Correspondence of elliptic curves and Mordell-Weil lattices of certain elliptic $K3$ surfaces

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Abstract

We study the Mordell-Weil lattice of certain elliptic K3 surfaces, related to the Kummer surface of a product abelian surface. Our aim is first to determine the precise structure of such a lattice, and second to give some explicit generators, in the case beyond rational elliptic surfaces.

1 Introduction

The main purpose of this paper is to study certain elliptic fibrations on the Kummer surface of a product abelian surface, both geometrically (equation-free) and with the use of equations, and to identify the elements of the Mordell-Weil lattice coming from algebraic cycles on the Kummer surface, especially from the correspondences of the factor elliptic curves.

We state here the main results in terms of Weierstrass equations, which should show some new feature of Mordell-Weil lattices for elliptic K3 surfaces, different from the well-studied case of rational elliptic surfaces. The background will be explained after the statements.

Theorem 1.1 *Let C_1, C_2 be two elliptic curves with the absolute invariant j_1, j_2, defined over an algebraically closed field k of characteristic $\neq 2, 3$. Let $F^{(1)}$ denote the elliptic curve over the rational function field $k(T)$*

$$y^2 = x^3 - 3\alpha x + (T + \frac{1}{T} - 2\beta), \tag{1.1}$$

where

$$\alpha = \sqrt[3]{j_1 j_2},\ \beta = \sqrt{(1-j_1)(1-j_2)}. \tag{1.2}$$

Assume that $j_1 \neq j_2$ *(i.e.* C_1, C_2 *are not isomorphic to each other). Then there is a natural isomorphism of* $\mathrm{Hom}(C_1, C_2)$ *to the Mordell-Weil lattice* $F^{(1)}(k(T))$, $\varphi \mapsto R_\varphi$, *such that the height of* R_φ $\langle R_\varphi, R_\varphi \rangle$ *is equal to* $2\deg(\varphi)$. *In other words, there is a natural isomorphism of lattices:*

$$\mathrm{Hom}(C_1, C_2)[2] \simeq F^{(1)}(k(T)). \tag{1.3}$$

Theorem 1.2 *Let* $F^{(2)}$ *denote the elliptic curve over* $k(t)$, *obtained from* $F^{(1)}$ *by the base change* $T = t^2$. *Assume that* $j_1 \neq j_2$. *Then the Mordell-Weil lattice* $F^{(2)}(k(t))$ *contains a sublattice of finite index* 2^h *(*$h = \mathrm{rkHom}(C_1, C_2)$*) which is naturally isomorphic to the direct sum of lattices*

$$\mathrm{Hom}(C_1, C_2)[4] \oplus A_2^*[2]^{\oplus 2} \tag{1.4}$$

where A_2^* *denotes the dual lattice of the root lattice* A_2 *(of rank 2).*

N.B. (1) The absolute invariant is normalized so that $j = 1$ for $y^2 = x^3 - x$. (2) Given a lattice L, we denote by $L[n]$ the lattice structure on L with the norm (or pairing) multiplied by n. (3) For the root lattices, we refer to [4].

Here we briefly mention some background of the above results; more details will be given later in §2 and §3.

Let $S^{(n)}$ be the associated elliptic surface to the elliptic curve $F^{(n)}(n = 1, 2)$. Then they are both K3 surfaces, and $S^{(2)}$ is isomorphic to the Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$ of the product of two elliptic curves. This elliptic fibration on the Kummer surface and $S^{(1)}$ are discovered by Inose ([5]) in search for the notion of isogeny between singular K3 surfaces ([6]).

More recently, Kuwata ([8]) has made a nice observation on Inose's results; he introduces elliptic K3 surfaces corresponding to $F^{(n)}(n \leq 6)$ defined by the base change $T = t^n$, and shows that their Mordell-Weil rank can become as large as 18, the maximum in case of $\mathrm{char}(k) = 0$. Inspired by the work of Inose and Kuwata, we ([15]) have made a preliminary study on these elliptic K3 surfaces from the viewpoint of Mordell-Weil lattices.

Now, given any $\varphi \in \mathrm{Hom}(C_1, C_2)$, the image of its graph under the rational map from $C_1 \times C_2$ to S (and to $S^{(1)}$) gives a curve on S (or $S^{(1)}$), and this image curve determines a rational point $P_\varphi \in F^{(2)}(k(t))$ (or $R_\varphi \in F^{(1)}(k(T))$)

by the formalism of Mordell-Weil lattices (see §2; compare [17]). The correspondence $\varphi \mapsto R_\varphi$ in Theorem 1.1 is "natural" in the sense that it is defined in this way. Theorems 1.1 and 1.2 are the refinement of some results announced in [15], covering also the case of arbitrary characteristic $\neq 2, 3$. As an application to the case of the higher rank, we mention the following:

Example 1.3 *Assume $j_1 \neq 0, j_2 = 0$ $(C_2 : y^2 = x^3 - 1)$. Then the Mordell-Weil lattice $F^{(6)}(k(t))$ of the elliptic curve over $k(t)$*

$$y^2 = x^3 + (t^{12} - 2\beta t^6 + 1), \quad \beta = 1 - j_1^2 \tag{1.5}$$

is of rank $r = h + 16$ and it contains a finite index sublattice isomorphic to the direct sum

$$L_0 \oplus E_8[2] \oplus D_4^*[4]^{\oplus 2}, \quad \mathrm{rk} L_0 = h \tag{1.6}$$

where E_8, D_4, A_2 are root lattices and $*$ means the dual lattice. If $\mathrm{char}(k) = 0$, *then we have ($h = 2$ or 0)*

$$L_0 = A_2[6d_0] \text{ or } 0 \tag{1.7}$$

$d_0(\geq 2)$ *being the degree of minimal isogeny $C_1 \to C_2$ (cf. §8). The generators of $k(t)$-rational points of this sublattice can be given explicitly in case $h = 0$ or if d_0 is small.*

This paper is organized as follows. In §2, we review the formalism of Mordell-Weil lattices which is our main tool. In the next sections, we study the elliptic fibrations on the Kummer surface of a product abelian surface. After reviewing the so-called double Kummer pencils (§3), we study the Inose's pencil, first by a geometric method (§4) and second by introducing equations (§5). With these preparations, we prove our main results (and Theorems 1.1 and 1.2) in §6. Some comments for the case $j_1 = j_2$ (§7) and examples (§8) are given. We hope to come back to the higher rank case in some other occasion.

It is my pleasure to dedicate this paper to Professor Jacob Murre on the occasion of his 75th birthday. It was reported at the workshop on Algebraic Cycles held at Lorentz Center, Leiden, in his honor. The paper has been prepared partly during my stay at the Max-Planck-Institut für Mathematik, Bonn, in summer 2004. I would like to thank MPI for the hospitality and excellent atmosphere, and my special thanks go to Professor Hirzebruch for everything he has done for me. Finally I thank the referee for his/her careful reading of the manuscripts and for useful suggestions.

2 Review of MWL-formalism

Let us make a review of the basic formalism of Mordell-Weil lattices, fixing some notation (cf. [14]).

Let E/K be an elliptic curve over the function field $K = k(C)$ of a smooth projective curve C/k. The base field k is an algebraically closed field of arbitrary characteristic (later we assume $\text{char}(k) \neq 2, 3$). Let $f : S \longrightarrow C$ be the asociated elliptic surface (the Kodaira-Néron model of E/K); S is a smooth projective surface over k and E is the generic fibre of f. The set of K-rational points of E, $E(K)$, is in a natural bijective correspondence with the set of the sections of f. For $P \in E(K)$, we use the same symbol P to denote the corresponding section $P : C \to S$ and the symbol (P) to denote the image curve in S; thus for example (O) denotes the image of the zero-section in S. Let $\text{Sing}(f)$ (resp. $\text{Red}(f)$) denote the set of singular fibres (reducible singular fibres) of f. It is known that, if $\text{Sing}(f) \neq \emptyset$, then $E(K)$ is finitely generated (Mordell-Weil theorem).

Let $\text{NS}(S)$ be the Néron-Severi group of S which is defined as the group of divisors on the surface S modulo algebraic equivalence; the class of a divisor D is denoted by $[D]$ (or simply by D if no confusion will arise). Let $T = T(f)$ denote the subgroup generated by the classes of the zero-section (O), any fibre F and all the irreducible components of reducible fibres which are disjoint from (O). Then we have a natural isomorphism

$$E(K) \simeq \text{NS}(S)/T. \tag{2.1}$$

The correspondence is given by $P \mapsto [(P)] \text{mod} T$, and the inverse correspondence is induced by the following map of the divisor group of S to $E(K)$:

$$\mu(D) = \mu_f(D) = sum(D|_E) \in E(K) \tag{2.2}$$

Namely, given a divisor D on S, restrict it to the generic fibre E and take the summation of its components ($\in E(\bar{K}), \bar{K}$ being the algebraic closure of K) with respect to the group law of E.

Now $\text{NS}(S)$ forms an indefinite integral lattice with respect to the intersection pairing, and T forms a sublattice, called the *trivial sublattice*, which has an orthogonal decomposition $T = U \oplus \sum_{v \in \text{Red}(f)} T_v$ where U is the unimodular rank 2 lattice generated by $(O), F$, and T_v is the lattice of rank $m_v - 1$ spanned by the irreducible components away from (O) of the reducible fibre $f^{-1}(v)$. Each T_v is a root lattice of type A, D, E, up to sign, by Kodaira [7].

Table 1: Values of local contribution

T_v^-	A_1	E_7	A_2	E_6	A_{b-1}	D_{b+4}
type of F_v	III	III^*	IV	IV^*	$I_b(b \geq 2)$	$I_b^*(b \geq 0)$
$contr_v(P)$	$1/2$	$3/2$	$2/3$	$4/3$	$i(b-i)/b$	$\begin{cases} 1 & (i=1) \\ 1+b/4 & (i>1) \end{cases}$
$contr_v(P,Q)$ $(i<j)$	$--$		$1/3$	$2/3$	$i(b-j)/b$	$\begin{cases} 1/2 & (i=1) \\ (2+b)/4 & (i>1) \end{cases}$

The key idea of Mordell-Weil lattices is to define the lattice structure on the Mordell-Weil group via the intersection theory on the surface as follows. There is a unique homomorphism

$$\nu : E(K) \longrightarrow N_{\mathbf{Q}} = \mathrm{NS}(S) \otimes \mathbf{Q} \tag{2.3}$$

satisfying the condition: for every $P \in E(K)$,

$$\nu(P) \perp T, \quad \nu(P) \equiv [(P)] \mathrm{mod} T_{\mathbf{Q}}. \tag{2.4}$$

Then $E(K)$ modulo torsion is embedded into the orthogonal complement of T in $N_{\mathbf{Q}}$, which is negative-definite by the Hodge index theorem. Therefore, by defining the *height pairing* on $E(K)$ by the formula:

$$\langle P, Q \rangle := -(\nu(P) \cdot \nu(Q)), \tag{2.5}$$

one obtains the structure of a positive-definite lattice on $E(K)/E(K)_{tor}$. It is called the *Mordell-Weil lattice* (abbreviated from now on as MWL) of the elliptic curve E/K or the elliptic surface $f : S \to C$.

The height formula takes the following explicit form:

$$\langle P, Q \rangle = \chi + (PO) + (QO) - (PQ) - \sum_{v \in \mathrm{Red}(f)} \mathrm{contr}_v(P, Q) \tag{2.6}$$

where χ is the arithmetic genus of S, (PQ) denotes the intersection number of the sections (P) and (Q), and $\mathrm{contr}_v(P, Q)$ is a local contribution at v. For later use, we copy the table from [14, (8.16)], Table 1, in which i, j are defined so that the section (P) (or (Q)) intersects the i-th (or j-th) irreducible component of the singular fibre $f^{-1}(v)$ under suitable numbering.

The determinant of Néron-Severi lattice and that of MWL are related by the formula:

$$\det \mathrm{NS}(S) = \det(E(K)/E(K)_{tor}) \cdot \det T/|E(K)_{tor}|^2. \tag{2.7}$$

Remark (1) Given the information of the trivial lattice T, it is easy to write down the explicit formula for $\nu(P) = (P) + \cdots$ satisfying the *Linear Algebra* condition (2.4). Indeed this is how the height formula (2.6) is derived in general. On the other hand, it is also possible to compute the height by applying the original definition (2.5), especially when P is given as $P = \mu(D)$ for some divisor D. This method gives an algorithm suited for computer calculation which can be used for checking theoretical computation.

(2) The structure of MWL is clarified in the case where S is a rational elliptic surface. In this case, the lattices in question form a hierarchy dominated by the root lattice E_8, the unique positive-definite even unimodular lattice of rank 8 (cf. [11]). Also it is easy in this case to give the generators of rational points, for example, and there are many interesting applications. Beyond this case, not much is known even in the next simplest case of elliptic K3 surfaces.

3 The Kummer pencils

In the subsequent sections, we study certain elliptic K3 surfaces related to the Kummer surface of a product abelian surface.

In general, let A be an abelian surface and let ι_A denote the inversion automorphism of A. We assume that $\mathrm{char}(k) \neq 2$. The Kummer surface $S = \mathrm{Km}(A)$ is a smooth K3 surface obtained from the quotient surface A/ι_A by resolving the 16 singular points corresponding to the points of order 2 on A. The Picard number is given by $\rho(S) = \rho(A) + 16$.

Now consider the case of a product abelian surface, i.e. $A = C_1 \times C_2$ where C_1, C_2 are elliptic curves. If we denote by h the rank of the free module $\mathrm{Hom}(C_1, C_2)$ of homomorphisms of C_1 to C_2, then

$$\rho(S) = h + 18, \quad h := \mathrm{rkHom}(C_1, C_2) \tag{3.1}$$

since we have $\rho(A) = 2 + h$. It is known that $H = \mathrm{Hom}(C_1, C_2)$ has the structure of a positive-definite lattice such that the norm of $\varphi \in H$ is $\deg(\varphi)$, the degree of the homomorphism (see the Remark below).

First we look at the Kummer pencil (cf. [6, §2]), i.e. the elliptic fibration

$$\pi_1 : S = \mathrm{Km}(C_1 \times C_2) \to \mathbf{P}^1 \tag{3.2}$$

induced from the projection of A to C_1. It has the 4 singular fibres of type I_0^*:

$$\pi_1^{-1}(\bar{v}_i) = 2F_i + \sum_{j \in I} A_{ij}. \tag{3.3}$$

Here we use the following notation. Let $I = \{0, 1, 2, 3\}$ and let $\{v_i | i \in I\} \subset C_1$ be the 2-torsion points (take v_0= the origin); similarly for $\{v'_j | j \in I\} \subset C_2$. We denote by $\bar{v}_i$ the image point of v_i under $C_1 \to C_1/\iota_1 = \mathbf{P}^1$. The curves $F_i, G_j \subset S (i, j \in I)$ are the image of $v_i \times C_2, C_1 \times v'_j$ under the rational map of degree two $A \to S$. Further A_{ij} denotes the exceptional curve corresponding to $v_i \times v'_j$. All the 24 curves $\{F_i, G_j, A_{ij}\}$ on S are smooth rational curves with self-intersection number -2 (i.e. -2-curves). The intersection numbers among these curves are given as follows:

$$\begin{cases} (F_i \cdot F_j) = -2\delta_{ij}, & (G_i \cdot G_j) = -2\delta_{ij}, \quad (F_i \cdot G_j) = 0, \\ (A_{ij} \cdot A_{kl}) = -2\delta_{ik}\delta_{jl}, & (F_i \cdot A_{kl}) = \delta_{ik}, \qquad (G_i \cdot A_{kl}) = \delta_{il}. \end{cases} \tag{3.4}$$

Note that each of the 4 curves G_j gives a section of π_1 since it intersects the fibre (3.3) with intersection mutiplicity 1. Take $G_0 = (O)$ as the zero-section. Then the other sections G_j are of order 2. The generic fibre of π_1 is isomorphic to the constant elliptic curve C_2 over $k(C_1) \supset k(\mathbf{P}^1)$, but of course not over $k(\mathbf{P}^1)$.

Proposition 3.1 *Let $\mathcal{E}$ denote the generic fibre of π_1. Then we have*

$$\mathcal{E}(k(\mathbf{P}^1)) \simeq \mathrm{Hom}(C_1, C_2) \oplus (\mathbf{Z}/2\mathbf{Z})^2 \tag{3.5}$$

i.e. the Mordell-Weil lattice $\mathcal{E}(k(\mathbf{P}^1))/(tor)$ is isomorphic to the lattice $H :=$ $\mathrm{Hom}(C_1, C_2)$ with norm $\varphi \mapsto \deg(\varphi)$.

Proof This should be wellknown if we ignore the lattice structure, but for the sake of completeness, let us first check the isomorphism of both side as groups. Take any $P \in \mathcal{E}(k(\mathbf{P}^1))$, and regard it as a section $\sigma : \mathbf{P}^1 \to S$. Its pullback to C_1, $\tilde{\sigma} : C_1 \to A = C_1 \times C_2$, is of the form $\tilde{\sigma}(u) = (u, \alpha(u))$ $(u \in C_1)$, where $\alpha : C_1 \to C_2$ is a morphism such that $\alpha(-u) = -\alpha(u)$. Hence we have $\alpha(u) = \varphi(u) + v'$ for some homomorphism $\varphi \in \mathrm{Hom}(C_1, C_2)$ and a 2-torsion point $v' \in C_2$. This establishes the bijection of both sides.

For any nonzero $\varphi \in \mathrm{Hom}(C_1, C_2)$, consider the image $\Gamma = \Gamma_\varphi$ of its graph under the rational map $A \to S$. Let $Q_\varphi = \mu(\Gamma_\varphi) \in \mathcal{E}(k(\mathbf{P}^1))$, with $\mu = \mu_f$ for $f = \pi_1$ defined by (2.2) and (3.2). It is easy to see that $\varphi \mapsto Q_\varphi$ is a homomorphism. We must prove that the height $\langle Q_\varphi, Q_\varphi \rangle$ is equal to $\deg(\varphi)$.

To prove this, we use the height formula (2.6) for $P = Q_\varphi$; note that $\chi = 2$ (for a K3):

$$\langle P, P \rangle = 4 + 2(PO) - \sum \mathrm{contr}_v(P).$$

For a moment, admit Lemma 3.2 below. Then the term $(PO) = (\Gamma \cdot G_0)$ is given by (3.6). On the other hand, Table 1 shows that we have $\mathrm{contr}_v(P) = 1$ iff the section $(P) = \Gamma$ meets a non-identity component of I_0^*-fibre. Thus the sum of local contribution is 3 (or 2 or 0) according to the case (a) (or (b) or (c)) of Lemma. This proves $\langle P, P \rangle = \deg(\varphi)$. *q.e.d.*

Lemma 3.2 *Given $\varphi \in \mathrm{Hom}(C_1, C_2)$, $\varphi \neq 0$, let $d = \deg(\varphi)$ and set $\varphi(v_i) = v'_{j(i)} (i \in I)$; let $n = n(\varphi)$ denote the number of distinct $j(i)$. Then the curve $\Gamma = \Gamma_\varphi$ is a (-2)-curve on S, and it satisfies $\Gamma \cdot A_{i,j} = \delta_{i,j(i)}$ and $\Gamma \cdot F_i = 0$ for all $i \in I$. The intersection number of Γ with G_i is described in the table (3.6) below according to the three cases (a) $n = 4$, (b) $n = 2$ or (c) $n = 1$, which can be also characterized by the following properties: (a) d is odd, (c) $\varphi = 2\varphi_1$ for some $\varphi_1 \in \mathrm{Hom}(C_1, C_2)$, (b) otherwise. (In case (b), we change ordering v'_j so that $\{j(i) | i \in I\} = \{0, 1\}$.) Then we have*

	(a)	(b)	(c)
$\Gamma \cdot G_0$	$(d-1)/2$	$(d-2)/2$	$(d-4)/2$
$\Gamma \cdot G_1$	$(d-1)/2$	$(d-2)/2$	$d/2$
$\Gamma \cdot G_2$	$(d-1)/2$	$d/2$	$d/2$
$\Gamma \cdot G_3$	$(d-1)/2$	$d/2$	$d/2$

(3.6)

Proof Let $\tilde{\Gamma}$ be the graph of φ in $A = C_1 \times C_2$. Then a 2-torsion point of A lies on $\tilde{\Gamma}$ if and only if it is of the form $v_i \times v'_{j(i)}$ for some $i \in I$. Under the rational map $A \to S$ of degree two, $\tilde{\Gamma}$ is mapped to Γ. Thus we have $\Gamma \cap A_{i,j} \neq \emptyset$ if and only if $j = j(i)$, in which case the intersection number is one. Hence the first assertion. As for the intersection of Γ with G_j, for example with G_0, we note that $\tilde{\Gamma} \cdot (C_1 \times v'_0)$ has degree $d = \deg(\varphi)$, of which $n' = 4/n$ simple intersection occur at the 2-torsion points. Hence we have $\Gamma \cdot G_0 = (d - n')/2$ on S, as asserted. We can argue similarly for other G_j. *q.e.d.*

Theorem 3.3 *The Néron-Severi lattice* $\mathrm{NS}(S)$ *of the Kummer surface* $S = \mathrm{Km}(C_1 \times C_2)$ *is generated by the 24 curves* F_i, G_j, A_{ij} *together with* $\Gamma_\varphi(\varphi \in \mathrm{Hom}(C_1, C_2))$*. Its determinant is given by*

$$\det \mathrm{NS}(S) = 2^4 \cdot \det \mathrm{Hom}(C_1, C_2). \tag{3.7}$$

Proof The first assertion follows from (2.1) (applied to $E = \mathcal{E}$) and (3.5), since both trivial lattice T and the torsion part $(\mathbf{Z}/2\mathbf{Z})^2$ are generated by curves belonging to the 24 curves. For (3.7), apply the formula (2.7), where we have $\det T = 4^4$ (as $T = U \oplus D_4^{\oplus 4}$) and $|E(K)_{tor}| = 2^2$. *q.e.d.*

Remark The prototype of the above arguments is the wellknown fact:

$$\mathrm{NS}(A) = T_0 \oplus T_0^\perp, \quad (T_0^\perp)[-1] \simeq \mathrm{Hom}(C_1, C_2)[2] \tag{3.8}$$

where $A = C_1 \times C_2$ and T_0 is the sublattice of $\mathrm{NS}(A)$ generated by $C_1 \times v_0'$ and $v_0 \times C_2$. It relates the correspondence theory of curves to geometry of surfaces, and its most remarkable application is Weil's proof of the Riemann hypothesis for curves over a finite field ([19]). At any rate, it shows that $\mathrm{Hom}(C_1, C_2)[2]$ (with $2\deg(\varphi)$ as the norm of φ) is a (positive-definite) integral lattice and $\det \mathrm{NS}(A) = 2^h \det \mathrm{Hom}(C_1, C_2)$. Note that $\mathrm{Hom}(C_1, C_2)$ itself is not necessarily an integral lattice.

Corollary 3.4 *Let* N_0 *denote the sublattice of* $\mathrm{NS}(S)$ *generated by the 24 curves* $\{F_i, G_j, A_{ij}\}$*. Then (i)* N_0 *is an indefinite lattice of rank 18 and* $\det 2^4$*. (ii) Assume that* C_1 *and* C_2 *are not isogenous to each other. Then we have* $\mathrm{NS}(S) = N_0$*, namely* $\mathrm{NS}(S)$ *is generated by the 24 curves* F_i, G_j, A_{ij}*.*

This is obvious from Theorem 3.3. It is classically wellknown (e.g.[12]).

Proposition 3.5 *The map* $\varphi \mapsto \Gamma_\varphi$ *induces a surjective homomorphism from* $\mathrm{Hom}(C_1, C_2)$ *to* $NS(S)/N_0$*.*

Proof It suffices to show that

$$\Gamma_{\varphi+\psi} \equiv \Gamma_\varphi + \Gamma_\psi \bmod N_0, \tag{3.9}$$

since the surjectivity follows from Theorem 3.3. Consider the divisor $D = \Gamma_{\varphi+\psi} - (\Gamma_\varphi + \Gamma_\psi) + \Gamma_0$ on S. The restriction of D to the generic fibre $\mathcal{E}$ of the Kummer pencil π_1 gives $\mu(D) = Q_{\varphi+\psi} - (Q_\varphi + Q_\psi) = O$, μ being the map (2.2). Hence D is algebraically equivalent to a sum of irreducible components of fibres of π_1, which proves (3.9). *q.e.d.*

4 Inose's pencil

Next we define Inose's pencil on $S = \mathrm{Km}(C_1 \times C_2)$. Take the following divisors on S:

$$\begin{cases} \Phi_1 = G_1 + G_2 + G_3 + 2(A_{01} + A_{02} + A_{03}) + 3F_0, \\ \Phi_2 = F_1 + F_2 + F_3 + 2(A_{10} + A_{20} + A_{30}) + 3G_0, \end{cases} \tag{4.1}$$

where F_i, G_j, A_{ij} are the -2-curves used in §3. They are disjoint and they have the same type as a singular fibre of type IV^*. Recall that a divisor on a K3 surface X is a fibre of some elliptic fibration on X if it has the same type as one of the Kodaira's list of singular fibres (cf.[7], [12]). Therefore there is an elliptic fibration, say $f : S \to \mathbf{P}^1$, such that $f^{-1}(0) = \Phi_1$ and $f^{-1}(\infty) = \Phi_2$. We call it *Inose's pencil* on the Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$. Note that the divisors Φ_1, Φ_2 are interchanged when the order of the factors C_1, C_2 is changed. Let E be the generic fibre of f; it will be identified with the elliptic curve $F^{(2)}/k(t)$ of Theorem 1.2 later.

Each of the 9 curves $A_{ij} (i, j \neq 0)$ defines a section of f; for instance, A_{33} intersects the fibre Φ_1 transversally at the simple component G_3. Let us choose $A_{33} = (O)$ as the zero-section. To avoid confusion, we let $Q_{ij} \in E(k(t))$ denote the section such that $(Q_{ij}) = A_{ij}$.

Throughout §4, we make the assumption:
(#) f has no other reducible fibres than Φ_1, Φ_2.

Lemma 4.1 *Under (#), the Mordell-Weil lattice $L = E(k(t))$ of the Inose's pencil has rank $4 + h$, and the 9 sections Q_{ij} form a sublattice L_1 of rank 4 isomorphic to $A_2^*[2]^{\oplus 2}$.*

Proof Under the assumption (#), the trivial lattice T is isomorphic to $U \oplus E_6^{\oplus 2}$, of rank 14, and hence the Mordell-Weil rank is equal to $\rho(S) - 14 = 4 + h$ by (2.1), (3.1). Also $E(k(t))$ is torsion-free by the height formula.

Let us compute the height of $Q = Q_{ij} (i, j \neq 0)$ by (2.6). The curve $(Q) = A_{ij}$ hits the singular fibre Φ_1 (of type IV^*) at a non-identity component iff $i = 1, 2$. By Table 1, the local contribution is equal to

$$\mathrm{contr}_v(Q) = 4/3 \text{ for } i = 1, 2, \text{ and } = 0 \text{ for } i = 3.$$

By replacing i by j, we get the corresponding value at the fibre Φ_2. Hence we have $\langle Q, Q \rangle = 4 - 4/3 - 4/3 = 4/3$. Similarly we can compute the height pairing $\langle Q, Q'\rangle$ for $Q \neq Q'$. Thus we see that both $\{Q_{11}, Q_{22}\}$ and $\{Q_{12}, Q_{21}\}$ span a mutually orthogonal sublattice (isomorphic to $A_2^*[2]$) with the Gram matrix $\begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$. *q.e.d.*

Now we turn our attention to the curves $\Gamma_\varphi (\varphi \in H)$ to study the remaining rank h part of the Mordell-Weil lattice of f. Let

$$P_\varphi = \mu_f(\Gamma_\varphi) \in E(k(t)), \tag{4.2}$$

with μ_f as in (2.2). To compute the height $\langle P_\varphi, P_\varphi\rangle$, we cannot directly apply the height formula (2.6) as before, because we do not know the local contribution terms. Thus we need to go back to the original definition (2.5).

Lemma 4.2 *Given $\varphi \in \mathrm{Hom}(C_1, C_2)$ with $d = \deg(\varphi)$, the height $\langle P_\varphi, P_\varphi\rangle$ has the following value:*

$$\langle P_\varphi, P_\varphi\rangle = \begin{cases} 2d^2 - d + 1 & (a) \\ 2d^2 - 2d + 4/3 & (b) \\ 2d^2 + d & (c) \end{cases} \tag{4.3}$$

in the respective case (a), (b), (c) for φ stated in Lemma 3.2.

Proof Set $P = P_\varphi$. Consider the case (c) where φ is divisible by 2 in H; in particular d is divisible by 4. Solving the linear algebra condition (2.4) for $(P) = \Gamma_\varphi$, we find by a direct computation that

$$\begin{aligned} \nu(P) = \quad & \Gamma_\varphi - 3d\Phi - \tfrac{3d}{2}(O) + d(F_1 + F_2 + G_1 + G_2) + 2dF_0 + (3d-1)G_0 \\ & + \tfrac{3d}{2}(A_{01} + A_{02}) + dA_{03} + 2d(A_{10} + A_{20}) + \tfrac{3d}{2}A_{30} \end{aligned} \tag{4.4}$$

where Φ denotes a fibre class of f. Then by (2.5), we compute $\langle P, P\rangle = -(\nu(P)^2)$ using Lemma 3.2 and verify that it is equal to $2d^2 + d$, as asserted. The other cases (a), (b) can be verified in the same way. *q.e.d.*

It follows from above that $P_\varphi \neq 0$ for $\varphi \neq 0$, but we cannot say that the map $\varphi \to P_\varphi$ is an injective map from $H = \mathrm{Hom}(C_1, C_2)$ to the Mordell-Weil lattice, since we only know (Prop.3.5) that the map $\varphi \to \Gamma_\varphi$ induces a group homomorphism $H \to \mathrm{NS}(S)/N_0$.

To remedy this situation, let us proceed as follows. Consider the orthogonal complement of L_1 in L (with the notation of Lemma 4.1):

$$L_1' = L_1^\perp. \tag{4.5}$$

Obviously the $\mathbf{Q}$-vector space $V = L \otimes \mathbf{Q}$ is an orthogonal direct sum of $V_1 = L_1 \otimes \mathbf{Q}$ and $V_1' = L_1' \otimes \mathbf{Q}$, although the lattice L itself is not in general equal to the direct sum $L_1 \oplus L_1'$. Decompose $P_\varphi \in V$ as a sum of the V_1-component and V_1'-component:

$$P_\varphi = [P_\varphi]^+ + [P_\varphi]^-, \quad [P_\varphi]^+ \in V_1, [P_\varphi]^- \in V_1'. \tag{4.6}$$

Lemma 4.3 *The V_1-component of $P_\varphi \in L$ is represented by the following element:*

$$[P_\varphi]^+ = \begin{cases} \frac{d}{2}(Q_{12}+Q_{21}) + \frac{d-1}{2}(Q_{11}+Q_{22}) & (a) \\ \frac{d-1}{2}(Q_{12}+Q_{21}+Q_{22}) + \frac{d}{2}Q_{11} & (b) \\ \frac{d}{2}(Q_{11}+Q_{12}+Q_{21}+Q_{22}) & (c) \end{cases} \tag{4.7}$$

In particular, it is an element of L_1 if and only if φ is in the case (c), i.e. $\varphi = 2\varphi_1$ for some $\varphi_1 \in \mathrm{Hom}(C_1, C_2)$.

Proof The first part is verified by a linear algebra computation. For the second part, note that in case (c), $d = \deg(\varphi) = 4\deg(\varphi_1)$ is divisible by 4. Thus $d/2$ is an integer, and $[P_\varphi]^+ \in L_1$. In case (a), d is odd, and we can easily see that $(Q_{12}+Q_{21})$ is not divisible by 2 in L. The case (b) can be treated in a similar way. *q.e.d.*

Lemma 4.4 *Depending on the case of $\varphi \in H$, we have*

$$\langle [P_\varphi]^+, [P_\varphi]^+ \rangle = \begin{cases} 2d^2 - 2d + 1 & (a) \\ 2d^2 - 3d + 4/3 & (b) \\ 2d^2 & (c) \end{cases} \tag{4.8}$$

and, for any $\varphi \in H$,

$$\langle [P_\varphi]^-, [P_\varphi]^- \rangle = d. \tag{4.9}$$

Proof Using Lemma 4.1, check first that both $(Q_{12}+Q_{21})$ and $(Q_{11}+Q_{22})$ have height 4 and they are orthogonal. By Lemma 4.3, we see for instance in case (a) that the "height" of $[P_\varphi]^+$ is equal to

$$(\frac{d}{2})^2 \cdot 4 + (\frac{d-1}{2})^2 \cdot 4 = 2d^2 - 2d + 1.$$

Other cases are similar, and this proves (4.8). Now it follows from (4.6) that

$$\langle [P_\varphi]^-, [P_\varphi]^- \rangle = \langle P_\varphi, P_\varphi \rangle - \langle [P_\varphi]^+, [P_\varphi]^+ \rangle$$

Comparing (4.3) and (4.8), we conclude (4.9) that the "height" of $[P_\varphi]^-$ is equal to d in all cases. *q.e.d.*

Proposition 4.5 *Let*

$$R_\varphi := 2[P_\varphi]^- \in L_1'. \tag{4.10}$$

Then the map $\varphi \mapsto R_\varphi$ gives an imbedding of the lattice $H[4]$ into L_1'.

Proof Let $N = \mathrm{NS}(S)$. By (2.1), we have $L \cong N/T$. Under this isomorphism, we have $L_1 \cong N_0/T$, since N_0 is generated by the 24 curves (cf. Cor.3.4) of which 15 (resp. 9) give generators of T (resp. L_1). It follows that $L/L_1 \cong N/N_0$, and the map $\varphi \mapsto P_\varphi$ induces a group homomorphism $H \to L/L_1$ by Proposition 3.5. On the other hand, the orthogonal projection $V \to V_1'$ induces the homomorphism $L/L_1 \to \frac{1}{2}L_1'$ sending P_φ modL_1 to $[P_\varphi]^-$. By composing the two maps, we obtain a homomorphism

$$H \to \frac{1}{2}L_1', \quad \varphi \mapsto [P_\varphi]^- \tag{4.11}$$

preserving the norm (or height) by (4.9). In other words, the map $\varphi \mapsto R_\varphi$ gives an injective homomorphism $H \to L_1'$ such that

$$\langle R_\varphi, R_\varphi \rangle = 4\deg(\varphi). \tag{4.12}$$

This proves the assertion. *q.e.d.*

5 Defining equation of Inose's pencil

Now we introduce the equations to make more detailed analysis. Suppose that the elliptic curve $C_l (l = 1, 2)$ is given by the Weierstrass equation:

$$C_l : y_l^2 = f_l(x_l) = x_l^3 + \cdots \quad = \prod_{k=1}^{3}(x_l - a_{l,k}). \tag{5.1}$$

The 2-torsion points of C_1, C_2 are given by $v_i = (a_{1,i}, 0), v_j' = (a_{2,j}, 0)$.

The function

$$t = y_2/y_1 \tag{5.2}$$

on $A = C_1 \times C_2$ is invariant under ι_A, and it defines an elliptic fibration on the Kummer surface S whose generic fibre is isomorphic to the plane cubic curve over $k(t)$ defined by

$$f_1(x_1)t^2 = f_2(x_2). \tag{5.3}$$

The following result is essentially due to Inose [5], for which we give a simplified proof bellow (cf. [15]):

Proposition 5.1 *The elliptic fibration on the Kummer surface S induced by $t = y_2/y_1$ is isomorphic to the Inose's pencil. The Weierstrass form of the cubic curve (5.3) is given by*

$$E^{(2)} : y^2 = x^3 - 3\alpha t^4 x + t^4(t^4 - 2\beta t^2 + 1) \tag{5.4}$$

where α, β are defined by (1.2), i.e. $\alpha = \sqrt[3]{j_1 j_2}$, $\beta = \sqrt{(1-j_1)(1-j_2)}$. There are two singular fibres of type IV^* at $t = 0$ and ∞, and the other singular fibres are given, in an abridged form, as follows: (i) $I_1 \times 8$ if $j_1 \neq j_2, j_1 j_2 \neq 0$, (ii) $II \times 4$ if $j_1 \neq j_2, j_1 j_2 = 0$, (iii) $I_2 \times 2 + I_1 \times 4$ if $j_1 = j_2 \neq 0, 1$, (iv) $I_2 \times 4$ if $j_1 = j_2 = 1$, (v) $IV \times 2$ if $j_1 = j_2 = 0$.

Proof To prove the first assertion, we claim that the divisor of the function t on S is equal to

$$(t) = \Phi_1 - \Phi_2 \tag{5.5}$$

where Φ_1, Φ_2 are the divisors defined by (4.1).

Indeed, by (5.1) and (5.2), we have

$$(t^2) = (f_2(x_2)/f_1(x_1)) = \sum_{k=1}^{3}(x_2 - a_{2,k}) - \sum_{i=1}^{3}(x_1 - a_{1,i}). \tag{5.6}$$

Since the function x_1 defines the first Kummer pencil π_1 (3.2), we have

$$(x_1 - a_{1,i}) = \pi_1^{-1}(v_i) - \pi_1^{-1}(v_0) = 2F_i + \sum_{j \in I} A_{i,j} - (2F_0 + \sum_{j \in I} A_{0,j}) \tag{5.7}$$

by (3.3). Writing down the corresponding fact for the the second Kummer pencil π_2, we have

$$(x_2 - a_{2,j}) = \pi_2^{-1}(v'_j) - \pi_2^{-1}(v'_0) = 2G_j + \sum_{i \in I} A_{i,j} - (2G_0 + \sum_{i \in I} A_{i,0}). \tag{5.8}$$

Then, by (4.1), (5.6), (5.7) and (5.8), we can easily check that

$$(t^2) = 2(\Phi_1 - \Phi_2). \tag{5.9}$$

This implies our claim (5.5), proving that the function t defines Inose's pencil.

Next, setting $T = t^2$, we consider the linear pencil of plane cubic curves

$$f_1(x_1)T = f_2(x_2). \tag{5.10}$$

The base points $(x_1, x_2) = (v_k, v'_l)$ define nine $k(T)$-rational points of the generic member, which can be transformed to a Weierstrass form over $k(T)$ such that one of the points, say (v_3, v'_3), is mapped to the point at infinity (cf. [2]). By carrying out the computation, one obtains an equation is of the form:

$$E^{(1)} : y^2 = x^3 + AT^2 x + T^2 B(T) \tag{5.11}$$

where A is a constant and $B(T)$ is a quadratic polynomial which depend on the coefficients of f_1, f_2. By replacing x, y, T by suitable constant multiples, they can be normalized so that

$$A = -3\alpha, B(T) = T^2 - 2\beta T + 1, \tag{5.12}$$

with $\alpha = \sqrt[3]{j_1 j_2}, \beta = \sqrt{(1 - j_1)(1 - j_2)}$ as in (1.2). (Note that the choice of the cube root or square root is irrelevant, since they give rise to isomorphic Weierstrass equations.)

Going back to (5.3), we see that the Weierstrass form of this plane cubic is given by $E^{(2)} = E^{(1)}|_{T=t^2}$ defined by (5.4). The singular fibres are easily determined by using [7] or [18]. (Also it is a simple consequence of the following lemma, since the map $t \mapsto T = t^2$ is a double cover ramified only at $t = 0$ and ∞.) *q.e.d.*

Lemma 5.2 *The elliptic surface corresponding to $E^{(1)}$ is a rational surface. It has two singular fibres of type IV at $T = 0$ and ∞, and other singular fibres are given as follows: (i) $I_1 \times 4$ if $j_1 \neq j_2, j_1 j_2 \neq 0$, (ii) $II \times 2$ if $j_1 \neq j_2, j_1 j_2 = 0$, (iii) $I_2 + I_1 \times 2$ if $j_1 = j_2 \neq 0, 1$, (iv) $I_2 \times 2$ if $j_1 = j_2 = 1$, (v) IV if $j_1 = j_2 = 0$.*

Proof The discriminant $\Delta(E^{(1)})$ is given by $T^4(B(T)^2 - 4\alpha^3 T^2)$ up to constant. Then the verification is a simple exercise using [7] or [18]. *q.e.d.*

Proposition 5.3 *The Mordell-Weil lattice $E^{(1)}(k(T))$ is isomorphic to $(A_2^*)^{\oplus 2}$ if $j_1 \neq j_2$, and to $A_2^* \oplus \langle 1/6 \rangle$, or $\langle 1/6 \rangle^{\oplus 2}$ or $A_2^* \oplus \mathbf{Z}/3\mathbf{Z}$ if $j_1 = j_2$, in the respective case (iii) or (iv) or (v). $E^{(1)}(k(T))$ is generated by the rational points of the form $x = aT, y = T(cT + d)$. If $j_1 \neq j_2$, there are 12 such points which are the 12 minimal vectors of height (or norm) $2/3$ in $(A_2^*)^{\oplus 2}$.*

Proof Assume $j_1 \neq j_2$. By the height formula (2.6), a point $P = (x, y) \in E^{(1)}(k(T))$ has the minimal norm $2/3$ if and only if $(PO) = 0$ and (P) passes through the non-identity component of each of the two reducible fibres of type IV^*. The first condition $(PO) = 0$ implies that the coordinates x, y of P are polynomial of degree ≤ 2 or 3 (cf. [14, §10]) and the second condition implies that their constant terms as well as the highest terms should vanish. Hence the result follows. The same method can be applied for the case $j_1 = j_2$. *q.e.d.*

We note some consequence of Proposition 5.1:

Corollary 5.4 *The Mordell-Weil rank* $r^{(2)} = \mathrm{rk}E^{(2)}(k(t))$ *is equal to* $4 + h$ *if* $j_1 \neq j_2$*, and to* $2 + h$ *(resp.* $h = 2$*) in case (iii) (resp. (iv) or (v)).*

Proof The rank $r^{(2)}$ is equal to the Picard number of S minus the rank of the trivial lattice (cf. (2.1)), so the verication is immediate. *q.e.d.*

Corollary 5.5 *The torsion subgroup of* $E^{(2)}(k(t))$ *is trivial in case (i)-(iv) and* $\mathbf{Z}/3\mathbf{Z}$ *in case (v).*

Proof This follows from the classification results of Shimada [13] (cf. also [10], [9]).

Corollary 5.6 *The condition (#) in §4 holds if and only if* $j_1 \neq j_2$*, i.e.* C_1 *and* C_2 *are not isomorphic to each other.*

6 MWL of Inose's pencil

We keep the notation in the previous sections. We assume the condition $j_1 \neq j_2$ in this section.

The Mordell-Weil lattice $L = E^{(2)}(k(t))$ obviously contains $E^{(1)}(k(T))$ with $T = t^2$, which can be identified with the sublattice L_1 of Lemma 4.1. Recall that the height of a point gets multiplied by the degree of the base change (see [14, Prop. 8.12]).

Let $\sigma : t \mapsto -t$ be the non-trivial automorphism of the quadratic extension $k(t)/k(T)$. It acts naturally on L and we have

$$L_1 = E^{(1)}(k(T)) = \{P \in L | P^\sigma = P\}. \tag{6.1}$$

On the other hand, letting $F^{(1)}$ denote the quadratic twist of $E^{(1)}/k(T)$ with respect to $k(t)/k(T)$ ($t^2 = T$):

$$F^{(1)} : y^2 = x^3 - 3\alpha T^4 x + T^5(T^2 - 2\beta T + 1), \tag{6.2}$$

we have

$$F^{(1)}(k(T)) \tilde{\to} \{P \in E^{(2)}(k(t)) | P^\sigma = -P\} =: L_1'' \subset L \tag{6.3}$$

where $Q = (x(T), y(T)) \in F^{(1)}(k(T))$ corresponds to $P = (x(t^2)/t^2, y(t^2)/t^3)$. Note that L_1'' is orthogonal to L_1 (and $L_1'' \cap L_1 = 0$) since there is no 2-torsion (in fact, L is torsion free under the assumption). Moreover, as is standard in this situation, we have for any $P \in L$

$$2P = (P + P^\sigma) + (P - P^\sigma), \quad (P + P^\sigma) \in L_1, \quad (P - P^\sigma) \in L_1''. \tag{6.4}$$

It follows that $L_1 + L_1'' \supset 2L$ and $L_1'' = L_1'$ is the orthogonal complement of L_1, (4.5). Also V_1 (or V_1') in §4 is the eigenspace with eigenvalue 1 (or -1) for the action of σ on $V = L \otimes \mathbf{Q}$.

Theorem 6.1 *Assume $j_1 \neq j_2$. Then the Mordell-Weil lattice $L = E^{(2)}(k(t))$ contains the sublattice $L_1 \oplus L_1'$ with finite index $I = 2^h$ where*

$$L_1 = E^{(1)}(k(T))[2] \cong A_2^*[2]^{\oplus 2}, \quad \mathrm{rk} L_1 = 4, \quad \det L_1 = \frac{2^4}{3^2}, \tag{6.5}$$

$$L_1' \cong F^{(1)}(k(T))[2] \cong H[4], \quad \mathrm{rk} L_1' = h, \quad \det L_1' = 2^{2h} \cdot \delta \tag{6.6}$$

in which h (or δ) denotes as before the rank (or det) of $H = \mathrm{Hom}(C_1, C_2)$.

Proof The only facts yet to be proven in the the above statements are the following:

$$(i) \quad I = 2^h, \quad (ii) \quad L_1' \cong H[4] \tag{6.7}$$

Letting ν be the index of $H[4]$ in L_1' in Proposition 4.5, we have

$$\det L_1' = 4^h \cdot \delta / \nu^2. \tag{6.8}$$

Hence we have

$$\det L = \det(L_1 \oplus L_1')/I^2 = 2^4/3^2 \cdot 4^h \delta/(\nu^2 I^2) \tag{6.9}$$

On the other hand, by applying (2.7) to $E = E^{(2)}, S = \mathrm{Km}(C_1 \times C_2)$ and $T = U \oplus E_6^2$, we have $\det NS(S) = \det L \cdot 3^2$, which gives by Theorem 3.3

$$\det L = 2^4 \cdot \delta / 3^2. \tag{6.10}$$

By comparing (6.9) and (6.10), we have

$$I \cdot \nu = 2^h. \tag{6.11}$$

The next lemma shows $I = 2^h$, and hence $\nu = 1$ by (6.11), which is equivalent to the claim (ii) in (6.7) . This proves both claims of (6.7). *q.e.d.*

Lemma 6.2 *The map $\varphi \mapsto P_\varphi$ induces an isomorphism*

$$H/2H \cong L/(L_1 + L_1'). \tag{6.12}$$

In particular, the index $I = [L : L_1 + L_1']$ is equal to 2^h.

Proof The map in question induces a surjective homomorphism of H to L/L_1 (as shown in the proof of Proposition 4.5), and hence $H \to L/(L_1 + L_1')$ is also a surjection. By (6.4), the latter map induces a surjection $H/2H \to L/(L_1 + L_1')$, which is also an injection by Lemma 4.3. *q.e.d.*

Theorem 6.3 *Let $S^{(1)}$ denote the elliptic surface associated with $F^{(1)}/k(T)$. Then it is a K3 surface, and it has two singular fibres of type II^* at $T = 0, \infty$, and the other singular fibres are given as follows: (i) $I_1 \times 4$ if $j_1 \neq j_2, j_1 j_2 \neq 0$, (ii) $II \times 2$ if $j_1 \neq j_2, j_1 j_2 = 0$, (iii) $I_2 + I_1 \times 2$ if $j_1 = j_2 \neq 0, 1$, (iv) $I_2 \times 2$ if $j_1 = j_2 = 1$, (v) IV if $j_1 = j_2 = 0$.*

The Mordell-Weil rank $r^{(1)} = \mathrm{rk} F^{(1)}(k(T))$ is equal to h if $j_1 \neq j_2$, and to $h - 1$ (resp. $h - 2 = 0$) in case (iii) (resp. (iv) or (v)).

Assume $j_1 \neq j_2$. Then the Mordell-Weil lattice $F^{(1)}(k(T))$ is isomorphic to $H[2] = \mathrm{Hom}(C_1, C_2)[2]$.

Proof The singular fibres are checked in the same way as in Lemma 5.2 for $E^{(1)}/k(T)$ which is the twist of $F^{(1)}/k(T)$, and it shows that $S^{(1)}$ is a K3 surface since the Euler number (or the order of the discriminant) is 24. As for the rank formula, it follows from Corollary 5.4 and

$$r^{(1)} = \mathrm{rk} E^{(2)}(k(t)) - \mathrm{rk} E^{(1)}(k(T)). \tag{6.13}$$

The final assertion is just a restatement of the fact $L_1' \cong H[4]$ proven in Theorem 6.1, (6.6), in view of the height behavior under the base change (here, of degree two). *q.e.d.*

Now Theorem 1.1 or 1.2 in the Introduction (§1) follow from the above Theorem 6.3 or 6.1. (Note that $F^{(1)}/k(T)$ in §1 and §6 are the same up to simple coordinate change. Also $F^{(2)}/k(t)$ and $E^{(2)}/k(t)$ are isomorphic.) They are formulated in terms of elliptic curves only, without reference to a K3 or Kummer surface, but the latter is essential for the proof as seen above.

7 Comments on the case $j_1 = j_2$

We have excluded the case $j_1 = j_2$ for the sake of simplicity in some of the above discussion. In this case, we have "extra" reducible fibres in Inose's pencil (see Proposition 5.1) and the Mordell-Weil rank drops. We can clarify this situation by the use of the curve Γ_φ (§3) for the " isomorphism correspondence".

Proposition 7.1 *Assume $j_1 = j_2$, i.e. C_1 and C_2 are isomorphic elliptic curves. Let $\varphi : C_1 \to C_2$ be any isomorphism. Then the curve Γ_φ (the image of the graph of φ in $A = C_1 \times C_2$ under the rational map $A \to S$) is an irreducible component of an extra reducible fibre.*

Proof We can assume that $C_1 = C_2$ and it is defined by (5.1). Recall that the elliptic fibration $f : S \to \mathbf{P}^1$ is given by the function (5.2):

$$t = y_2/y_1. \tag{7.1}$$

Suppose $\varphi : (x_1, y_1) \mapsto (x_2, y_2)$ is an automorphism of C_1.

(i) If $\varphi = id$ is the identity, then we have $t = y_2/y_1 = 1$ on its graph. Hence the curve Γ_{id} is contained in the fibre over $t = 1$, $f^{-1}(1)$. Similarly, if $\varphi = -id$ is the inversion, we have $y_2 = -y_1$ so that $t = -1$. Hence $\Gamma_{-id} \subset f^{-1}(-1)$. In this way, we get two reducible fibres of type I_2 at $t = 1, -1$ for general value of j_1, namely for $j_1 \neq 0, 1$.

(ii) Let $C_1 : y_1^2 = x_1^3 - x_1 (j_1 = 1)$ and suppose $\varphi : (x_1, y_1) \mapsto (-x_1, \pm i y_1)$. Then we have $t = \pm i$ ($i = \sqrt{-1}$). In this case, we get four reducible fibres of type I_2 at $t = 1, -1, i, -i$.

(iii) Let $C_1 : y_1^2 = x_1^3 - 1$ ($j_1 = 0$) and suppose $\varphi : (x_1, y_1) \mapsto (\omega x_1, \pm y_1)$ ($\omega^3 = 1$). Then we have $t = \pm 1$. In this case, we get two singular fibres of type IV at $t = 1, -1$. The three curves Γ_φ for three values of ω give the three irreducible components for type IV-fibre.

This completes the proof. *q.e.d.*

8 Examples

First, the general case of Theorem 6.1 and 6.3 is very simple.

Example 8.1 *Assume that C_1, C_2 are non-isogenous elliptic curves. Then*

$$F^{(2)}(k(t)) \cong A_2^*[2], \quad F^{(1)}(k(T)) = \{0\} \tag{8.1}$$

The generators of $k(t)$-rational points are given by Proposition 5.3 by setting $T = t^2$.

Next a special case of Theorem 6.3 implies:

Example 8.2 *Suppose that $C_2 : y^2 = x^3 - 1$ ($j_2 = 0$) and $j_1 = j$ is arbitrary. We have $\alpha = 0, \beta = \sqrt{1-j}$, and $F^{(1)}$ has the equation:*

$$F_j = F^{(1)} : y^2 = x^3 + T^5(T^2 - 2\beta T + 1), \quad j = 1 - \beta^2. \tag{8.2}$$

Then (1) F_j has a $k(T)$-rational point ($\neq O$) if and only if $j = j(C_1)$ for some elliptic curve C_1 isogenous (but not isomorphic) to C_2, and thus (2) $F_j(k(T)) \neq \{O\}$ holds only for countably many values of $j \in k$.

Let us write down further properties of the elliptic curve $F_j/k(T)$. For simplicity, assume $\mathrm{char}(k) = 0$ and let $F_j(k(T)) \neq \{O\}$.

(3) The Mordell-Weil lattice $F_j(k(T))$ is isomorphic to $A_2[d_0]$ where $d_0 = \deg(\varphi_0)$ is the minimal isogeny $\varphi_0 : C_1 \to C_2$. (A_2 is the root lattice.)

(4) The minimal height is $2d_0 \geq 4$.

(5) There is a rational point P of height $\langle P, P \rangle = 4$ if and only if C_1 has degree 2 isogeny to C_2. In this case, $P = (\xi, \eta)$ is an "integral point" with both $\xi, \eta \in k[T]$ with $\deg(\xi) = 4, \deg(\eta) = 6$. (This follows from the height formula.)

(6) Such C_1 is unique up to isomorphism and one has $j = j(C_1) = 125/4$. Then F_j is given by

$$F_j : y^2 = x^3 + T^5(T^2 - 11\sqrt{-1}T + 1). \tag{8.3}$$

which is equivalent (up to coordinate change) to the following equation:

$$y^2 = x^3 + T^5(T^2 - 11T - 1). \tag{8.4}$$

(7) There are three different methods to find an integral point $P = (\xi, \eta)$ of height 4 of the form in (5). [N.B. The resulting integral point is essentially the same by the uniqueness in (5) or (6).]
(i) straightforward computer search ([3]),
(ii) explicit computation of $P_\varphi = \mu(\Gamma_\varphi)$ for the degree two isogeny φ, explained in the present paper (the result was announced in [15]),
(iii) use the idea of "Shafarevich partner" ([16]). We outline the third method below, because this simple device can be useful in more general situation.

(8) Take a rational elliptic surface with 4 singular fibres $I_1 \times 2, I_5 \times 2$ (see e.g. [1]). Assume that the fibres of type I_5 are at $T = 0, \infty$. Write down the minimal Weierstrass equation of the generic fibre as follows:

$$Y^2 = X^3 - 3\xi X - 2\eta, \quad \xi, \eta \in k[T] \tag{8.5}$$

Then the discriminant is equal to $\xi^3 - \eta^2$ up to constant, but at the same time it should take the form $cT^5(T - v_1)(T - v_2)$ by the data of singular fibres. This shows that $P = (\xi, \eta)$ gives rise to an integral point of required form.

Note that this argument can be reversed to prove the existence and uniqueness of the rational elliptic surface with singular fibres $I_1 \times 2, I_5 \times 2$, from the knowledge of such an integral point of height 4.

References

[1] Beauville, A.: Les familles stables de courbes elliptiques sur $\mathbf{P}^1$, C. R. Acad. Sci. Paris, 294 (I), 657–660 (1982).

[2] Cassels, J.W.S.: Lectures on Elliptic Curves, Cambridge Univ. Press (1991).

[3] Chahal, J., Meijer, M., Top, J.: Sections on certain $j = 0$ elliptic surfaces, Comment. Math. Univ. St. Pauli 49, 79–89 (2000).

[4] Conway, J., Sloane, N.: Sphere Packings, Lattices and Groups, Springer-Verlag (1988); 2nd ed.(1993); 3rd ed.(1999).

[5] Inose, H. : Defining equations of singular K3 surfaces and a notion of isogeny, Intl. Symp. on Algebraic Geometry/ Kyoto 1977, Kinokuniya, 495–502 (1978).

[6] Inose, H., Shioda, T.: On singular K3 surfaces, in: Complex Analysis and Algebraic Geometry, Iwanami Shoten and Cambridge Univ. Press, 119–136 (1977).

[7] Kodaira, K.: On compact analytic surfaces II-III, Ann. of Math. 77, 563-626(1963); 78, 1-40(1963); Collected Works, III, 1269-1372, Iwanami and Princeton Univ. Press (1975).

[8] Kuwata, M.: Elliptic K3 surfaces with given Mordell-Weil rank, Comment. Math. Univ. St. Pauli 49, 91–100 (2000).

[9] Nishiyama, K.: On Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups, Japan J. Math. 22, 293–347 (1996).

[10] Oguiso, K.: On Jacobian fibrations on the Kummer surfaces of product of non-isogenous elliptic curves, J. Math. Soc. Japan 41, 651–680 (1989).

[11] Oguiso, K., Shioda, T.: The Mordell–Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Pauli 40, 83–99 (1991).

[12] Piateckii-Shapiro, I., Shafarevich, I.R.: A Torelli theorem for algebraic surfaces of type K3, Math. USSR Izv. 5, 547–587 (1971).

[13] Shimada, I.: On elliptic K3 surfaces, Michigan Math. J. 47 (2000), 423–446.

[14] Shioda, T.: On the Mordell-Weil lattices, Comment. Math. Univ. St. Pauli 39, 211– 240 (1990).

[15] — : A note on K3 surfaces and sphere packings, Proc.Japan Acad. 76A, 68–72 (2000).

[16] — : Elliptic surfaces and Davenport-Stothers triples, Comment. Math. Univ. St. Pauli 54, 49–68 (2005).

[17] — : Classical Kummer surfaces and Mordell-Weil Lattices, Proc. KIAS Conf. Algebraic Geometry 2004 (to appear).

[18] Tate, J: Algorithm for determining the type of a singular fiber in an elliptic pencil, SLN 476, 33-52 (1975).

[19] Weil, A.: Variétés abéliennes et courbes algébriques, Hermann, Paris (1948/1973).

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