then we have $k(S) = k(X, Y, t)$ with the relation

$$Y^2 = f_5(x_1)f_5(x_2) = x_1x_2 \prod_{i=1}^{4}(x_1 - d_i)(x_2 - d_i), \tag{2.4}$$

which can be rewritten, under (2.2), as

$$E : Y^2 = t \prod_{i=1}^{4}(X - (d_i + \frac{t}{d_i})). \tag{2.5}$$

This equation defines an elliptic curve $E$ over $k(t)$, which gives an elliptic fibration on the K3 surface $S = \mathrm{Km}(J)$:

$$f : S \to \mathbf{P}^1 \tag{2.6}$$

# 3 The twisted rational elliptic surface

First we consider the quadratic twist $\mathcal{E}$ of the elliptic curve $E$ with respect to $k(\sqrt{t})$:

$$\mathcal{E} : Y^2 = \prod_{i=1}^{4}(X - (d_i + \frac{t}{d_i})). \tag{3.1}$$

By an elementary algorithm as in [1, Ch.8], we can transform it into the Weierstrass form:

$$\mathcal{E}_{\mathcal{W}} : y^2 = x^3 + a_4(t)x + a_6(t) \tag{3.2}$$

where $a_4(t), a_6(t)$ are polynomials in $t$ of degree 4 or 6 with coefficients in $k_0 = \mathbf{Q}_0(d_1, d_2, d_3, d_4)$. (Here $\mathbf{Q}_0$ denotes the prime field in $k$.) We do not write down $a_4(t), a_6(t)$ here, but they can be derived easily from (4.5). The discriminant $\Delta(\mathcal{E}_{\mathcal{W}})$ is equal, up to a constant, to

$$\delta(t) = \prod_{i<j}(t - d_i d_j)^2. \tag{3.3}$$

Therefore the elliptic surface associated with (3.2) is a rational elliptic surface, and it has 6 singular fibres of type $I_2$ at $t = d_i d_j$ (since $a_4 \neq 0$ there) provided that the six values $d_i d_j (i < j)$ are distinct.

**Lemma 3.1** *Given $d_1, d_2, d_3, d_4$ with $d_i \neq d_j (i < j)$ and $\prod_{i=1}^4 d_i = 1$, we have the following three cases:*
*(i) the six values $d_i d_j (i < j)$ are distinct; (ii) there is exactly one pair such that $d_i d_j = d_k d_l$; (iii) there are two such pairs.*

*In terms of the coefficients $c_i$ of (2.1), the above correspond to the three cases: (i) $c_1 \neq \pm c_3$ ; (ii) $c_1 = \pm c_3 \neq 0$ ; (iii) $c_1 = c_3 = 0$.*

**Lemma 3.2** *In case $c_1 = c_3$ or $c_1 = -c_3$, the curve $C$ admits an involutive automorphism*

$$\phi : (x, y) \mapsto (\frac{1}{x}, \frac{y}{x^3}) \text{ or } (\frac{-1}{x}, \frac{\sqrt{-1}y}{x^3}), \tag{3.4}$$

*and the quotient curve $C/\phi$ is an elliptic curve. Hence the Jacobian variety $J$ is isogenous to a product of elliptic curves.*

The proof of these lemmas is immediate, and we omit it.

**Proposition 3.3** *The rational elliptic surface defined by (3.2) has the following singular fibres: (i) $I_2 \times 6$, (ii) $I_2 \times 4 + I_4$, or (iii) $I_2 \times 2 + I_4 \times 2$, according to the three cases in Lemma 3.1. The structure of the Mordell-Weil lattice on $\mathcal{E}_{\mathcal{W}}(k(t))$ is accordingly isomorphic to the following:*
*(i) $A_1^{*\oplus 2} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$, (ii) $\langle \frac{1}{4} \rangle \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$, (iii) $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$.*

*Proof* The first part follows from Lemma 3.1. Then the second part follows from [6] where No.42, No.60 and No.74 are the relevant cases for (i), (ii) and (iii). *q.e.d.*

# 4 Explicit generators via the height formula

Can one write down the generators of the Mordell-Weil group explicitly? Yes, we can. To present a clear prescription for it, we proceed as follows. By (3.2) and (3.3), the RHS of (3.2) decomposes, at $t = d_1 d_2$, as $(x-\alpha)^2(x+2\alpha)$, where the computation shows that $\alpha$ is given by

$$N(d_1, d_2|d_3, d_4) = \frac{(d_1-d_3)(d_1-d_4)(d_2-d_3)(d_2-d_4)}{12\ d_3 d_4} \neq 0. \tag{4.1}$$

Now by the height formula (see [9] for what follows), we have

$$\langle P, P\rangle = 2\chi + 2(PO) - \sum_v \mathrm{contr}_v(P)\ (\chi = 1) \tag{4.2}$$

for any $P \in \mathcal{E}_W(k(t))$, with $\mathrm{contr}_v(P) = 1/2$ or $0$ for type $I_2$. Thus, for $P$ 2-torsion, we have

$$0 = \langle P, P\rangle = 2 + 2\cdot 0 - \frac{1}{2}\times 4 - 0\times 2, \tag{4.3}$$

which means that the $x$-coordinate $x(P) = A + Bt + Ct^2$ is a degree 2 polynomial which takes the value $N(d_i, d_j|d_k, d_l)$ at $t = d_i d_j$ for 4 distinct pairs $(ij)$.

By solving the linear equations in $A, B, C$:

$$A + B(d_i d_j) + C(d_i d_j)^2 = N(d_i, d_j|d_k, d_l) \tag{4.4}$$

for $i = 1, 2$ and $j = 3, 4$, we find (under the condition (2.2)) that

$$A = C = \frac{1}{12}\{(d_1+d_2)(d_3+d_4) - 2(d_1d_2 + d_3d_4)\},$$

$$B = \frac{1}{12}\{(d_1+d_2)(d_3+d_4)(d_1d_2+d_3d_4) - 2d_1d_2(d_3^2+d_4^2) - 2d_3d_4(d_1^2+d_2^2)\}.$$

We denote this 2-torsion point $P$ by $T_1$, i.e. $T_1 = (x(T_1), 0)$ with $x(T_1) = A + Bt + Ct^2$ where $A, B, C$ are determined above.

By permuting the indices $\{1, 2, 3, 4\}$, we obtain two more points $T_2, T_3$ and we have $\{O, T_1, T_2, T_3\} \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. It should be remarked that we can recover the Weierstrass equation (3.2) from these data, since it is equal to

$$y^2 = (x - x(T_1))(x - x(T_2))(x - x(T_3)). \tag{4.5}$$

Next we determine the rational points of height 1/2. The height formula says this time that

$$\frac{1}{2} = \langle P, P \rangle = 2 + 2 \cdot 0 - \frac{1}{2} \times 3 - 0 \times 3. \tag{4.6}$$

By solving the linear equations (4.4) for $i = 1, j = 2, 3, 4$, we obtain a rational point $P = Q_1$ with ($x(Q_1)$ omitted and)

$$y(Q_1) = \frac{(d_1 - d_2)(d_1 - d_3)(d_1 - d_4)}{8d_1^2}(t - d_1d_2)(t - d_1d_3)(t - d_1d_4). \tag{4.7}$$

By permutations of indices, we get four points $Q_1, Q_2, Q_3, Q_4$ of height 1/2. One can check that they are stable under translation by 2-torsions, i.e. 2-torsors.

Similarly, solving (4.4) for $(ij) = (23), (24), (34)$, we obtain $P = R_1$ with

$$y(R_1) = \frac{(d_1 - d_2)(d_1 - d_3)(d_1 - d_4)}{8}(t - d_2d_3)(t - d_2d_4)(t - d_3d_4). \tag{4.8}$$

By permutations, we get four points $R_1, R_2, R_3, R_4$ of height 1/2. One can check again that they are 2-torsors.

Now we can state the following result on explicit generators.

**Theorem 4.1** *In the general case (i), the Mordell-Weil group $\mathcal{E}_{\mathcal{W}}(k(t))$ is generated by $Q_1, R_1, T_1, T_2$, where $Q_1, R_1$ are the rational points of height 1/2 such that $\langle Q_1, R_1 \rangle = 0$ and $T_1, T_2$ generate the torsion group. In the special case (ii), $Q_1$ (and $R_1$) becomes a rational point of height 1/4, and $Q_1, T_1, T_2$ generate the Mordell-Weil group. In the very special case (iii), $Q_1$ (and $R_1$) reduces to a torsion point of order 4, and $Q_1, T_1$ generate the Mordell-Weil group isomorphic to $\mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.*

*Proof* The case (i) is proven above in view of Proposition 3.3. The verification of the case (ii) and (iii) will be left as an exercise to the reader.

# 5 The elliptic Kummer surface

Let us go back to the elliptic fibration on the Kummer surface $\mathrm{Km}(J)$ defined by (2.5).

First the Weierstrass normal form of (2.5) is given by the twist of (3.2):

$$E_W : y^2 = x^3 + t^2 a_4(t)x + t^3 a_6(t), \tag{5.1}$$

whose discriminant $\Delta(E_W)$ is equal to $t^6\delta(t)$ up to a constant (cf. (3.3)). Hence we have:

**Proposition 5.1** *The elliptic K3 surface defined by (5.1) has the two singular fibres of type $I_0^*$ at $t = 0$ and $\infty$, in addition to the semi-stable fibres (i) $I_2 \times 6$, (ii) $I_2 \times 4 + I_4$, or (iii) $I_2 \times 2 + I_4 \times 2$, as in Proposition 3.3.*

Thus the trivial lattice $T \subset \mathrm{NS}(S)$ is given by

$$T = U \oplus D_4^{\oplus 2} \oplus \begin{cases} A_1^{\oplus 6} & (i) \\ A_1^{\oplus 4} \oplus A_3 & (ii) \\ A_1^{\oplus 2} \oplus A_3^{\oplus 2} & (iii) \end{cases} \tag{5.2}$$

where $U$ is a rank two unimodular lattice spanned by the fibre class and zero-section. In particular, we have

$$\mathrm{rk}\ T = 16, 17 \text{ or } 18, \quad \det T = 2^{10}. \tag{5.3}$$

Now we consider the Mordell-Weil lattice $E_W(k(t))$. Its rank is given by the wellknown formula

$$r := \mathrm{rk}\ E_W(k(t)) = \rho(S) - \mathrm{rk}\ T \tag{5.4}$$

where $\rho(S)$ is the Picard number of $S = \mathrm{Km}(J)$. Recall that $\rho(\mathrm{Km}(A)) = \rho(A) + 16$ for any abelian surface, and that $\rho(A)$ is equal to the rank of $\mathrm{End}(A)^{sym}$ which is the symmetric part of the endomorphism algebra of $A$ (cf. [4]). In the case under consideration, we have by (5.3):

$$r = \rho(J) - \begin{cases} 0 & (i) \\ 1 & (ii) \\ 2 & (iii) \end{cases} \tag{5.5}$$

It follows from Lemma 3.1 and 3.2 that $J$ is isogenous to a product of two elliptic curves $C_1 \times C_2$ in case (ii) and to a self-product $C_1 \times C_1$ in case (iii). It implies that $\rho(J) \geq 2$ in case (ii) and $\rho(J) \geq 3$ in case (iii). Hence we have:

**Proposition 5.2** *The Mordell-Weil lattice $E_W(k(t))$ has always a positive rank $r$ given by (5.5). In particular, the Kummer surface* $\mathrm{Km}(J)$ *of any genus two curve has an infinite group of automorphisms preserving the elliptic fibration (2.6).*

**Corollary 5.3** *(to Proposition 5.1) The torsion subgroup of $E_W(k(t))$ is $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ in all three cases (i), (ii), (iii), at least if* $\mathrm{char}(k) = 0$.

*Proof* This follows from Proposition 5.1 in view of the Shimada's list [8] of singular fibres and torsion group for elliptic K3 surfaces. *q.e.d.*

In the next sections, we construct some rational points (sections) of infinite order in $E_W(k(t))$ by two different methods: the first one is to make use of correspondence on the curve, while the second depends on the nature of the height formula and symmetric functions (Galois theory).

# 6 The use of correspondence of the curve

To study the sections on an elliptic K3 surface $S$, we go back to the canonical isomorphism $E(K) \simeq \mathrm{NS}(S)/T$ where $K = k(t)$, which is the source of the formula (5.4) (cf. [9]). Given any divisor $D$ on $S$, its class in the Néron-Severi group $\mathrm{NS}(S)$ modulo $T$ determines a point $P \in E(K)$ by the rule: take the restriction of $D$ to the generic fibre $E$ of $f : S \to \mathbf{P}^1$, and then sum up the points on $E(\bar{K})$ to have a point $P \in E(K)$. This method has been used recently to study the Mordell-Weil lattice related to the Kummer surface of a product abelian surface (cf. [10]).

We apply this idea to the following situation. Starting from a correspondence $\Gamma$ on the curve $C$, i.e. $\Gamma \subset C \times C$, we take its image under the rational map $C \times C \to J \to S = \mathrm{Km}(J)$, and obtain a rational point $P \in E(K)$ as above. In particular, given an automorphism $\varphi : C \to C$, we take its graph $\Gamma = \Gamma_\varphi$ and obtain a point $P_\varphi$.

**Proposition 6.1** *In case $\varphi = id$, the identity automorphism of $C$, the rational point $P_\varphi = (x, y) \in E_W(k(t))$ is given by the following:*

$$x = \frac{1}{48}(3 - 2c_2t + (38 + 3c_2^2 - 8c_1c_3)t^2 - 2c_2t^3 + 3t^4). \tag{6.1}$$

*Proof* In this case, $\Gamma_\varphi = \Delta_C$ is the diagonal in $C \times C$. In the notation of §2, we have then $(x_1, y_1) = (x_2, y_2)$ so that

$$X = 2x_1, t = x_1^2, Y = y_1^2 = f_5(x_1) = x_1(x_1^4 + \cdots + 1). \tag{6.2}$$

It follows that $x_1 = \pm\sqrt{t}$, and that $X = 2x_1, Y = f_5(x_1)$. Thus we have the two points $P = (X, Y) = (2\sqrt{t}, f_5(\sqrt{t}))$ and its conjugate $P'$ on the curve $E$ defined by (2.5) over $k(t)$. The point $P$ defines a point $(X, Y) = (2\sqrt{t}, (\sqrt{t})^4 + c_1(\sqrt{t})^3 + \cdots + 1)$ on the curve $\mathcal{E}$, (3.1), which are transformed to the point $Q \in \mathcal{E}_{\mathcal{W}}(k(\sqrt{t}))$ defined by (3.2). The sum $P + P' \in E(k(t))$ corresponds to $Q - Q'$, and an explicit computation gives the rational point $P_\varphi$ ($\varphi = id$) on $E_W(k(t))$, of the form $(x, y)$ with $\deg(x) = 4, \deg(y) = 6$; explicitly the $x$-coordinate of this point is given by (6.1). *q.e.d.*

The ring of correspondence on $C$ ([11]) is isomorphic to the ring of endomorphisms of the Jacobian variety End$(J)$, and the latter is isomorphic to $\mathbf{Z}$ for a general curve $C$, generated by the identity. The above method can be applied to other correspondence (at least of relatively small degree), but we do not go into this since the structure of End$(J)$ is not so simple in general.

On the other hand, it should be remarked how the use of correspondence clarifies the occurrence of singular fibres of type $I_4$ in the special case (ii) or (iii) of Proposition 3.3.

**Proposition 6.2** *In the situation of Lemma 3.2, the graph of the automorphism $\phi$ is mapped to an irreducible component of the singular fibre (of type $I_4$) over $t = 1$ or $t = -1$.*

*Proof* Suppose $\phi$ is the first automorphism defined by (3.4) in case $c_1 = c_3$. The generic point of the graph $\Gamma_\phi$ is $(x_1, y_1) \times (x_2, y_2)$ where $x_2 = 1/x_1$. By (2.3), we have

$$X = x_1 + \frac{1}{x_1}, \quad t = x_1 x_2 = 1. \tag{6.3}$$

Hence $\Gamma_\phi$ is mapped into the fibre over $t = 1$. Since the six branch points of the double cover $C \to \mathbf{P}^1$ are stable under $\phi$, we may assume for example $d_2 = 1/d_1$. Then we have $d_1 d_2 = 1 = d_3 d_4$ by (2.2). Thus the two $I_2$-fibres over $t = d_1 d_2$ and $d_3 d_4$ in general has the confluence at $t = 1$ and the resulting fibre is of type $I_4$ by Proposition 3.3 or 5.1. The other case $c_1 = -c_3$ is similar. *q.e.d.*

# 7 An explicit generator of height 1

According to the classical theory recalled in §1, we should have rational points $P \in E_W(k(t))$ of height 1. (The point given by Proposition 6.1 has height 4 in general.) The height formula (4.2) (with $\chi = 2$ for K3)

$$1 = \langle P, P \rangle = 4 + 2 \cdot 0 - \frac{1}{2} \times 6, \tag{7.1}$$

suggests a possibility of a section $P = (x, y)$ where $x = \sum_{n=0}^{4} A_n t^n$ is a degree 4 polynomial such that

$$\sum_{n=0}^{4} A_n (d_i d_j)^n = N'(d_i, d_j | d_k, d_l) \tag{7.2}$$

for all six $i, j (i < j)$. Here the RHS is the $x$-coordinates of the node of the degenerate Weierstrass cubic (5.1) at $t = d_i d_j$; explicitly we have

$$N'(d_1, d_2 | d_3, d_4) = (d_1 d_2) N(d_1, d_2 | d_3, d_4). \tag{7.3}$$

Now we can uniquely solve the 6 linear equations (7.2) in the 5 unknown $A_n$ in the same way as in §4, and the result is expressed in terms of the elementary symmetric functions $c_n$ of $d_1, \ldots, d_4$ (see (2.1)) as follows:

$$A_0 = A_4 = \frac{1}{4}, \; A_1 = A_3 = -\frac{c_2}{6}, \; A_2 = \frac{c_1 c_3 + 2}{12}. \tag{7.4}$$

In this way, we find a beautiful $k(t)$-rational point:

**Proposition 7.1** *There is a rational point $P_1 \in E_W(k(t))$ with*

$$x(P_1) = \frac{1}{4}t^4 - \frac{c_2}{6}t^3 + \frac{c_1 c_3 + 2}{12}t^2 - \frac{c_2}{6}t + \frac{1}{4}, \; y(P_1) = \frac{1}{8}\prod_{i<j}(t - d_i d_j). \tag{7.5}$$

*which has height 1. The duplicated point $2P_1$ of height 4 is equal (up to sign) to the point $P_{id}$ in Proposition 6.1 arising from the identity correspondence.*

*Proof* The point (7.5) does not meet the singular point of the cuspidal cubic (5.1) at $t = 0$ or $\infty$, but it does meet the node at the six value $t = d_i d_j$ because it is so arranged by (7.2) above. Hence the height of $P_1$ is equal to 1 by (7.1). We can check that this is true even in the confluent cases (ii) and (iii). A direct computation shows $2P_1 = \pm P_{id}$. *q.e.d.*

**Theorem 7.2** *The above point $P_1$ of height 1 is a generator of the Mordell-Weil group $E_W(k(t))$ modulo the 2-torsion subgroup, for any genus two curve $C$ with* $\mathrm{End}(J) = \mathbf{Z}$

*Proof* If $\mathrm{End}(J) = \mathbf{Z}$, we have $r = \rho(J) = 1$, and we are in the case (i) with six $I_2$-fibres. Suppose $P$ is a generator modulo torsion of the rank 1 lattice and $P_1 = nP$ for some positive integer $n$. Then the height $\langle P, P\rangle$ of $P$ is equal to $1/n^2$. On the other hand, it follows from the formula (4.2) that it is an integer or a half integer. Therefore we have $n = 1$ which implies that $P_1 = P$ is a generator. *q.e.d.*

**Corollary 7.3** *If* $\mathrm{End}(J) = \mathbf{Z}$ *(or $\rho(J) = 1$), the Néron-Severi lattice of the Kummer surface* $\mathrm{NS}(\mathrm{Km}(J))$ *has* $\mathrm{rk} = 17$ *and* $\det = 2^6$.

*Proof* This is wellknown (cf.[7]), but it follows also from the above consideration. In fact, the index $I$ of the narrow MWL $E(K)^0$ in $E(K)$ is equal, in general, to the index $[N : T \oplus L]$ where $L = T^{\perp}$ (cf. [9]). In our case, $L \simeq E(K)^0$ is generated by $2P_1$ of height 4. Hence $\det L = 4$ and $I = 2|E(K)_{tor}| = 2^3$. Thus $\det N$ is equal to $\det(T)\det(L)/I^2 = 2^{10} \cdot 4/2^6 = 2^6$. *q.e.d.*

Finally let us consider the elliptic K3 surface $\tilde{S}$ which is obtained by the base change $\mathbf{P}^1_u \to \mathbf{P}^1_t$, $u \mapsto t = u^2$, whose generic fibre $\tilde{E}$ is defined by (3.2) over $k(u)$:

$$\tilde{E} = \mathcal{E}_{\mathcal{W}} \otimes_{k(t)} k(u) : y^2 = x^3 + a_4(u^2)x + a_6(u^2). \qquad (7.6)$$

**Theorem 7.4** *The elliptic K3 surface $\tilde{S}$ has the semi-stable singular fibres only: (i) $I_2 \times 12$, (ii) $I_2 \times 8 + I_4 \times 2$, or (iii) $I_2 \times 4 + I_4 \times 4$, according to the three cases in Lemma 3.1. The Mordell-Weil group $\tilde{E}(k(u))$ contains with finite index the subgroup generated by $E(k(t))$ and $\mathcal{E}_{\mathcal{W}}(k(t))$. In particular, if* $\mathrm{End}(J) = \mathbf{Z}$*, it contains 3 independent rational points $\tilde{P}_1, \tilde{Q}_1, \tilde{R}_1$ of respective height 2,1,1 which are mutually orthogonal.*

*Proof* This follows easily from Proposition 3.3, Theorems 4.1 and 7.2, because the height is multiplied by the degree of the base change (cf.[9]). *q.e.d.*

This result might be of some use in the study of supersingular K3 surfaces in positive characteristic, since the elliptic fibration is always semi-stable.

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