

AN INTERESTING ELLIPTIC SURFACE OVER AN ELLIPTIC CURVE

TETSUJI SHIODA AND MATTHIAS SCHÜTT

To Michel Raynaud

ABSTRACT. We study the elliptic modular surface attached to the commutator subgroup of the modular group. This has an elliptic curve as base and only one singular fibre. We employ an algebraic approach and then consider some arithmetic questions.

1. INTRODUCTION

Let Γ' be the commutator subgroup of the modular group $\Gamma = SL(2, \mathbb{Z})$. It is known that Γ' is a congruence subgroup of Γ of index 12 with

$$-\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \in \Gamma', \quad -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \Gamma'.$$

Let $S = S(\Gamma')$ denote the elliptic modular surface attached to Γ' in the sense of [9]. This elliptic surface has the remarkable property that it has only one singular fibre. In Kodaira's notation [2], this fibre has type I_6^* (see [9, Ex. 5.9]).

From an analytic viewpoint, S has been studied by Stiller in [13]. Here we follow a more algebraic approach. We then consider some arithmetic questions. Throughout the paper, k will denote an algebraically closed field of characteristic $\neq 2, 3$.

2. THE ELLIPTIC MODULAR CURVE ASSOCIATED TO Γ'

Let $B = B(\Gamma')$ denote the modular curve attached to Γ' . Since $\Gamma(6) \subset \Gamma' \cdot \{\pm 1\}$, this elliptic curve is closely related to the modular curve $B(6)$

Date: May 14, 2007.

2000 Mathematics Subject Classification. 14G35, 14J27.

Key words and phrases. elliptic modular surface, commutator subgroup, Tate Conjecture.

We thank the Dipartimento di Matematica "Frederico Enriques" of Milano University for kind hospitality and stimulating atmosphere. The first author was funded by Grant-in-Aid for Scientific Research No. 17540044. The second author gratefully acknowledges support from the MCRTN network "Arithmetic Algebraic Geometry" and from DFG under grant "Schu 2266/2-1". We thank the anonymous referee for helpful comments.

with level 6-structure. In fact,

$$\Gamma' \cdot \{\pm 1\} = \Gamma^2 \cap \Gamma^3$$

with the subgroups of squares resp. cubes in Γ [5]. Hence we first recall the classical formulae for the elliptic curves with level 2 and level 3-structure (employing Igusa's notation [1]).

The elliptic curve with level 2-structure is given in *Legendre form* with a parameter $\lambda \in k - \{0, 1\}$: *Legendre form* with parameter $\lambda \neq 0, 1$:

$$E_\lambda/k(\lambda) : y^2 = x(x-1)(x-\lambda),$$

$$j(E_\lambda) = 256(\lambda^2 - \lambda + 1)^3/\lambda^2(\lambda - 1)^2,$$

$$(1) \quad j(E_\lambda) - 12^3 = \eta^2.$$

where $\eta = 8(\lambda + 1)(\lambda - 2)(2\lambda - 1)/\lambda(\lambda - 1)$.

The elliptic curve with level 3-structure takes the *Hessian form*

$$E_\mu/k(\mu) : X^3 + Y^3 + Z^3 - 3\mu XYZ = 0$$

with parameter $\mu \in k, \mu^3 \neq 1$. Transformation to Weierstrass form leads to

$$E_\mu/k(\mu) : y^2 = x^3 - 27\mu(\mu^3 + 8)x + 54(\mu^6 - 20\mu^3 - 8).$$

Setting $\xi = 3\mu(\mu^3 + 8)/(\mu^3 - 1)$ gives

$$(2) \quad j(E_\mu) = \xi^3.$$

Combining (1) and (2) in case $j = j(E_\lambda) = j(E_\mu)$, we derive the defining equation of the modular curve B :

$$(3) \quad B/k : \eta^2 = \xi^3 - 12^3.$$

We choose the point at ∞ as origin of the group law and denote it by o_B .

3. THE ELLIPTIC MODULAR SURFACE ATTACHED TO Γ'

There is a remarkable elliptic surface over B which has constant discriminant and yet variable moduli (i.e. non-constant j -invariant) with only one singular fibre. Namely, consider the elliptic curve

$$(4) \quad E : y^2 = x^3 - 27\xi x - 54\eta$$

over the function field $k(B)$. It is immediate that E has discriminant

$$\Delta = 2^6 3^9 (\xi^3 - \eta^2) = 6^{12}.$$

Hence the associated elliptic surface S has no singular fibre over $B - \{o_B\}$. On the other hand, we have

$$j(E) = \xi^3,$$

so that $\text{ord}_{o_B}(j) = -6$. Since $\text{ord}_{o_B}(\Delta) = 12$, the singular fibre over o_B has type I_6^* (cf. [15]).

As modular curve resp. surface for Γ' , such B and S are naturally unique. In particular, for $k = \mathbb{C}$, we have

$$B \otimes \mathbb{C} = \overline{\mathbb{H}}/\Gamma', \quad S \otimes \mathbb{C} = S(\Gamma')$$

with \mathbb{H} the upper half plane. Note also that the fundamental group of the pointed curve $B(\mathbb{C}) - \{o_B\}$ is isomorphic to Γ' .

However, S is not the unique elliptic surface over B with only one singular fibre up to isomorphism. In fact, twisting its defining equation (4) over the 2-torsion points of B , we obtain three more such surfaces. These are also modular and mutually isomorphic (cf. [13]). Among these surfaces, S is distinguished by the constant discriminant Δ . Any elliptic surface over an elliptic curve with constant discriminant and exactly one singular fibre is, up to isomorphism, obtained from S via purely inseparable base change.

4. THE ELLIPTIC MODULAR SURFACE OF LEVEL 6

We already pointed out that there is a close relation between Γ' and the principal congruence subgroup $\Gamma(6)$. In this section, this will be made explicit.

In the notation of Section 2, consider the diagram of extensions

$$\begin{array}{ccc} & \mathbb{Q}(\lambda, \mu) & \\ & \nearrow & \nwarrow \\ \mathbb{Q}(\lambda, \xi) & & \mathbb{Q}(\eta, \mu) \\ & \nwarrow & \nearrow \\ & \mathbb{Q}(\eta, \xi) & \end{array}$$

It defines a commutative diagram of isogenies of elliptic curves

$$\begin{array}{ccc} & B(6) & \\ & \swarrow & \searrow \\ A & & \tilde{B} \\ & \searrow & \swarrow \\ & B & \end{array}$$

where the \swarrow are the duplication maps and the \searrow are 3-isogenies.

Then the base change of the elliptic curve E over $k(B)$ from B to $B(6)$ gives, appropriately twisted, the elliptic modular surface of level 6. At the same time, we obtain the identification of $\mathbb{Q}(B(6)) = \mathbb{Q}(\lambda, \mu)$ with $\mathbb{Q}(s, t)$ satisfying $s^2 = t^3 - 12^3$. This is perhaps slightly more natural than the construction given in [6].

5. THE CUSP FORMS ASSOCIATED TO S

We recall some notable properties of elliptic modular surfaces (cf. [9]). Let $\Gamma'' \subset \Gamma$ of finite index such that $-1 \notin \Gamma''$. Consider the complex elliptic modular surface $S = S(\Gamma'')$ attached to Γ'' :

- (i) The holomorphic 2-forms on the elliptic modular surface S correspond to the cusp forms of weight 3 with respect to Γ'' . (This resembles the correspondence of holomorphic 1-forms on the modular curve $B(\Gamma'')$ with the cusp forms of weight 2 with respect to Γ'' .) In particular, the geometric genus p_g of S equals the dimension of the \mathbb{C} -vector space of cusp forms $\mathcal{S}_3(\Gamma'')$.
- (ii) S is extremal: The Néron-Severi group has maximal rank $\rho(S) = h^{1,1}(S)$, while the Mordell-Weil group has rank zero.

For the commutator subgroup Γ' , we have $g = p_g = 1$. Thus, the Hodge diamond of $S = Y(\Gamma')$ reads

$$\begin{array}{ccccc} & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \end{array}$$

We shall now determine the precise cusp forms corresponding to our model S/\mathbb{Q} , given by (3) and (4). For some prime ℓ , we consider the ℓ -adic Galois representations associated to $H_{et}^1(S, \mathbb{Q}_\ell)$ and to the transcendental lattice

$$T_S = NS(S)^\perp \subset H^2(S, \mathbb{Z}).$$

By the above properties, both representations are two-dimensional and correspond to some cusp forms with rational Fourier coefficients a_n resp. b_n . Furthermore, $H_{et}^3(S, \mathbb{Q}_\ell) = H_{et}^1(S, \mathbb{Q}_\ell)(1)$ by Poincaré duality, so the corresponding L -series agree up to a shift. The remaining cohomology is algebraic. It gives rise to one-dimensional Galois representations which are easily understood.

Recall Dedekind's eta-function

$$\eta(\tau) = q \prod_{n \in \mathbb{N}} (1 - q^n) \quad (q = e^{2\pi i \tau / 24}).$$

It is classically known that

$$\mathcal{S}_2(\Gamma') = \mathbb{C} \eta(\tau)^4 \quad \text{and} \quad \mathcal{S}_3(\Gamma') = \mathbb{C} \eta(\tau)^6.$$

As a consequence, the cusp forms corresponding to $H_{et}^1(S, \mathbb{Q}_\ell)$ and T_S are twists of the above forms, depending on the chosen model S/\mathbb{Q} (which implicitly also includes the choice of the model B/\mathbb{Q}).

Note that $H_{et}^1(S, \mathbb{Q}_\ell) = H_{et}^1(B, \mathbb{Q}_\ell)$. Hence we can use the known modularity of B to deduce

$$(5) \quad L(H_{et}^1(S, \mathbb{Q}_\ell), s) = L(\eta(\tau)^4, s).$$

Classically, this follows from the fact that B has CM by $\mathbb{Q}(\sqrt{-3})$ (cf. [12, II.10.6]). Alternatively, since B has conductor 36, it can be derived from the fact that

$$\mathcal{S}_2(\Gamma_0(36)) = \mathbb{C} \eta(6\tau)^4.$$

In terms of L -series, there is no need to distinguish between these two cusp forms since $L(\eta(\tau)^4, s) = L(\eta(6\tau)^4, s)$.

On the other hand, we consider T_S . Note that S has good reduction outside $\{2, 3\}$. As this prescribes the ramification of $H_{et}^*(S, \mathbb{Q}_\ell)$, there are only a few possible twists of $\eta(\tau)^6$. We have to take both quadratic and quartic twists into consideration, since $\eta(\tau)^6$ has complex multiplication by $\mathbb{Q}(\sqrt{-1})$. The quartic twisting can be achieved in terms of the corresponding Grössencharacter of $\mathbb{Q}(\sqrt{-1})$ which has conductor (2) and ∞ -type 2 (cf. [7]).

An observation of the twisting characters in question shows that they are determined by their values at (the primes in $\mathbb{Z}[\sqrt{-1}]$ above) 5 and 13. To determine the cusp form corresponding to T_S , it thus suffices to know the Fourier coefficients at 5 and 13.

At a prime $p > 3$, the coefficient b_p can be computed with the Lefschetz fixed point formula. We use the modularity result of (5) and Poincaré duality as mentioned, plus the fact that the one-dimensional (i.e. algebraic) representations involved are trivial. In other words, $NS(S)$ is generated by divisors over \mathbb{Q} . In the present case, this follows from Tate's algorithm [15] which shows that all components of the I_6^* fibre of S are defined over \mathbb{Q} . As a result, the Lefschetz fixed point formula for S reads

$$\#S(\mathbb{F}_p) = 1 + 12p + b_p + p^2 - (1 + p)a_p.$$

Since we know a_p , we can calculate b_p from the number of points of S over \mathbb{F}_p . Counting points with a machine, we obtain $b_5 = -6$ and $b_{13} = 10$. Up to the Euler factor at 3, this gives

$$(6) \quad L(T_S, s) = L(\eta(\tau)^6, s).$$

Theorem 1. *Up to the Euler factors at 2 and 3, we have*

$$\zeta(S/\mathbb{Q}, s) = \frac{\zeta(s) \zeta(s-1)^{12} L(\eta(\tau)^6, s) \zeta(s-2)}{L(\eta(\tau)^4, s) L(\eta(\tau)^4, s-1)}.$$

6. THE RANK OF E DEPENDING ON THE CHARACTERISTIC

We shall use Theorem 1 to determine the rank of the elliptic curve E of (4) in positive characteristic. In other words, we are concerned with the Mordell-Weil rank and the Picard number of the corresponding surface S and look for supersingular primes. Recall that S has good reduction at the primes $p > 3$.

Lemma 2. *Let k be an algebraically closed field of characteristic $p > 3$. Then*

$$\rho(S/k) = \begin{cases} 12 & \text{if } p \equiv 1 \pmod{4}, \\ 14 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

The leitmotif to prove the lemma is to consider the ζ -function of S/\mathbb{F}_p . This is obtained from $\zeta(X/\mathbb{Q}, s)$ by considering the local Euler factors at p . From the associated Grössencharacter, we know that the factors corresponding to $\eta(\tau)^6$ have eigenvalues

$$(7) \quad \begin{cases} \pi^2, \bar{\pi}^2 & \text{if } p \equiv 1 \pmod{4} \text{ splits as } p = \pi\bar{\pi} \text{ in } \mathbb{Z}[2\sqrt{-1}], \\ p, -p & \text{if } p \equiv 3 \pmod{4}, p > 3. \end{cases}$$

As a consequence, Lemma 2 in its entirety would follow from the Tate Conjecture [14], but this is only known in some few cases (cf. [16, Thm. (5.6)]). Nevertheless, the first case of the lemma follows from (7), since then the eigenvalues are not p times a root of unity, a necessity for algebraic classes (cf. [9, App. C, Ex. 10]).

Now, by the relation of the Picard number and the Mordell-Weil rank (cf. [10, Cor. 5.3]), Lemma 2 is equivalent to the following:

$$(8) \quad \text{rank } E(k(B)) = \begin{cases} 0 & \text{if } p \equiv 1 \pmod{4}, \\ 2 & \text{if } p \equiv 3 \pmod{4}, p > 3. \end{cases}$$

To prove the second case above, we use the fact that S can be derived from an elliptic K3 surface via base change. The advantage of this approach is that the Tate Conjecture is known for elliptic K3 surfaces.

Let X be the elliptic K3 surface over \mathbb{P}^1 , given in Weierstrass form

$$(9) \quad E' : y^2 = x^3 - 27(t^2 + 12^3)x - 54t(t^2 + 12^3)^4.$$

This has singular fibres of type I_2^* over ∞ and IV^* over the two square roots of -12^3 . The idea is to pull-back via a base change of degree 3 which is ramified exactly above these cusps. In terms of the affine modular curve B , such a map is given by the following projection:

$$\begin{aligned} B &\rightarrow \mathbb{A}^1 \\ (\xi, \eta) &\mapsto \eta. \end{aligned}$$

The base change replaces t in the Weierstrass equation (9) of X by η . Then we use the relation between η and ξ from the defining equation (3) of B . After minimalizing, we obtain exactly the modular surface S as in (4). Projectively, the pull-back corresponds to the following commutative diagram:

$$\begin{array}{ccc} S & \rightarrow & X \\ \downarrow & & \downarrow \\ B & \rightarrow & \mathbb{P}^1. \end{array}$$

In other words, the elliptic curve $E/k(B)$ is obtained from $E'/k(t)$ by this base change. Let us identify $E' = E$ below. As a consequence, we have the injection $E(k(t)) \hookrightarrow E(k(B))$ for any characteristic $\neq 2, 3$. Hence, to deduce the second claim of (8), it suffices to prove the corresponding statement for $E(k(t))$.

Lemma 3. *If k has characteristic $p \equiv 3 \pmod{4}$, $p > 3$, then*

$$\text{rank } E(k(t)) = 2.$$

The proof starts by considering X/\mathbb{Q} which is known to be modular. This provides us with the ζ -function of X/\mathbb{F}_p for any $p > 3$. Then we apply the (known) Tate Conjecture.

Over \mathbb{C} , X is a singular K3 surface. Since $E(\mathbb{C}(t)) = 0$ (cf. Cor. 9), X is extremal, and the discriminant of $NS(X)$ is -36 . By [3, Ex. 1.6], the transcendental lattice T_X (as a Galois module) is associated to a newform of weight 3 with complex multiplication by $\mathbb{Q}(\sqrt{-1})$. (By construction, this is exactly $\eta(\tau)^6$.) Hence, if $p \equiv 3 \pmod{4}$, $p > 3$, the roots of the Euler factor at p are p and $-p$ as in (7).

We now consider the reduction of X modulo p . Recall that the Tate Conjecture is known for elliptic K3 surfaces in characteristic $p > 3$ [16, Thm. (5.6)]. Due to the eigenvalues, this predicts that the Picard number (over the algebraic closure) increases by two upon reducing. Since the fibre configuration stays unchanged, this implies Lemma 3. As explained, Lemma 2 follows.

In more detail, the above argument gives the following

Corollary 4. *Fix $p \equiv 3 \pmod{4}$, $p > 3$. Let $r \in \mathbb{N}$ and $q = p^r$. Then*

$$\text{rank } E(\mathbb{F}_q(B)) = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases}$$

In particular, the Tate Conjecture holds for S/\mathbb{F}_q .

As the Tate conjecture implies the Artin-Tate conjecture ([4], cf. [9, App. C, Ex. 10]), we have also established the following:

Proposition 5. *For any $p \equiv 3 \pmod{4}$, $p > 3$, we have*

$$\det NS(X) = \det NS(S) = -p^2.$$

7. MORDELL-WEIL LATTICES.

Let us recall two general formulas from the theory of Mordell-Weil lattices [10]. For a moment, suppose E is an elliptic curve over $K = k(B)$ (B/k any curve) such that the associated elliptic surface $f : S \rightarrow B$ has at least one singular fibre. We identify the K -rational points $P \in E(K)$ with the sections $\sigma : B \rightarrow S$ and denote the image curve $Im(\sigma) \subset S$ by the symbol (P) .

(1) Height formula: For any $P \in E(K)$, we have

$$(10) \quad \langle P, P \rangle = 2\chi(S) + 2(PO) - \sum_v \text{contr}_v(P),$$

where $\chi(S)$ is the arithmetic genus of S and (PO) denotes the intersection number of the section (P) and the zero-section (O) on the elliptic surface S ([10, Thm. 8.2]). The term $\text{contr}_v(P)$ is a local contribution at $v \in B$ such that the fibre at v is a reducible singular fibre; its value is given by [10, (8.16)].

(2) Determinant formula: Let M be the Mordell-Weil lattice $E(K)/E(K)_{\text{tor}}$. By [10, Thm. 8.7], we have

$$(11) \quad \det M/|E(K)_{\text{tor}}|^2 = \pm \det NS(S)/\det V_S$$

where V_S denotes the trivial lattice generated by the zero-section (O) and fibre components.

Going back to the previous situation, we have (regardless of the characteristic $\neq 2, 3$)

$$(12) \quad V_S = \langle 1 \rangle \oplus \langle -1 \rangle \oplus D_{10}[-1].$$

Here the root lattice D_{10} corresponds to the singular fibre of type I_6^* . For any lattice L , $L[n]$ denotes the lattice with the pairing multiplied by n .

Theorem 6. *Let E/K , $K = k(B)$ be as in (4) with $p = \text{char}(k) \neq 2, 3$. Then*

- (i) $E(K) = 0$ if $p = 0$ or $p \equiv 1 \pmod{4}$.
- (ii) $E(K) \cong \mathbb{Z}^{\oplus 2}[\frac{p}{2}]$ if $p \equiv 3 \pmod{4}$.

Lemma 7. $E(k(B))$ is torsion-free.

Indeed, if there is a torsion point $P \neq O$ in $E(k(B))$, then (10) implies the relation:

$$0 = \langle P, P \rangle = 2\chi + 2(PO) - \text{contr}_{o_B}(P)$$

where $\chi = 1$ and $\text{contr}_{o_B}(P) = 0, 1$ or $\frac{5}{2}$ for a singular fibre of type I_6^* . But this is impossible, since there is no integer $(PO) \geq 0$ satisfying the above relation.

Then the assertion (i) of Theorem 6 follows in view of (8). To prove the assertion (ii), note first that the rank two lattice $M = E(K)$ has $\det M = (p/2)^2$, by Prop. 5, (11) and (12). The following lemma implies that M is similar to the square lattice, which will complete the proof of (ii).

Lemma 8. *The elliptic surface S is endowed with an automorphism of order 4:*

$$\phi : (x, y, \xi, \eta) \mapsto (-x, iy, \xi, -\eta), \quad i = \sqrt{-1}.$$

It acts as an automorphism of order 4 on the transcendental lattice T_S in case $k = \mathbb{C}$, and on $E(K)$ in the supersingular case $p \equiv 3 \pmod{4}$.

First consider the case $k = \mathbb{C}$. Let ω_S be the unique holomorphic 2-form on S up to constants and suppose $\phi^*(\omega_S) = \alpha\omega_S$. Since the fibre involution ϕ^2 acts as multiplication by -1 , we have $\alpha^2 = -1$ and $\alpha = \pm i$. As ω_S is contained in $T_S \otimes \mathbb{C}$, this proves the first assertion. In this case, $H^2(S, \mathbb{Z})$ contains $NS(S) \oplus T_S$ as a finite index subgroup, and we have $NS(S) = V_S$ because $NS(S)/V_S \cong E(K) = 0$ by [10, Thm. 1.3] and the above Theorem 6 (i). The action of ϕ^* on V_S is trivial except that it exchanges two simple components of the I_6^* fibre. Hence it is of order 2.

Next we pass to $S(p)$ which denotes the reduction of $S \pmod{p}$. Let $k(p) = \overline{\mathbb{F}}_p$. In order to consider $S(p)/k(p)$, we work with étale cohomology, and choose $\ell \equiv 1 \pmod{4}$ so that $i \in \mathbb{Q}_\ell$. Then $T_S \otimes \mathbb{Q}_\ell$ is identified as $\pm i$ -eigenspace of ϕ^* on $H_{et}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ by what we have seen above. Now assume $p \equiv 3 \pmod{4}$. Then $NS(S(p))$ spans this cohomology group over \mathbb{Q}_ℓ by Lemma 2 and the usual comparison theorem. Since the trivial lattice V_S stays the same as in characteristic 0, ϕ^* acts on the Mordell-Weil group $E(K) \cong NS(S(p))/V_S$ as an automorphism of order 4. Since the height pairing is invariant under automorphisms (cf. [10, Prop. 8.12]), ϕ^* is compatible with the lattice structure. This completes the proof of Lemma 8 and hence Theorem 6 (ii).

Corollary 9. *We have*

(i) $E(k(t)) = 0$ if $p = 0$ or $p \equiv 1 \pmod{4}$, and

(ii) $E(k(t)) \cong \mathbb{Z}^{\oplus 2}[\frac{p}{6}]$ if $p \equiv 3 \pmod{4}$.

Indeed, (i) is obvious since $E(k(t)) \subset E(k(B))$. As for (ii), there is an injection of Mordell-Weil lattices $E(k(t)) \hookrightarrow E(k(B))[\frac{1}{3}]$ by [10, Prop. 8.12]. Then equality follows from the determinant formula (11) applied to X .

8. SIMILARITY OF LATTICES.

We keep the notation S, X to denote the elliptic surface over the elliptic curve B and the elliptic K3 surface, defined by (4) and (9). In this section, we shall compare two kinds of lattices. On the one hand, we consider the transcendental lattices T_X, T_S . On the other hand, we consider the *supersingular reduction lattices* $L_X(p), L_S(p)$ which is defined for a supersingular prime p as follows: With respect to the natural injection $NS(X/\mathbb{Q}) \hookrightarrow NS(X/\overline{\mathbb{F}}_p)$, we take the orthogonal complement of its image:

$$L' = NS(X/\overline{\mathbb{Q}})^\perp \subset NS(X/\overline{\mathbb{F}}_p).$$

We define $L_X(p) = L'[-1]$, and likewise for S . It was conjectured in [11] that the positive-definite lattices T_X and $L_X(p)$ are similar for a singular K3 surface X and a supersingular prime p . This conjecture extends to any surface with transcendental lattice of rank two.

In the present situation, it is easy to verify this:

Proposition 10. *The lattices $T_S, T_X, L_S(p), L_X(p)$ are all similar to the square lattice $L_0 = \mathbb{Z}^{\oplus 2}$. In particular, the above-mentioned conjecture holds for X and for S . More precisely, the scaling factor for each of these lattices is determined as follows:*

$$\begin{aligned} T_S &= L_0[2], & L_S(p) &= L_0[2p], \\ T_X &= L_0[6], & L_X(p) &= L_0[6p]. \end{aligned}$$

In fact, all the lattices $T_S, T_X, L_S(p), L_X(p)$ are rank two positive-definite lattices which admit a lattice automorphism of order 4 by Lemma 8 and its analogue for X . Hence they are similar to the square lattice $L_0 = \mathbb{Z}^{\oplus 2}$.

As for the scaling factor, it suffices to determine the discriminants of these lattices. We have

$$\det T_X = -\det NS(X) = -\det V_X = 4 \cdot 3^2 = 6^2,$$

since $NS(X)/V_X \cong E(k(t)) = 0$, and similarly, $\det T_S = 2^2$. Next, $\det L_S(p) = (2p)^2$ and $\det L_X(p) = (6p)^2$ follow from Theorem 6 (ii) resp. Cor. 9 (ii) and the following

Lemma 11. *Let $k = \overline{\mathbb{F}}_p$. The lattice $L_S(p)$ is isomorphic to the narrow Mordell-Weil lattice $E(k(B))^0$, and similarly, $L_X(p)$ is isomorphic to $E(k(t))^0$.*

Indeed we have $NS(S) = V_S$ for $S/\overline{\mathbb{Q}}$, and the trivial lattice V_S does not change under reduction mod p . Hence $L_S(p)$ agrees with the *essential lattice* defined in [10, §8]. Thus the claim follows from [10, Thm. 8.9]. The same argument applies to X .

Remark 12. *In a recent preprint [8], Shimada studies transcendental lattices T_X and supersingular reduction lattices $L_X(p)$ of singular $K3$ surfaces X . In the case at hand, i.e. for the singular $K3$ surface X in (9) and $p \equiv 3 \pmod{4}$ ($p > 3$), Thm. 2 (L) in [8] gives an alternative proof that $L_X(p) = T_X[p]$.*

9. BEHAVIOUR OF MW UNDER BASE CHANGE

We start with a general question. Throughout we work over an algebraically closed field. Let S be an elliptic surface over an elliptic curve C . Denote the Mordell-Weil rank of S by r . Let

$$n_C : C \rightarrow C$$

be multiplication by n . Write $S^{(n)}$ for the pullback of S under n_C and $r^{(n)} = \text{rk } MW(S^{(n)})$.

Question 13. *How does $r^{(n)}$ behave as $n \rightarrow \infty$?*

By construction, $r^{(n)}/n^2$ is bounded. However, we will only be able to do better in the special situation where S is extremal. Recall that S is called extremal if its Picard number $\rho(S)$ is maximal and if its Mordell-Weil group $MW(S)$ is finite, i.e. $r = 0$.

Lemma 14. *If S is extremal, then so is $S^{(n)}$.*

We will use the following invariants of an elliptic surface S with elliptic base C : Let χ denote the arithmetic genus. In the case at hand, χ equals the geometric genus p_g . The Euler number is $e = 12\chi$. This gives $b_2 = 12\chi + 2$ and $h^{1,1} = 10\chi + 2$. Finally,

$$\rho = r + 2 + \sum_{v \in C} (m_v - 1)$$

where r is the Mordell-Weil rank as before and m_v denotes the number of components of the fibre at $v \in C$.

Let us prove Lemma 14 in characteristic zero. By assumption, $\rho(S) = h^{1,1} = 10\chi + 2$. Since n_C is unramified, $e(S^{(n)}) = n^2 e(S) = 12n^2\chi$ and

$$\begin{aligned} \rho(S^{(n)}) &= r^{(n)} + 2 + n^2 \sum_{v \in C} (m_v - 1) \\ &\geq 2 + n^2 \sum_{v \in C} (m_v - 1) = 2 + 10n^2\chi. \end{aligned}$$

On the other hand, Lefschetz' bound reads

$$\rho(S^{(n)}) \leq h^{1,1}(S^{(n)}) = 2 + 10n^2\chi.$$

Hence $\rho(S^{(n)}) = h^{1,1}(S^{(n)})$ and $r^{(n)} = 0$.

In case of positive characteristic, the assumption is $\rho(S) = b_2(X)$, and the same argument applies using Igusa's bound $\rho \leq b_2$.

REFERENCES

- [1] Igusa, J.-I.: *Fibre Systems of Jacobian Varieties: III. Fibre Systems of Elliptic Curves*, Amer. J. Math. **81**, no. 2 (1959), 453–476.
- [2] Kodaira, K.: *On compact analytic surfaces II, III*, Ann. of Math. (2) **77** (1963), 563–626; *ibid.* **78** (1963), 1–40.
- [3] Livné, R.: *Motivic Orthogonal Two-dimensional Representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$* , Israel J. of Math. **92** (1995), 149–156.
- [4] Milne, J. S.: *On a conjecture of Artin and Tate*, Ann. of Math. **102** (1975), 517–533.
- [5] Newman, M.: *The structure of some subgroups of the modular group*, Illinois J. Math. **6** (1962), 480–487.
- [6] Rubin, K., Silverberg, A.: *Mod 6 representations of elliptic curves*, in: *Automorphic forms, automorphic representations, and arithmetic* (Fort Worth 1996), Proc. Sympos. Pure Math. **66**, no. 1 (1999), 213–220.
- [7] Schütt, M.: *Hecke eigenforms with rational coefficients and complex multiplication*, preprint(2005), arXiv: [math.AG/0511228](https://arxiv.org/abs/math/0511228).
- [8] Shimada, I.: *Transcendental lattices and supersingular reduction lattices of a singular K3 surface*, preprint(2006), arXiv: [math.AG/0611208](https://arxiv.org/abs/math/0611208).
- [9] Shioda, T.: *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.
- [10] Shioda, T.: *On the Mordell-Weil lattices*, Comm. Math. Univ. St. Pauli **39** (1990), 211–240.
- [11] Shioda, T.: *The elliptic K3 surfaces with a maximal singular fibre*, C. R. Acad. Sci. Paris, Ser. I, **337** (2003), 461–466.
- [12] Silverman, J. H.: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Math. **151**, Springer (1994).
- [13] Stiller, P.: *On the classification of elliptic surfaces with $q = 1$* , Manuscripta Math. **60** (1988), 299–321.
- [14] Tate, J.: *Algebraic cycles and poles of zeta functions*, in: *Arithmetical Algebraic Geometry* (Proc. Conf. Purdue Univ., 1963), 93–110, Harper & Row (1965).
- [15] Tate, J.: *Algorithm for determining the type of a singular fibre in an elliptic pencil*, in: *Modular functions of one variable IV* (Antwerpen 1972), SLN **476** (1975), 33–52.
- [16] Tate, J.: *Conjectures on algebraic cycles in ℓ -adic cohomology*, in: *Motives* (Seattle 1991), Proc. Sympos. Pure Math. **55**, no. 1 (1994), 71–83.

DEPARTMENT OF MATHEMATICS, RIKKYO UNIVERSITY, NISHI-IKEBUKURO, TOSHIMAKU, TOKYO 171, JAPAN

E-mail address: shioda@rikkyo.ac.jp

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, 1 OXFORD STREET, CAMBRIDGE, MA 02138, USA

E-mail address: mschuett@math.harvard.edu