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Fundamental Invariants of Weyl Groups and Excellent Families of Elliptic Curves

by

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1. Introduction

As is well-known, the ring of invariants of a finite reflection group is a polynomial ring (Chevalley's theorem: see e.g. [B]). To be more precise, if G is such a group acting on a finite-dimensional vector space V over a field F (with coordinates $u_1, \cdots, u_n$), then the ring of invariants in $F[u_1, \cdots, u_n]$ is a graded polynomial ring generated by n homogeneous elements, say $p_1, \cdots, p_n$:

$$F[u_1, \cdots, u_n]^G = F[p_1, \cdots, p_n] , \tag{1.1}$$

provided that the order of G is not divisible by char(F). The set of the weights (or degrees) of the *fundamental invariants* $p_1, \cdots, p_n$ is uniquely determined by G.

Geometrically, this can be translated as follows: the group G acts on the affine space A^n (with coordinates $u_1, \cdots, u_n$) in such a way that the quotient space A^n/G becomes an affine space again (with coordinates $p_1, \cdots, p_n$). In other words, $X = A^n$ is realized as a finite Galois covering of another $Y = A^n$ with Galois group G:

$$\pi: A^n \to A^n/G \simeq A^n . \tag{1.2}$$

Further there is a compatible action of the multiplicative group G_m on the affine spaces A^n both upstairs and downstairs.

The most classical example of the above situation is the case where $G = \mathscr{S}_n$ is the symmetric group on n letters $u_1, \cdots, u_n$. Then the fundamental theorem on symmetric functions says that

$$F[u_1, \cdots, u_n]^{\mathscr{S}_n} = F[\varepsilon_1, \cdots, \varepsilon_n] ,$$

where ε_i denotes the i-th elementary symmetric function of $u_1, \cdots, u_n$. (F can be any field in this case, and indeed it can be replaced by the ring of integers $\boldsymbol{Z}$.) The action of G restricted to the $(n-1)$-dimensional subspace $\sum_{i=1}^n u_i = 0$ of V identifies $\mathscr{S}_n$ with $W(A_{n-1})$, the Weyl group of the root system of type A_{n-1} (cf. [B], [CS]), and (1.1) reduces to

$$F[u_2, \cdots, u_n]^{W(A_{n-1})} = F[\varepsilon_2, \cdots, \varepsilon_n] . \tag{1.3}$$

In general, the Weyl group $W(R)$ of a root system R is a finite reflection group, for which the weights of the fundamental invariants are well-known: they are the exponents increased by 1 (cf. [B]). For example, the weights for $W(R)$ are given as follows when R is of type A, D, E:

$$(1.4)\qquad \begin{cases} W(A_{n-1}) & \{2, 3, \cdots, n\} \qquad (n\geq 2) \\ W(D_n) & \{2, 4, \cdots, 2(n-1), n\} \qquad (n\geq 4) \\ W(E_6) & \{2, 5, 6, 8, 9, 12\} \\ W(E_7) & \{2, 6, 8, 10, 12, 14, 18\} \\ W(E_8) & \{2, 8, 12, 14, 18, 20, 24, 30\}\,. \end{cases}$$

Now let us introduce the notion of an *excellent family* with Galois group G. For the sake of simplicity, we take $F=\boldsymbol{Q}$ in what follows, but a suitable modification works in a more general case. Suppose that $\{X_\lambda\}$ is a family of algebraic varieties (possibly with some extra structure) depending on the parameter

$$\lambda=(p_1, \cdots, p_n)\in A^n\,.$$

The generic member X_λ is a variety defined over the field $k_0=\boldsymbol{Q}(\lambda)$ which is a purely transcendental extension of $\boldsymbol{Q}$ of dimension n. Furthermore, suppose given $\mathscr{C}(X_\lambda)$, a suitable group of algebraic cycles on X_λ, such that (i) it spans a finite dimensional vector space isomorphic to V and (ii) the Galois group $\mathrm{Gal}(k/k_0)$ acts on it (k is the algebraic closure of k_0). Let ρ_λ denote the Galois representation:

$$(1.5)\qquad \rho_\lambda : \mathrm{Gal}(k/k_0)\to \mathrm{Aut}(\mathscr{C}(X_\lambda)) \subsetneq \mathrm{Aut}(V)\,.$$

Let $\mathscr{K}_\lambda$ be the Galois extension of k_0 corresponding to $\mathrm{Ker}(\rho_\lambda)$, i.e. with Galois group $\mathrm{Gal}(\mathscr{K}_\lambda/k_0)=\mathrm{Im}(\rho_\lambda)$. It is equal to the smallest extension of k_0 such that $\mathscr{C}(X_\lambda)$ is generated by $\mathscr{K}_\lambda$-rational cycles. We call $\mathscr{K}_\lambda$ the *splitting field* of $\mathscr{C}(X_\lambda)$.

Now recall that G is a finite reflection group acting on V so that it is a subgroup of $\mathrm{Aut}(V)$.

DEFINITION. We call $\{X_\lambda\}$ an *excellent family* with Galois group G if the following conditions hold:

(1) the image of ρ_λ is equal to G.

(2) there is a $\mathrm{Gal}(k/k_0)$-equivariant evaluation map

$$s\colon \mathscr{C}(X_\lambda)\to k\,,$$

(3) there exists a basis $\{Z_1, \cdots, Z_n\}$ of $\mathscr{C}(X_\lambda)$ such that if we set $u_i=s(Z_i)$, then $u_1, \cdots, u_n$ are algebraically independent over $\boldsymbol{Q}$, and

(4)

$$\boldsymbol{Q}[u_1, \cdots, u_n]^G=\boldsymbol{Q}[p_1, \cdots, p_n]\,.$$

It follows from the definition that the splitting field $\mathscr{K}_\lambda$ is then equal to $\boldsymbol{Q}(u_1, \cdots, u_n)$, a purely transcendental extension of $\boldsymbol{Q}$, such that

$$\mathrm{Gal}(\mathbf{Q}(u_1, \cdots, u_n)/\mathbf{Q}(p_1, \cdots, p_n)) \simeq G .$$

We call $u_1, \cdots, u_n$ the *splitting variables* or *variables upstairs* (in view of (1.2)).

Example 1. The classical case (1.3) for $G=\mathscr{S}_n=W(A_{n-1})$ can be interpreted as follows. Let X_λ be the 0-dimensional variety defined by the equation

$$x^n+\varepsilon_2 x^{n-2}+\cdots+(-1)^n\varepsilon_n=0 \tag{1.6}$$

over $k_0=\mathbf{Q}(\lambda)$, $\lambda=(\varepsilon_2, \cdots, \varepsilon_n)$. Let u_i $(i=1, \cdots, n)$ be the n roots of (1.6). Then we have

$$X_\lambda \otimes_{k_0} k=\{u_1, \cdots, u_n\} \qquad (u_1+\cdots+u_n=0)$$

and each u_i defines a 0-cycle $[u_i]$ on X_λ. Take

$$\mathscr{C}(X_\lambda)=\sum_{i=1}^{n} \mathbf{Z}\cdot[u_i] \bmod \sum_{i=1}^{n}[u_i]=0$$

and define the evaluation map by s: $[u_i] \to u_i$, extended by linearity; obviously this is a Galois equivariant map. The Galois group $\mathrm{Gal}(k/k_0)$ acts as permutations of $[u_i]$, so the image of ρ_λ is equal to $G=\mathscr{S}_n$. In view of (1.3), $\{X_\lambda\}$ forms an excellent family with Galois group $\mathscr{S}_n$.

Example 2. In the recent work on Mordell-Weil lattices (see especially [S5], [S6]), it has been shown that certain family of elliptic curves over a rational function field (or that of rational elliptic surfaces) forms an excellent family with Galois group $W(E_r)$ $(r=6, 7, 8)$. For example, for the case $r=8$, consider the elliptic curve E_λ

$$y^2=x^3+x\left(\sum_{i=0}^{3} p_{20-6i}t^i\right)+\left(\sum_{j=0}^{3} p_{30-6j}t^j+t^5\right),$$

over $k_0(t)$ where we set $k_0=\mathbf{Q}(\lambda)$ with

$$\lambda=(p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30}) .$$

Let

$$\mathscr{C}(E_\lambda)=E_\lambda(k(t))$$

be the Mordell-Weil group of $k(t)$-rational points on E_λ, k being the algebraic cosure of k_0; by the theory of Mordell-Weil lattices (MWL), it has the structure of the root lattice E_8 for generic λ. Further there is a natural evaluation map

$$sp_\infty : E(k(t)) \to G_a(k)=k ,$$

(the specialization homomorphism at the singular fibre over $t=\infty$). Then the fundamental theorems for type E_8 (cf. [S5], §8) show that $\{E_\lambda\}$ forms an excellent family of elliptic curves with Galois group $W(E_8)$ (cf. §4 below).

This fact and its variant for E_6 and E_7 are the motivating examples for introducing the notion of an *excellent family of elliptic curves*. In view of its potential applications to arithmetic, algebra and geometry, ..., as shown for E_r in [S5] or [S6], it will be an important problem to establish the existence of an excellent family of elliptic curves with Galois group G for a wider class of Weyl groups G.

The purpose of this paper is to construct such a family when $G=W(L)$ is the Weyl group of (the root system in) a root lattice L, which is a sublattice of E_8 of relatively high rank. More precisely, we consider the Mordell-Weil lattice of a rational elliptic surface, the structure of which has been classified into 74 types (see [OS] and the next section). Among them, there are exactly 31 "admissible" types for which the narrow Mordell-Weil lattice L is a root lattice of positive rank. In this paper, we treat about half of these admissible types; we construct an excellent family for each type where the rank of L is greater than 4 and for a few more types. This will extend the previous work [S5] for $L=E_8$, E_7, E_6, D_4, A_2 and the recent one [U] for D_5. The remaining cases will be treated in a forthcoming paper [S7], where the configuration of singular fibres can be more complicated as the rank of L gets smaller. Together with it, the existence of excellent families of elliptic curves (over the rational function field) will be established for all admissible types (see also [S8]).

The paper is organized as follows. First, in §2, we state the main results and exhibit an excellent family of elliptic curves for each type in question (Theorem 1). Then we give a necessary and sufficient condition for the nondegeneracy of MWL (Theorem 2). In §3, we explain a general idea of proof and then our strategy for finding (or constructing) a good candidate for such a family; there are two main ideas: *magic of weights* and *degeneration of Mordell-Weil lattices*. In §4, we review the results for the case $L=E_8$, E_7, E_6 from [S5]. The rest of the paper is devoted to verifying that the families given in §2 are excellent. Naturally this process requires a case by case examination, but during the course of it, we obtain further useful informations. Thus, in each case, the results will include:

(I) classifying data and the defining equation
(II) discriminant and singular fibres
(III) minimal (or short) vectors in the MWL
(IV) the fundamental algebraic equation
(V) explicit formula for the fundamental invariants of the Weyl group
(VI) generators of the Mordell-Weil group
(VII) non-degeneracy condition of MWL
(VIII) a $\boldsymbol{Q}$-split example.

Finally it should be remarked that, once an excellent family is given, we have some standard applications such as (i) construction of Galois extensions (or representations) of $\boldsymbol{Q}$ with Galois group $W(L)$ or (ii) deformation of rational double points, etc. (cf. [S6], [S4]). Apart from these, the defining equations of excellent families (given in §2 and [S7]) could be regarded as a new kind of *normal forms of elliptic curves* for which the generators of the rational points of infinite order have explicit description, just as the classical Legendre (or Hessian) normal form describes elliptic curves with torsion points of order 2 (or 3). As such, it is not hard to imagine that they should have some further interesting applications.

2. Main results

First we fix the notation used throughout the paper, recalling some facts on Mordell-Weil lattices (MWL); we refer to [S1], [S2], [OS] for more details.

General notation:

k:	an algebraically closed field
$K=k(t)$:	the rational function field over k
E/K:	an elliptic curve over K
$E(K)$:	the Mordell-Weil group, i.e. the group of K-rational points of E with the identity O, which is finitely generated under a mild assumption
$\langle P, Q\rangle$:	the height pairing on $E(K)$, defined in [S1, I] or [S2], §8. This is a symmetric bilinear pairing which is positive-definite modulo torsion.
the MWL $E(K)$:	We abuse this terminology, the Mordell-Weil lattice $E(K)$, to mean the structure of Mordell-Weil group given with the height pairing.
$E(K)^0$:	the narrow Mordell-Weil lattice of E/K; this is a certain subgroup of finite index in $E(K)$, which becomes an even integral lattice with respect to the height pairing.
$f: S \to C = P^1$:	the associated elliptic surface (the Kodaira-Néron model) of E/K. A K-rational point $P \in E(K)$ is identified with a section of f.
(P):	the curve on S determined by a section P, esp. (O) is the zero-section viewed as a curve on S.
R:	the set of $v \in C$ such that $f^{-1}(v)$ is a reducible singular fibre of f (cf. [K], [N], [T]).
$\Theta_{v,j}$:	irreducible components of $f^{-1}(v)$ $(v \in R)$, with $j=0$ corresponding to the identity component
T_v:	the lattice generated by $\Theta_{v,j}$ $(j>0)$ with the sign changed; this is a root lattice of type A, D, E determined by the type of reducible fibre $f^{-1}(v)$ (cf. [S2], (7.6)).
$T=\bigoplus_{v \in R} T_v$:	the trivial lattice

Now we are interested in the case where $L=E(K)^0$ is a root lattice of positive rank. Note that this will be the case only if the elliptic surface S is a rational elliptic surface. In fact, the minimal norm of $E(K)^0$ has, in general, a lower bound 2χ, where χ is the arithmetic genus of S (see [S2], Th. 8.7). Since the minimal norm of a root lattice is 2, we must have $\chi=1$, which implies that S is a rational elliptic surface.

The structure of the Mordell-Weil lattice of a rational elliptic surface has been determined in [S2], §10, and classified in [OS]. Namely, given such an $E/K=k(t)$, let

$$(2.1)\qquad \begin{cases} T = \bigoplus_{v\in R} T_v \subset E_8 & \text{(trivial lattice)} \\ L = E(K)^0 & \text{(narrow MWL)} \\ M = E(K) & \text{(MWL)}. \end{cases}$$

By [S2], Th. 10.3 (with the notation modified), L is equal to the orthogonal complement of T in E_8 (which depends on the embedding of T in E_8), and M is the direct sum of the dual lattice L^* of L and the torsion subgroup $E(K)_{tor}\simeq T'/T$, T' being the primitive closure of T in E_8. Further, by the main result of [OS], the structure of the triple $\{T, L, M\}$ can be classified into 74 types (No. 1, $\cdots$, No. 74). Among them, there are exactly 31 "admissible" types for which the narrow Mordell-Weil lattice L is a root lattice of positive rank:

$$L=E_8, E_7, E_6, D_6, D_5, A_5, D_4\oplus A_1, A_4, D_4, A_3\oplus A_1, A_2\oplus A_2, \cdots, A_1 .$$

The main aim of this paper is to prove the following theorem, i.e. to construct explicitly an excellent family of elliptic curves for each admissible type with $\mathrm{rk}\, L>4$ and for a few more types derived from the case $L=A_4$. In the following statement, $W(L)$ denotes the Weyl group of a root lattice L, and for the fundamental invariants $p_w, \cdots$ of $W(L)$, the subscripts $w, \cdots$ will indicate the weights of the invariants.

THEOREM 1 (Existence). *For each admissible type $\{T, L, M\}$ below with L a root lattice, there exists an excellent family of elliptic curves (over the rational function field) with Galois group $W(L)$. More precisely, let E_λ be the elliptic curve defined below by a generalized Weierstrass equation with r parameters $\lambda=(p_w, \cdots)$ ($r=\mathrm{rk}\, L$). Then, for λ generic over $\mathbf{Q}$, (2.1) holds as isomorphy of lattices, where $E=E_\lambda$ and $K=k(t)$, k being the algebraic closure of $k_0=\mathbf{Q}(\lambda)$, and $\{E_\lambda\}$ defines an excellent family of elliptic curves with Galois group $W(L)$. In other words, for λ generic, we have*

(i) *the image of the Galois representation*

$$\rho_\lambda: \mathrm{Gal}(k/k_0)\to \mathrm{Aut}(E(k(t))^0)\simeq \mathrm{Aut}(L)$$

is exactly $W(L)$.

(ii) *Let $u_i=sp_v(Q_i)$ for a suitable choice of rational points $Q_i\in E(k(t))$ and a place v of $k(t)$ (sp_v: the specialization map). Then $u_1, \cdots, u_r$ are algebraically independent over $\mathbf{Q}$, and*

$$\mathbf{Q}[u_1, \cdots, u_r]^{W(L)}=\mathbf{Q}[p_w, \cdots] .$$

(iii) *In particular, the coefficients $p_w, \cdots$ of the elliptic curve E_λ form a set of fundamental invariants of the Weyl group $W(L)$, while $\mathscr{K}_\lambda=\mathbf{Q}(u_1, \cdots, u_r)$ gives the splitting field of MWL, i.e. the smallest extension of k_0 such that $E_\lambda(k(t))=E_\lambda(\mathscr{K}_\lambda(t))$.*

No. 1: $T=\{0\}, L=E_8, M=E_8$

$$y^2 = x^3 + x\left(\sum_{i=0}^{3} p_{20-6i}t^i\right) + \left(\sum_{j=0}^{3} p_{30-6j}t^j + t^5\right)$$

$$\lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30})$$

No. 2: $T = A_1,\ L = E_7,\ M = E_7^*$

$$y^2 = x^3 + x(p_{12} + p_8 t + t^3) + \left(\sum_{i=0}^{4} p_{18-4i}t^i\right)$$

$$\lambda = (p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18})$$

No. 3: $T = A_2,\ L = E_6,\ M = E_6^*$

$$y^2 = x^3 + x\left(\sum_{i=0}^{2} p_{8-3i}t^i\right) + \left(\sum_{i=0}^{2} p_{12-3i}t^i + t^4\right)$$

$$\lambda = (p_2, p_5, p_6, p_8, p_9, p_{12})$$

No. 4: $T = A_1^{\oplus 2},\ L = D_6,\ M = D_6^*$

$$y^2 = x^3 + x^2 r_6 + x(p_8 t + p_4 t^2 + t^3) + (p_{10}t^2 + p_6 t^3 + p_2 t^4)$$

$$\lambda = (p_2, p_4, p_6, r_6, p_8, p_{10})$$

No. 5: $T = A_3,\ L = D_5,\ M = D_5^*$

$$y^2 + p_5 xy = x^3 + x^2(p_4 t) + x(p_8 t^2 + p_2 t^3) + (p_6 t^4 + t^5)$$

$$\lambda = (p_2, p_4, p_5, p_6, p_8)$$

No. 6: $T = A_2 \oplus A_1,\ L = A_5,\ M = A_5^*$

$$y^2 + p_3 xy + p_5 ty = x^3 + x^2(p_2 t) + x(p_4 t^2 + t^3) + (p_6 t^3)$$

$$\lambda = (p_2, p_3, p_4, p_5, p_6)$$

No. 7: $T = A_1^{\oplus 3},\ L = D_4 \oplus A_1,\ M = D_4^* \oplus A_1^*$

$$y^2 = x^3 + x^2(p_6 + p_2 t) + xt(t - p_4)(t - q_4) + q_2 t^2 (t - p_4)^2$$

$$\lambda = (p_2, p_4, q_4, p_6, q_2)$$

No. 8: $T = A_4,\ L = A_4,\ M = A_4^*$

$$y^2 + p_5 xy + p_3 t^2 y = x^3 + x^2(p_4 t) + x(p_2 t^3) + t^5, \quad \lambda = (p_2, p_3, p_4, p_5)$$

No. 9: $T = D_4,\ L = D_4,\ M = D_4^*$

$$y^2 = x^3 + x(p_4 - t^2) + (q_6 + q_4 t + q_2 t^2), \quad \lambda = (q_2, q_4, p_4, q_6)$$

No. 10: $T = A_3 \oplus A_1,\ L = A_3 \oplus A_1,\ M = A_3^* \oplus A_1^*$

$$y^2 + p_3 xy = x^3 + x^2(p_2 t) + x(p_4 t^2 + t^3) + q_2 t^4, \quad \lambda = (p_2, p_3, p_4, q_2)$$

No. 11: $T=A_2^{\oplus 2}, L=A_2^{\oplus 2}, M=A_2^{*\oplus 2}$

$$y^2+p_2xy+p_3ty=x^3+x(q_2t^2)+(q_3t^3+t^4)\,, \quad \lambda=(p_2,p_3,q_2,q_3)$$

No. 15: $T=A_5, L=A_2\oplus A_1, M=A_2^*\oplus A_1^*$

$$y^2+q_2p_3xy+p_3t^2y=x^3+x^2(p_2q_2t)+x(p_2+q_2)t^3+t^5$$

$$\lambda=(p_2,p_3,q_2)$$

No. 16: $T=D_5, L=A_3, M=A_3^*$

$$y^2+p_3t^2y=x^3+x^2(p_4t)+x(p_2t^3)+t^5\,, \quad \lambda=(p_2,p_3,p_4)$$

No. 26: $T=D_6, L=A_1^{\oplus 2}, M=A_1^{*\oplus 2}$

$$y^2=x^3+x^2(p_2q_2t)-x(p_2+q_2)t^3+t^5\,, \quad \lambda=(p_2,q_2)$$

No. 27: $T=E_6, L=A_2, M=A_2^*$

$$y^2+p_3t^2y=x^3+x(p_2t^3)+t^5\,, \quad \lambda=(p_2,p_3)$$

No. 43: $T=E_7, L=A_1, M=A_1^*$

$$y^2=x^3+x(p_2t^3)+t^5\,, \quad \lambda=p_2$$

REMARK. (i) An excellent family is not unique in general (see §6 where two such families are given for No. 6). But the existence of an explicit family is sufficient for usual arithmetic applications such as (a) construction of Galois representations (or Galois extensions) with Galois group $W(L)$ or (b) construction of elliptic curves over $\boldsymbol{Q}(t)$ such that $E(\boldsymbol{Q}(t))^0\simeq L$.

(ii) Actually Theorem 1 is valid for every admissible type $\{T, L, M\}$; the remaining cases will be proven in the paper [S7] (in preparation). As for the cases No. 12, No. 17, ... of [OS] missing above and in [S7], the narrow Mordell-Weil lattice L is either a non-root lattice or $\{0\}$.

Next we describe the condition for the Mordell-Weil lattice to be non-degenerate when the parameter λ is specialized. It is known that the ramification locus of the quotient map π (see (1.2)) is defined by $\delta(\lambda)=0$ where $\delta(\lambda)$ denotes the square of the basic anti-invariant of $W(L)$ ([B] Ch. 5, §5.4, Prop. 5).

THEOREM 2 (Nondegeneracy of MWL). *Suppose $\{E_\lambda\}$ is an excellent family of elliptic curves as above. Let $\lambda\to\lambda'$ be any specialization and let k' be an algebraically closed field containing $\boldsymbol{Q}(\lambda')$. Then the MWL is nondegenerate, i.e.,*

$$E_{\lambda'}(k'(t))\simeq E_\lambda(k(t))=M\,,$$

if and only if

$$\delta(\lambda')\neq 0 \quad \text{and} \quad J(\lambda')\neq 0\,,$$

where $J(\lambda)$ is a certain invariant of $W(L)$, called the frame invariant (*cf.* 3.3(a)), *whose* *explicit form will be given later in each case.*

Both invariants $\delta(\lambda)$ and $J(\lambda)$ can be expressed as simple polynomials (such as a product of certain linear forms) in terms of the splitting variables $u=(u_1, \cdots, u_r)$ via $\lambda=\pi(u)$, the latter relation being in effect given by the explicit formula of the fundamental invariants of $W(L)$ (Thoerem 1 (ii) or the step (V) in 3.1).

COROLLARY 3 (Construction of Q-split examples). *Given an admissible type T, L, M with $rk\, L=r$, take any r-tuple $u^0=(u_i^0)\in Q^r$ satisfying $\delta(u^0)\neq 0$ and $J(u^0)\neq 0$, and let $\lambda^0=\pi(u^0)\in Q^r$. Then $E=E_{\lambda^0}$ is an elliptic curve defined over $Q(t)$ with rank r such that*

$$E(Q(t))^0\simeq L\,,\quad E(Q(t))\simeq M\,.$$

Moreover a set of explicit generators $\{Q_i\}$ of $E(Q(t))$ can be given in terms of the prescribed values u_i^0.

3. Idea of proof

3.1. Verification. The first three cases (No. 1, No. 2, No.3) have been treated in [S5], which will serve as a guide post for other cases. Given a good candidate of an excellent family $E=E_\lambda$ over $K=k(t)$ as in Theorem 1, we can verify its excellence with the help of the theory of Mordell-Weil lattices. This will be done in the following steps:

(I) The data to be achieved is the triple T, L, M, where T is a sublattice of E_8 which is a direct sum of root lattices of type A, D or E; L is a root lattice orthogonal to T and M is equal to the dual lattice L^* upto a torsion subgroup.

(II) Determine the reducible fibres of the associated elliptic surface

$$f: S\to P^1\,;$$

for this, we can freely use the results of Kodaira, Néron, Tate ([K], [N], [T]). Confirm that the trivial lattice so obtained is isomorphic to the given lattice T. In most cases, the structure of $L=T^\perp$ is unique, but in certain cases where $T^\perp$ has two possibilities, L is determined by the 2-torsion part of M (cf. [OS]).

Once the lattice structure of L, M is fixed, we have a complete information on the minimal norm, the number of minimal vectors and generators of M. In most cases, this is elementary, or we can refer to [CS], Ch. 4.

(III) Determine minimal (or short) vectors in $E(K)\simeq M$. An element $P\in E(K)$ is called a *minimal vector* if its norm $\langle P, P\rangle$ takes minimal positive value, and a *short vector* if $(PO)=0$ (cf. [S2], Lemma 10.2). The following formula for the height pairing will be constantly used:

$$\langle P, P\rangle=2+2(PO)-\sum_{v\in R} contr_v(P) \tag{3.1}$$

$$\langle P, Q\rangle=1+(PO)+(QO)-(PQ)-\sum_{v\in R} contr_v(P, Q) \tag{3.2}$$

See [S2], Th. 8.6 for the notation and the value of $contr_v(P)$ (p. 229).

(IV) Get the *fundamental algebraic equation* which describes minimal (or certain short) vectors (cf. [S5], [S6]). The relation of roots and coefficients of this equation should give (V).

(V) Explicit formula for the fundamental invariants of the Weyl group $W(L)$. This will prove that the elliptic curve E/K defines an excellent family for the case in question, hence Theorem 1 in this case.

Furthermore we give:

(VI) Generators of the Mordell-Weil group $M=E(K)$

(VII) Nondegeneracy condition of MWL (Theorem 2)

(VIII) A Q-split example, i.e. an example of elliptic curve $E/Q(t)$ such that $E(Q(t))=M$, together with explicit generators.

Thus, once a candidate of an excellent family is explicitly given, we can prove or disprove its excellence by the above method. The reader might then be interested in knowing how we find a good candidate of an excellent family for a given admissible type. This is indeed the hardest and perhaps the most interesting step, though logically it is not necessary for the proof of Theorem 1. There are two main ideas for this:

(A) Magic of weights, (B) Degeneration of Mordell-Weil lattices

3.2. Magic of weights. Given a triple T, L, M, we have to find some elliptic curve $E/K=k(t)$ for which the trivial lattice (the direct sum of root lattices T_v, $v \in R$ ranging over the reducible fibres) is equal to T, and $L=E(K)^0$ and $M=E(K)$. By assumption, L is a root lattice with the Weyl group $W(L)$. Let $wt(L)=\{w_1, \cdots, w_r\}$ be the set of weights of the fundamental invariants of $W(L)$, which will be called in short the *weight-set* of L.

We start from the known cases (**No. 1**, **No. 2**, **No. 3**). In the case **No. 1**, we have $L=E_8$ and $T=\{0\}$, and the defining equation

$$y^2=x^3+\cdots+t^5$$

is a weighted homogeneous one with total weight 30. Thus x, y, t have weights 10, 15, 6 respectively, and they determine the weights of other coefficients p_w, *etc.*, which coincide with the weight-set $wt(E_8)$ (cf. (1.4)). (N.B. This equation has been studied in some other context, i.e. as the semi-universal deformation of E_8-singularity. The method of MWL is closely related to that of Milnor lattice in the singularity theory; see [S4])

Let us consider a general Weierstrass equation

$$y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6\,, \tag{3.3}$$

where each a_d is a polynomial of degree at most d in t:

$$a_d=\sum_{i=0}^{d} a_{d,i}t^i\,.$$

Suppose that (3.3) has the same weighted homogeneity (of total degree 30) as before. Then each a_d has weight $5d$ so that the weight of $a_{d,i}$ is equal to $5d-6i$. Omitting $a_{d,i}$ with negative weight, we obtain the following table, which will be called the *weight-table of E_8 type*:

(3.4)

5	15	10	20	30
	9	4	14	24
	3		8	18
			2	12
				6
				0

Now assume that the weight-set $wt(L)$ of a given root lattice can be embedded into the above weight-table (3.4). For example, for **No. 5** and **No. 8**, we have $wt(D_5)=\{2, 4, 5, 6, 8\}$ and $wt(A_4)=\{2, 3, 4, 5\}$, both of which satisfy this assumption. Then, in the equation (3.3), we leave only those terms $a_{d,i}t^i$ with the relevant weights and t^5. In this way, we get a candidate for an excellent family of elliptic curves with Galois group $W(L)$. Then we apply the method described in 3.1. First, in the step (II), we compute the discriminant to determine the trivial lattice, which turns out to be (like magic!) the desired lattice T in many cases, and then we can complete the proof of excellence by following other steps in 3.1. In such a case, we say that *magic of weights works*. For example, this is the case for **No. 5, 8, 16, 27, 43**, as listed in Theorem 1 (see §4, §8 below). We note that this magical phenomenon has been first observed for **No. 5** in [U] other than the cases **No. 1, 2, 3, 9** treated in [S5].

Of course, "magic" does not always work. For instance, although the weight-set $wt(A_5)$ can be embedded into the weight-table (3.4), the resulting family does not give an excellent family for **No. 6**, since it has $T=A_3$, $L=D_5$ rather than the desired $T=A_2\oplus A_1$, $L=A_5$. Another drawback is that the weight-table of type E_8 (3.4) is rather restrictive in the sense that each weight occurs with multiplicity at most 1, and so it is impossible to embed a weight-set $wt(L)$ having some multiple weights.

However we can remedy this to certain extent by considering the variant for E_7 and E_6. The defining equation for **No. 2, 3**

$$y^2=x^3+xt^3+\cdots, \qquad \text{or} \quad y^2=x^3+\cdots+t^4$$

has total weight 18 or 12 (the Coxeter number), and the weight-table of type E_7 or E_6 is given as follows:

(3.5)

3	9	6	12	18
	5	2	8	14
	1		4	10
			0	6
				2

(3.6)

2	6	4	8	12
	3	1	5	9
			2	6
				3
				0

For example, the weight-set $wt(D_6)=\{2, 4, 6, 6, 8, 10\}$ (for **No. 4**) embeds into (3.5), and $wt(A_5)=\{2, 3, 4, 5, 6\}$ (for **No. 6**) embeds into both (3.5) and (3.6). The resulting families for these cases turn out to be excellent. Incidentally, note that we get 2 distinct families for **No. 6**, which shows the non-uniqueness of an excellent family of elliptic curves in general.

3.3. Degeneration of Mordell-Weil lattices. Another main idea for constructing a family with a new triple $\{T', L', M'\}$ is to consider a suitable specialization of some established family with $\{T, L, M\}$ where L' is a lattice of smaller rank than L. In general, the reducible fibres of an elliptic surface will "increase" under specialization so that the original trivial lattice T will be enlarged to a new trivial lattice T'; accordingly the Mordell-Weil lattice L (or M) will degenerate to L' (or M') of smaller rank (if not equal).

The problem here is how to specialize parameters in the original family in order to achieve a desired new triple $\{T', L', M'\}$, in particular, how to control parameters to produce a desired set of reducible fibres. The key ideas are the following:

(a) the *frame* and the *frame invariant*

(b) *vanishing roots* (analogy of vanishing cycles).

The former will be sufficient for proving Theorem 1, while the latter will play a key role in [S7].

(a) Given an excellent family $\mathscr{F}=\{E_\lambda\}$ with the data $\{T, L, M\}$, we have $T=\bigoplus_{v\in R} T_v$ for λ generic (see the notation at the beginning of §2). We call T the *frame* of the family $\mathscr{F}$.

When we specialize λ to some λ', the trivial lattice T' for the elliptic surface $f: S_{\lambda'}\to P^1$ has a similar description: $T'=\bigoplus_{v'\in R'} T'_{v'}$. The given specialization defines a map of R into R', say $\phi: R\to R'$.

DEFINITION. The frame is said to be *preserved* or *unbroken* under the specialization in question if the map $\phi: R\to R'$ is injective and if, for each $v\in R$, we have $T_v\simeq T'_{v'}$ with $v'=\phi(v)$. Otherwise the frame is said to be *broken* under the specialization in question.

There is an invariant of the Weyl group $W(L)$, say $J(\lambda)$, such that the frame T is broken under a specialization $\lambda\to\lambda'$ if and only if $J(\lambda')=0$. Such an invariant will be called a *frame invariant* of the family.

N.B. The type of the root lattice T_v determines the type of the reducible fibre $f^{-1}(v)$ if the rank of T_v is greater than 2, but this is not true for lower rank. Thus $T_v=A_1$ corresponds to the singular fibre of type I_2 or III, and $T_v=A_2$ to that of type I_3 or IV. It is natural to regard the frame to be preserved when a singular fibre of type I_2 (or I_3) is specialized to that of type III (or IV), leaving the lattice structure T_v to be unchanged. For this, see e.g. **No. 4**.

Observe that several excellent families in Theorem 1 are related to each other via specialization where the frame is broken. For instance, for **No. 8**, we have $T=A_4$, $L=A_4$, $M=A_4^*$ and

$$E_\lambda: y^2+p_5xy+p_3t^2y=x^3+x^2(p_4t)+x(p_2t^3)+t^5$$
$$\lambda=(p_2, p_3, p_4, p_5).$$

In this case, the successive specializations

$$\boxed{\text{No. 8}} \xrightarrow{p_5\to 0} \boxed{\text{No. 16}} \xrightarrow{p_4\to 0} \boxed{\text{No. 27}} \xrightarrow{p_3\to 0} \boxed{\text{No. 43}}$$

correspond to the following change of the frames:

$$T=A_4\to D_5\to E_6\to E_7.$$

Other example is given by

$$\boxed{\text{No. 26}} \xleftarrow{p_3\to 0} \boxed{\text{No. 15}} \xrightarrow{p_2\to 0} \boxed{\text{No. 27}}.$$

(b) Now we turn to a more systematic way to study the degeneration of MWL of an excellent family, by using parameters upstairs (or splitting variables) $\{u_1, \cdots, u_r\}$ rather than that of parameters downstairs λ (cf. (1.2)). To fix the idea, assume that there is a singular fibre of additive type over $t=\infty$ and let

$$sp_\infty: E(k(t))\to G_a(k)=k$$

be the specialization homomorphism which was denoted sp'_∞ in [S5].

For λ generic, it defines an isomorphism of $E(k(t))/(tor)\simeq L^*$ onto $Zu_1+\cdots+Zu_r$ where $u_i=sp_\infty(P_i)$ for suitable generators $\{P_i\}$ of $E(k(t))$ modulo torsion. We choose $\{P_i\}$ among minimal or short vectors. In particular, the "roots" in the root lattice L ($\subset L^*$) are mapped under sp_∞ to certain Z-linear combinations of $u_1, \cdots, u_r$, say $\alpha_1, \cdots, \alpha_N$, N being the number of roots in L. Let

$$\delta(\lambda)=\prod_{j=1}^{N}\alpha_j\in Z[u_1, \cdots, u_r];$$

note that this is an invariant of the Weyl group $W(L)$ which is equal to the square of the basic anti-invariant up to a constant.

(N.B. If $\{P_i\}$ are only assumed to be independent elements generating a subgroup of finite index in $E(k(t))$, then we can modify the above by replacing Z by Q so that α_j, $\delta(\lambda)$ have Q-coefficients.)

Now the proof of Theorem 2 (stated at the end of §2) reduces to the following:

LEMMA 3.1. *Suppose that the frame is unbroken under a given specialization* $\lambda\to\lambda'$, *i.e.* $J(\lambda')\neq 0$. *Then the Mordell-Weil lattice does not degenerate* ($E_\lambda(k(t))\simeq E_{\lambda'}(k'(t))$) *if and only if* $\delta(\lambda')\neq 0$.

Proof. Let T', L' denote the trivial lattice and the narrow MWL for $E'=E_{\lambda'}$.

The T for generic λ is naturally embedded into T' under a given specialization, and the MWL does not degenerate if and only if $T'=T$. First assume this. Then we have $L'=L$, so any root $P\in E(K)^0=L$ specializes to a root $P'\in E'(K)^0=L'$. Hence the section (P') is disjoint from the zero section (O), and in particular, we have $sp_\infty(P')\neq 0$, which implies $\delta(\lambda')\neq 0$.

Conversely, assume $T'\neq T$; by assumption, there is a new reducible fibre $f^{-1}(v)$, $v\in R'-\phi(R)$. Take an irreducible component, say Θ, other than the identity component. It has the self-intersection number -2, and is orthogonal to T as well as (O) and any fibre in $NS(S_{\lambda'})$. Thus it defines a root of L, which vanishes under the specialization in question, which implies $\delta(\lambda')=0$. q.e.d.

We can reformulate the above result by introducing the following terminology. It is obvious that any specialization upstairs

$$u=(u_1, \cdots, u_r)\to u'=(u'_1, \cdots, u'_r)$$

uniquely determines a specialization downstairs $\lambda\to\lambda'$. A root α_i of L will be called a *vanishing root* if it vanishes under the said specialization. In spirit, this is very close to the idea of *vanishing cycles* in the deformation of singularities (cf. [S4]).

COROLLARY 3.2 (Principle of vanishing roots). *Under any specialization $u\to u'$ with a vanishing root, the Mordell-Weil lattice $M=E_\lambda(k(t))$ degenerates.*

The mode of degeneration of MWL depends upon the behavior of vanishing roots. For instance, if there is only one vanishing root (upto sign) under a frame-preserving specialization, then the rank of MWL decreases by one.

This principle will be used in [S7] to construct a new excellent family from known ones. Actually it can be used in a more general situation where the narrow MWL may be no longer a root lattice: indeed this method is applicable for the existence proof of all the 74 types of T, L, M in [OS], which will be treated elsewhere (cf. [S8]).

4. Review for No. 1, No. 2, No. 3 ($L=E_8, E_7, E_6$)

Here let us briefly review the results for No. 1, 2, 3 from [S5] for later reference. (For instance, (No. 1, V) will refer to No. 1, (V).) Unless otherwise mentioned, λ is assumed to be generic, i.e. its components are algebraically independent over $\boldsymbol{Q}$.

4.1. No. 1 ($L=E_8$).

(I) **No. 1**: $T=\{0\}$, $L=E_8$, $M=E_8$ (cf. [S5], §4, §8)

$$y^2=x^3+x\left(\sum_{i=0}^{3} p_{20-6i}t^i\right)+\left(\sum_{j=0}^{3} p_{30-6j}t^j+t^5\right)$$

$$\lambda=(p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30})\,.$$

(II) For any λ, the elliptic surface $f: S_\lambda\to \boldsymbol{P}^1$ has an irreducible singular fibre of type II (a rational curve with a cusp) at $t=\infty$. For general λ, it has no reducible fibres at all, since the discriminant $\Delta(t)$ is a polynomial of degree 10 in t which has no multiple factors in general. In particular, for λ generic over $\boldsymbol{Q}$, the trivial lattice is $\{0\}$ so that $L=M=E_8$, as required.

(III) The root lattice E_8 has minimal norm 2 and the number of minimal vectors (= roots in this case) is 240. Accordingly there are 240 rational points $P\in E(k(t))$ with $\langle P, P\rangle=2$, or equivalently (by (3.1)), with $(PO)=0$, and they are of the form (cf. [S5], §10, Lemma 10.9)

$$P=(x, y)\,, \quad x=gt^2+at+b, y=ht^3+ct^2+dt+e\,.$$

(IV) By the specialization map

$$sp_\infty: E(k(t))\to f^{-1}(\infty)^\#=\boldsymbol{G}_a$$

(# denotes the smooth part), the above P is mapped to

$$u:=sp_\infty(P)=g/h \quad ([S5], \text{Lemma } 8.2)\,.$$

Substitute x, y into the defining equation to get the relations among $g, h, a, \cdots, e$. Then an explicit elimination leads to an algebraic equation of degree 240 in u:

$$u^{240}+60p_2u^{238}+\cdots=0\,;$$

the left hand side $\Phi(u, \lambda)$ is a monic polynomial in u with coefficients in $\boldsymbol{Z}[\lambda]=\boldsymbol{Z}[p_2, \cdots, p_{30}]$, which was called the universal polynomial of type E_8 ([S5], Th. 8.3.). The above equation (and the similar ones in the sequel) will be called the *fundamental algebraic equation* for the case under consideration. This terminology should be justified by the next step.

(V) Let $\{P_1, \cdots, P_8\}$ be a basis of $E(k(t))\simeq E_8$ in the sense of root systems, and let $u_i=sp_\infty(P_i)$. Then the 240 roots of the fundamental algebraic equation are given by $u_1, \cdots, u_8$ and certain integral linear combinations of them. Then the relation of roots and coefficients for u^w, w ranging over the weights of $W(E_8)$ in (1.4), gives an explicit expression of p_w as a fundamental invariant of weight w for $W(E_8)$ (see [S5], Th. ($\dot{E}_8$), p. 681; note that the notation p_i or q_j there correspond to p_{20-6i} or p_{30-6j} here). It follows that $\{E_\lambda\}$ forms an excellent family for No. 1 (see [S5], Th. 8.3, 8.4).

(VI) Generators. The above $\{P_1, \cdots, P_8\}$ forms a set of generators of the Mordell-Weil group $E(k(t))=E(\boldsymbol{Q}(u_1, \cdots, u_8)(t))$. Each P_i is of the form

$$P_i=(t^2/u_i^2+a_it+b_i, t^3/u_i^3+\cdots+e_i)$$

where $a_i, \cdots, e_i$ are rational functions in $u_1, \cdots, u_8$ ([S5], Th. 8.5).

(VII) Nondegeneracy condition. The MWL of a specialized elliptic curve $E_{\lambda'}$ is nondegenerate ($\simeq E_8$) if and only if $\delta(\lambda')\neq 0$, where $\delta(\lambda)=\Phi(0, \lambda)$. (Since $T=0$, the frame invariant can be taken as the constant 1.)

(VIII) For an example of $E(\boldsymbol{Q}(t))\simeq E_8$, together with explicit generators, see [S3] or [S5], p. 685.

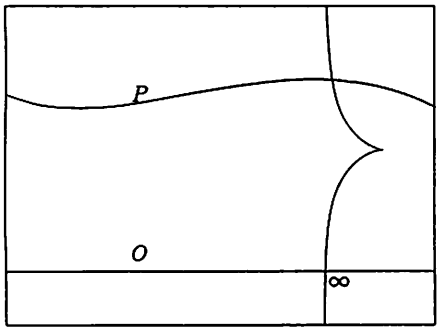

Figure No. 1

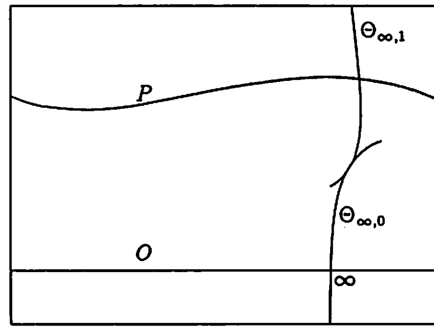

Figure No. 2

4.2. No. 2 ($L=E_7$).

(I) **No. 2:** $T=A_1, L=E_7, M=E_7^*$ (cf. [S5], §9).

$$y^2=x^3+x(p_{12}+p_8t+t^3)+\left(\sum_{i=0}^{4} p_{18-4i}t^i\right)$$

$$\lambda=(p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}) .$$

(II) For any λ, there is a reducible singular fibre of type III at $t=\infty$, consisting of two smooth rational curves tangent at a single point; thus $T_\infty=A_1$. For general λ, there are no other reducible fibres so that the trivial lattice T is A_1, and we have $L=E_7$, $M=E_7^*$, as required.

(III) The lattice E_7^* has minimal norm 3/2 and the number of minimal vectors is 56. Accordingly there are 56 rational points $P\in E(k(t))$ with $\langle P, P\rangle=3/2$; by (3.1), this is the case if and only if the section (P) is disjoint from the zero-section (O) and intersects the non-identity component $\Theta_{\infty,1}$ of $f^{-1}(\infty)$. Such a P is given by

$$P=(x, y), \qquad x=at+b, \quad y=ct^2+dt+e$$

([S5], Lemma 9.1).

(IV) In this case, we use the specialization map sp'_∞ which is the map:

$$sp_\infty: E(k(t))\to f^{-1}(\infty)^\# = G_a\times Z/2Z,$$

composed with the projection to the G_a-component. Then we have for the above P

$$sp'_\infty(P)=-c \qquad \text{([S5], Lemma 9.2)}.$$

By the elimination method as in (No. 1, VI), we obtain the fundamental algebraic equation for No. 2:

$$\Phi(c, \lambda)=c^{56}-36p_2c^{54}+\cdots=0 .$$

(V) Noting that E_7^* is generated by minimal vectors (as easily checked), we can choose a basis $\{P_1, \cdots, P_7\}$ of $E(k(t))\simeq E_7^*$ among minimal vectors. Let

$u_i = -sp'_\infty(P_i)$. As in (No. 1, V), the relation of roots and coefficients for the above equation gives the explicit formula of p_w as the fundamental invariants of $W(E_7)$ (see [S5], Th. (E_7), p. 680; the p_i or q_j there correspond to p_{14-4i} or p_{18-4j} here). We conclude that our family defines an excellent family for No. 2 ([S5], Th. 9.3, 9.4).

(VI) The above $\{P_1, \cdots, P_7\}$ generate the MWL $E(k(t))$. We have

$$P_i = (a_i t + b_i,\ u_i t^2 + d_i t + e_i) \qquad (i = 1, \cdots, 7)$$

where a_i, b_i, d_i, e_i are rational functions of $u_1, \cdots, u_7$ ([S5], Th. 9.5).

(VII) The MWL of a specialized elliptic curve $E_{\lambda'}$ is nondegenerate ($\simeq E_7^*$) if and only if $\delta(\lambda') \neq 0$, where $\delta(\lambda)$ is the product of all the 126 roots α_v of E_7, expressed as integral linear combinations of $u_1, \cdots, u_7$ (it was denoted by δ_0 in [S5].) The frame invariant can be taken as the constant 1, since the frame is unbroken ($T_\infty = A_1$) for any λ'.

(VIII) An example of $E(\mathbf{Q}(t)) \simeq E_7^*$, together with explicit generators, can be found in [S5], p. 684.

4.3. No. 3 ($L = E_6$).

(I) **No. 3:** $T = A_2$, $L = E_6$, $M = E_6^*$ (cf. [S5], §10)

$$y^2 = x^3 + x\left(\sum_{i=0}^{2} p_{8-3i} t^i\right) + \left(\sum_{i=0}^{2} p_{12-3i} t^i + t^4\right)$$

$$\lambda = (p_2, p_5, p_6, p_8, p_9, p_{12}).$$

(II) For any λ, there is a reducible singular fibre of type IV at $t = \infty$, consisting of 3 smooth rational curves transversally meeting at a single point; thus $T_\infty = A_2$. For general λ, there are no other reducible fibres so that the trivial lattice T is A_2, and we have $L = E_6$, $M = E_6^*$.

(III) The lattice E_6^* has 54 minimal vectors of minimal norm 4/3, which divides into two orbits under the Weyl group $W(E_6)$. Accordingly there are 54 rational points $P \in E(k(t))$ with $\langle P, P \rangle = 4/3$. By (3.1), they are characterized by the condition that $(PO) = 0$ and that the section (P) intersects one of the two non-identity components $\Theta_{\infty,i}$ $(i = 1, 2)$ of $f^{-1}(\infty)$; the two $W(E_6)$-orbits correspond to $i = 1$ or 2. Thus there are 27 P's such that

$$P = (x, y), \qquad x = at + b, \quad y = t^2 + dt + e$$

([S5], Lemma 10.1).

(IV) As in (No. 2, IV), we define the specialization map sp'_∞ as the composite of the map:

$$sp_\infty : E(k(t)) \to f^{-1}(\infty)^\# = \mathbf{G}_a \times \mathbf{Z}/3\mathbf{Z},$$

and the projection to the $\mathbf{G}_a$-component. Then we have for the above P

$$sp'_\infty(P) = -\frac{a}{2} \qquad ([S5], \text{ Lemma } 10.2).$$

By the elimination method, we obtain the fundamental algebraic equation for No. 3:

$$\Phi(a, \lambda)=a^{27}+12p_2a^{25}+\cdots=0 .$$

(V) In this case, we can choose a basis $\{P_1, \cdots, P_6\}$ of $E(k(t))\simeq E_6^*$ among 27 minimal vectors of the above form. Let $u_i=-2sp'_\infty(P_i)$. Again the relation of roots and coefficients for the above equation gives the explicit formula of p_w as the fundamental invariants of $W(E_6)$ (see [S5], Th. (E_6), p. 679; the p_i or q_j there correspond to p_{8-3i} or p_{12-3j} here). It follows that our family defines an excellent family for No. 3 ([S5], Th. 10.3, 10.4).

(VI) The above $\{P_1, \cdots, P_6\}$ are generators of the MWL $E(k(t))$ such that

$$P_i=(u_it+b_i, t^2+d_it+e_i) \qquad (i=1, \cdots, 6)$$

where b_i, d_i, e_i are rational functions of $u_1, \cdots, u_6$ ([S5], Th. 10.5).

(VII) A specialized elliptic curve $E_{\lambda'}$ has a nondegenerate MWL ($\simeq E_6^*$) if and only if $\delta(\lambda')\neq 0$, where $\delta(\lambda)$ is the product of all the 72 roots α_v of E_6, expressed as integral linear combinations of $u_1, \cdots, u_6$ (it was denoted by δ_0 in [S5].) The frame invariant can be taken as the constant 1, since the frame is unbroken ($T_\infty=A_2$) for any λ'.

(VIII) Examples of $E(\mathbf{Q}(t))\simeq E_6^*$, together with explicit generators, can be found in [S5], p. 683 or [S6], p. 487.

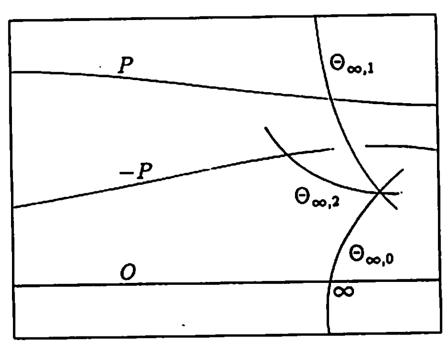

Figure No. 3

5. No. 4 and No. 5 ($L=D_6, D_5$)

5.1. No. 4 ($L=D_6$).

(I) No. 4: $T=A_1^{\oplus 2}, L=D_6, M=D_6^*$

$$E_\lambda: y^2=x^3+x^2r_6+x(p_8t+p_4t^2+t^3)+(p_{10}t^2+p_6t^3+p_2t^4)$$

$$\lambda=(p_2, p_4, p_6, r_6, p_8, p_{10})$$

(II) The discriminant is computed as follows:

$$\Delta = 16r_6^2(p_8^2 - 4p_{10}r_6)t^2 + (-64p_8^3 + r_6(\cdots))t^3 + \cdots - 64t^9 .$$

Let us see that the singular fibres over $t=0$ and $t=\infty$ are reducible with two irreducible components:

$$f^{-1}(v) = \Theta_{v,0} + \Theta_{v,1} \qquad (v = 0, \infty)$$

under the condition

$$J(\lambda) := p_8^2 - 4p_{10}r_6 \neq 0 .$$

Indeed, the singular fibre at $t=\infty$ is of type III for any λ, just as in No. 2. On the other hand, the singular fibre at $t=0$ is of type either I_2 or III according to whether $r_6 \neq 0$ or $=0$ under the said condition. For, if $r_6 \neq 0$, then $\mathrm{ord}_{t=0}(\Delta)=2$ and the above Weierstrass equation reduces to a nodal cubic $y^2 = x^3 + x^2 r_6$ (with the node at (0, 0)) as $t \to 0$. By the wellknown algorithm ([N], [T]), this shows that we have a singular fibre of type I_2 (two smooth rational curves meeting transversally at 2 points). Next, if $r_6 = 0$ (but $J \neq 0$), then $\mathrm{ord}_{t=0}(\Delta) = 3$ and the defining equation reduces to the cuspidal cubic $y^2 = x^3$ (with the cusp at (0, 0)) as $t \to 0$. This shows that the singular fibre is of type III.

For λ generic, there are no other reducible fibres (this is easily checked by looking at some special case; cf. example in (VIII)) and so the trivial lattice is $T = A_1^{\oplus 2}$, which implies that $L = D_6$, $M = D_6^*$ ([S2], Th. 10.4, or [OS]). Note that the above J is an example of what we called the frame invariant in §3, 3.3(a), with the property that the frame is unbroken precisely when $J \neq 0$.

(III) First recall basic facts on the root lattice D_6 and its dual lattice D_6^* (cf. [CS]). A standard realization of D_6 is the sublattice of $\boldsymbol{Z}^6 \subset \boldsymbol{R}^6$ consisting of those $(x_1, \cdots, x_6)$ with $x_1 + \cdots + x_6 \in 2\boldsymbol{Z}$. Then we have $D_6 \subset \boldsymbol{Z}^6 \subset D_6^*$ with each index 2. The minimal norm of D_6^* is 1 and the minimal vectors are the unit vectors $(1, 0, \cdots, 0)$, etc. up to sign (12 in number). There are $2^6 = 64$ vectors of norm 3/2 such as $(1/2, \cdots, 1/2)$, any one of which generates $D_6^*/\boldsymbol{Z}^6$.

The Weyl group $W(D_6)$ is genrated by the permutations of coordinates x_i and the sign change at 2 coordinates x_i, x_j. It acts transitively on the set of 12 minimal vectors as well as the set of 60 roots of D_6 (such as $(\pm 1, \pm 1, 0, \cdots, 0)$), but the set of vectors of norms 3/2 in D_6^* are divided into 2 orbits under $W(D_6)$. The weights of the fundamental invariants are $\{2, 4, 6, 6, 8, 10\}$, as reflected in the chosen parameter $\lambda = (p_2, p_4, p_6, r_6, p_8, p_{10})$.

By the formula of the height pairing (3.1), $P \in M = E(K)$ has norm

$$\langle P, P \rangle = 2 + 2(PO) - \begin{cases} \frac{1}{2} & (P\Theta_{0,1}) = 1 \\ 0 & \text{otherwise} \end{cases} - \begin{cases} \frac{1}{2} & (P\Theta_{\infty,1}) = 1 \\ 0 & \text{otherwise} . \end{cases}$$

Here $(P\Theta_{v,1}) = 1$ means that the section (P) passes through the non-identity component

$\Theta_{v,1}$ of $f^{-1}(v)$. Hence we have $\langle P, P\rangle=1$ if and only if $(PO)=0$ and (P) passes through the non-identity components at $t=0$ and $t=\infty$. In terms of the Weierstrass equation, this means that $P=(x, y)$ as in (No. 2, III) should satisfy $b=e=0$ to meet the node or cusp $(0, 0)$ at $t=0$, i.e.

$$P=(at, ct^2+dt)\,.$$

(IV) Using the same notation as in (No. 2, IV), we have $sp'_\infty(P)=-c$ for such a P. Now the above P satisfies the defining equation if and only if

$$a=c^2-p_2\,,\quad 2cd=a^3+ap_4+p_6\,,\quad d^2=p_{10}+ap_8+a^2r_6\,.$$

Eliminating a, d, we obtain the fundamental algebraic equation for No. 4:

$$\begin{aligned}\Phi(c,\lambda)=c^{12}&-6p_2c^{10}+c^8(15p_2^2+2p_4)\\&+c^6(2p_6-4r_6-20p_2^3-8p_2p_4)\\&+c^4(-4p_8+15p_2^4+12p_2^2p_4+p_4^2-6p_2p_6+8p_2r_6)\\&+c^2(-4p_{10}-6p_2^5-8p_2^3p_4-2p_2p_4^2+6p_2^2p_6+2p_4p_6+4p_2p_8-4p_2^2r_6)\\&+(p_2^3+p_2p_4-p_6)^2=0\,.\end{aligned}$$

(V) The 12 roots of this equation can be written as $\pm u_i$ $(i=1,\cdots,6)$. Writing down the relation of roots and coefficients, we obtain the following formulas:

$$\begin{cases}p_2 &= \varepsilon'_2/6\\ p_4 &= -(15p_2^2-\varepsilon'_4)/2\\ p_6 &= p_2^3+p_2p_4-\varepsilon_6\\ r_6 &= (-20p_2^3-8p_2p_4+2p_6+\varepsilon'_6)/4\\ p_8 &= (15p_2^4+12p_2^2p_4+p_4^2-6p_2p_6+8p_2r_6-\varepsilon'_8)/4\\ p_{10} &= (-6p_2^5-8p_2^3p_4-2p_2p_4^2+6p_2^2p_6+2p_4p_6+4p_2p_8-4p_2^2r_6+\varepsilon'_{10})/4\,.\end{cases}$$

Here ε'_{2d} denotes the d-th elementary symmetric function of $u_1^2,\cdots,u_6^2$ and $\varepsilon_6=u_1\cdots u_6$, which are obviously invariants of $W(D_6)$.

This shows that if $p_2, p_4, p_6, r_6, p_8, p_{10}$ are algebraically independent over $\boldsymbol{Q}$, so are $\varepsilon'_2,\cdots,\varepsilon'_{10}$ and ε_6, and hence both of them give a set of the fundamental invariants of $W(D_6)$, since they have the right weights. It follows that $u_1,\cdots,u_6$ are also algebraically independent over $\boldsymbol{Q}$, and that the above formulas express p_i and r_6 explicitly as the fundamental invariants of $W(D_6)$. Passing to the quotient fields, this implies that the extension $\boldsymbol{Q}(u_1,\cdots,u_6)/\boldsymbol{Q}(p_2,\cdots,p_{10})$ is a Galois extension with Galois group $W(D_6)$, and therefore the Galois representation ρ_λ has the image $W(D_6)$ with the splitting field $\mathscr{K}_\lambda=\boldsymbol{Q}(u_1,\cdots,u_6)$. Thus we have proven the excellency of our family.

(VI) The 6 rational points

$$Q_i=((u_i^2-p_2)t, u_it^2+d_it)\qquad(i=1,\cdots,6)$$

where d_i is rationally determined by u_i over $\boldsymbol{Q}(\lambda)$ as in (IV), generate the sublattice of $E(K)\simeq D_6^*$ of index 2 corresponding to Z^6. The full MWL is generated by Q_i and one more rational point of the form

$$Q_0=(at+b, ct^2+dt+e) \qquad \text{with} \quad c=\frac{1}{2}(u_1+\cdots+u_6).$$

(VII) The MWL is nondegenerate if and only if $\delta\neq 0$ and $J\neq 0$. In terms of the splitting variables $u_1, \cdots, u_6$, we have

$$\delta=\prod_{i<j}(\pm u_i \pm u_j)$$

and

$$J=\pm\prod\frac{1}{2}(u_1+\cdots+u_5-u_6)$$

where the latter product is taken over 16 linear forms obtained from the written one under $W(D_6)/\{\pm 1\}$.

(VIII) *Example* 1. Let $u_i=i-1$ for $i=1, \cdots, 6$, i.e.

$$\{u_i\}=\{0, 1, 2, 3, 4, 5\}.$$

Determine $p_2, \cdots, p_{10}$ by the formulas in (V), which define the elliptic curve $E/\boldsymbol{Q}(t)$:

$$y^2=x^3+\frac{2475}{32}x^2+\left(\frac{152625}{256}t-\frac{2849}{24}t^2+t^3\right)x+\frac{1313825}{512}t^2$$
$$-\frac{137335}{432}t^3+\frac{55}{6}t^4$$

Then we have $T=A_1^{\oplus 2}$, $L=D_6$, $M=D_6^*$ for this example by checking the above non-degeneracy condition (or by directly checking that $\Delta(t)$ has order 2 zero at $t=0$, and 7 simple zeroes at $t\neq 0, \infty$).

The generators of $E(\boldsymbol{Q}(t))\simeq D_6^*$ are given as follows. First the 6 points

$$Q_1=\left(\frac{-55}{6}t, -60t\right), \quad Q_2=\left(\frac{-49}{6}t, t^2+\frac{855}{16}t\right),$$
$$Q_3=\left(\frac{-31}{6}t, 2t^2+\frac{315}{8}t\right), \quad Q_4=\left(\frac{-1}{6}t, 3t^2-\frac{795}{16}t\right),$$
$$Q_5=\left(\frac{41}{6}t, 4t^2-\frac{405}{4}t\right), \quad Q_6=\left(\frac{95}{6}t, 5t^2+\frac{2835}{16}t\right).$$

generate the index 2 subgroup ($\simeq Z^6$). Observe that t^2-coefficient of the y-coordinate of Q_i is 0, 1, 2, 3, 4, 5 as prescribed. Next, corresponding to $c=\frac{1}{2}(u_1+\cdots+u_6)=15/2$, we find the following point:

$$Q_0=\left(\frac{565}{12}t+\frac{675675}{64},\frac{15}{2}t^2+\frac{116295}{16}t+\frac{557431875}{512}\right).$$

These points are related by $2Q_0=Q_1+\cdots+Q_6$, and the full Mordell-Weil group $E(\mathbf{Q}(t))$ is generated by $\{Q_0, Q_1, \cdots, Q_5\}$.

Example 2. If one prefers some examples of the same type but without denominators (both in the defining equation and in the coordinates of generators), here is an example. Take

$$\{u_i\}=\{0, 2, 4, 6, 8, 18\}.$$

We leave it as an exercise to verify the above statement.

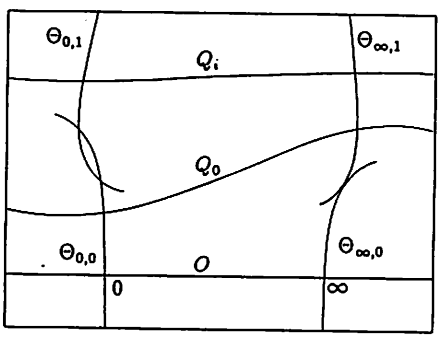

Figure No. 4

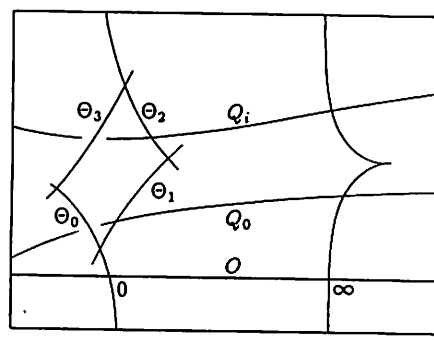

Figure No. 5

5.2. No. 5 ($L=D_5$).

(I) **No. 5:** $T=A_3, L=D_5, M=D_5^*$ (cf. [U] for details)

$$y^2+p_5xy=x^3+x^2(p_4t)+x(p_8t^2+p_2t^3)+(p_6t^4+t^5)$$

$$\lambda=(p_2, p_4, p_5, p_6, p_8).$$

(II) The discriminant is given as follows:

$$\Delta=p_5^4(p_8^2-p_5^2p_6)t^4+\cdots-432t^{10}.$$

It follows that $\operatorname{ord}_{t=0}(\Delta)=4$ if $p_5\neq 0$ and $J_1:=p_8^2-p_5^2p_6\neq 0$, in which case the singular fibre at $t=0$ is of type I_4:

$$f^{-1}(0)=\Theta_0+\Theta_1+\Theta_2+\Theta_3,$$

where 4 smooth rational curves Θ_i meet in a cyclic way like "#" (Θ_0 is the identity component).

At $t=\infty$, there is a singular fibre of type II, as in (No. 1, II). For general λ, there are no other reducible fibres, and hence the trivial lattice is $T=A_3$, and we have $L=D_5$, $M=D_5^*$ by [OS].

(III) The lattice D_5^* has 10 minimal vectors of minimal norm 1 and 32 vectors with norm 5/4. In view of (3.1), for $P \in M = E(K)$, $\langle P, P\rangle = 1$ holds if and only $(PO) = 0$ and (P) meets Θ_2. The minimal vectors P have the following form:

$$P = \left(\frac{t^2}{u^2}, \frac{t^3}{u^3} + ct^2\right), \qquad c = \frac{u^5 + p_2 u^3 + p_4 u - p_5}{2u^2},$$

where $u = sp_\infty(P)$ should satisfy the equation in (IV).

(IV) The fundamental algebraic equation is a polynomial of degree 10 in u:

$$\Phi(u, \lambda) = u^{10} + (2p_2)u^8 + (2p_4 + p_2^2)u^6 + (-4p_6 + 2p_2 p_4)u^4 + (-4p_8 + p_4^2)u^2 - p_5^2 = 0\,.$$

(V) Let $\pm u_i$ $(i = 1, \cdots, 5)$ be the roots of this equation. By the relation of roots and coefficients, we have:

$$\begin{cases} p_2 = -\varepsilon'_2/2 \\ p_4 = (\varepsilon'_4 - p_2^2)/2 \\ p_5 = u_1 \cdots u_5 \\ p_6 = (\varepsilon'_6 + 2p_2 p_4)/4 \\ p_8 = (p_4^2 - \varepsilon'_8)/4\,. \end{cases}$$

Here ε'_{2d} denotes the d-th elementary symmetric function of $u_1^2, \cdots, u_5^2$. This gives an explicit formula of the fundamental invariants of $W(D_5)$, and we deduce that the family in question is an excellent family for No. 5 by the same argument as (No. 4, V).

(VI) The 5 rational points

$$Q_i = \left(\frac{t^2}{u_i^2}, \frac{t^3}{u_i^3} + c_i t^2\right) \qquad (i = 1, \cdots, 5)$$

(c_i is determined as in (III)) generate a sublattice of $E(K) \simeq D_5^*$ of index 2. The full MWL is generated by Q_i and one more rational point of the form

$$Q_0 = \left(\frac{t^2}{u_0^2} + at, \frac{t^3}{u_0^3} + ct^2 + dt\right) \qquad \text{with} \quad u_0 = \frac{1}{2}(u_1 + \cdots + u_5)\,.$$

(VII) By (II), the frame invariant is given by $J(\lambda) = p_5 J_1$. Both p_5 and J_1 are expressed as the product of linear forms in u_i which correspond to the short vectors of norm 1 or 5/4 in D_5^*, taken modulo $\{\pm 1\}$. Thus

$$J_1 = p_8^2 - p_5^2 p_6 = \pm \prod \frac{1}{2}(u_1 + \cdots + u_5)\,,$$

where the product runs over 16 transforms of $\frac{1}{2}(u_1 + \cdots + u_5)$ under $W(D_5)$ modulo $\{\pm 1\}$.

(VIII) For an explicit example, see [U].

6. Two families for No. 6 ($L=A_5$)

6.1. No. 6 (family of type E_7).

(I) **No. 6:** $T=A_2\oplus A_1$, $L=A_5$, $M=A_5^*$

$$y^2+p_3xy+p_5ty=x^3+x^2(p_2t)+x(p_4t^2+t^3)+(p_6t^3)$$

$$\lambda=(p_2, p_3, p_4, p_5, p_6)$$

Observe that the weight-set of A_5 is {2, 3, ,4, 5, 6} and that this equation is obtained by magic of weights from the weight-table of type E_7 (cf. §3, 3.2).

(II) The discriminant is as follows:

$$\Delta=-p_3^3(-p_3^2p_4p_5+p_2p_3p_5^2-p_5^3+p_3^3p_6)t^3+(-27p_5^4+p_3(\cdots))t^4+\cdots-64t^9\,.$$

The singular fibre at $t=\infty$ is of type III for any λ, as in No. 2. The singular fibre at $t=0$ is reducible with 3 irreducible components:

$$f^{-1}(0)=\Theta_{0,0}+\Theta_{0,1}+\Theta_{0,2}$$

under the condition

$$J(\lambda):=-p_3^2p_4p_5+p_2p_3p_5^2-p_5^3+p_3^3p_6\neq 0\,.$$

Indded, it is of type either I_3 or IV according to whether $p_3\neq 0$ or $=0$ under the said condition. For, if $p_3\neq 0$, then $\mathrm{ord}_{t=0}(\Delta)=3$ and the defining equation reduces to a nodal cubic $y^2+p_3xy=x^3$ (with the node at $(0, 0)$) as $t\to 0$, which implies ([N], [T]) that the singular fibre is of type I_3 (3 smooth rational curves forming a triangle). Next, if $p_3=0$ (but $J\neq 0$), then $\mathrm{ord}_{t=0}(\Delta)=4$ and the defining equation reduces to the cuspidal cubic $y^2=x^3$ as $t\to 0$. This shows that the singular fibre is of type IV.

For λ generic, there are no other reducible fibres and so the trivial lattice is $T=A_2\oplus A_1$, which implies that $L=A_5$, $M=A_5^*$ ([OS]).

(III) The lattice A_5^* has 12 minimal vectors of minimal norm 5/6 which are divided into 2 orbits under the Weyl group $W(A_5)=\mathcal{S}_6$. The sum of 6 vectors in each orbit is zero, and any 5 vectors among them give a set of free generators of A_5^*.

By (3.1), $P\in M=E(K)$ satisfies $\langle P, P\rangle=5/6$ if and only $(PO)=0$ and (P) meets the non-identity components at $t=0$ and at $t=\infty$; the two components $\Theta_{0,i}$ $(i=1, 2)$ correspond to the 2 orbits. The minimal vectors P have the following form:

$$P=(at, ct^2+dt)\,.$$

By the defining equation, a, c, d should satisfy

$$a=c^2\,,\quad d=0\quad\text{or}\quad d=-ap_3-p_5\,,\quad -a^3+2cd-a^2p_2+acp_3-ap_4+cp_5-p_6=0\,.$$

(IV) Substituting $a=c^2$, $d=0$ into the third relation, we obtain the fundamental algebriac equation:

$$\Phi(c, \lambda)=c^6+c^4p_2-c^3p_3+c^2p_4-cp_5+p_6=0\ ,$$

which is nothing but the generic algebraic equation of degree 6 (cf. (1.6)).

(V) Let $c=u_i$ $(i=1, \cdots, 6)$ be the 6 roots of this equation. By the relation of roots and coefficients (in the standard sense), p_d is equal to the d-th elementary symmetric function of u_i $(i=1, \cdots, 6)$ for $d=2, \cdots, 6$ and $u_1+\cdots+u_6=0$. In other words, we have:

$$p_2=\sum_{i<j} u_i u_j\ , \quad \cdots, \quad p_6=\prod_i u_i\ .$$

This represents an explicit formula of the fundamental invariants of $W(A_5)=\mathscr{S}_6$, and we conclude that the family in question is an excellent family for No. 6.

(VI) The 6 rational points

$$Q_i=(u_i^2 t, u_i t^2) \qquad (i=1, \cdots, 6)$$

form a $W(A_5)$-orbit of minimal vectors, and they generate $E(K)\simeq A_5^*$. Note that

$$Q_1+\cdots+Q_6=0 \qquad \text{(in } E(K)\text{, of course)}$$

corresponding to $u_1+\cdots+u_6=0$, since the specialization homomorphism $sp'_\infty: E(K)\to\sum_i \mathbf{Z}u_i$ is an injective map for λ generic (cf. [S5], Th. 9.5).

(VII) The nondegeneracy condition of MWL is stated in Theorem 2, where the frame invariant is now given by $J(\lambda)$ in (II). In terms of u_i, it is expressed as a product of 15 linear forms (corresponding to short vectors of norm 4/3 in a $W(A_5)$-orbit):

$$J=\prod_{i<j}(u_i+u_j)\ ,$$

while $\delta(\lambda)$ is equal to the (ordinary) discriminant of the equation $\Phi(c, \lambda)=0$, i.e.

$$\delta=\left\{\prod_{i<j}(u_i-u_j)\right\}^2 .$$

Therefore the MWL is nondegenerate if and only if $u_i\neq\pm u_j$ for any $i<j$.

(VIII) *Example.* Take $u_i=i-1$ $(i=1, \cdots, 5)$ and $u_6=-10$, which satisfies the above nondegeneracy condition. Then

$$p_2=-65\ , \quad p_3=-300\ , \quad p_4=-476\ , \quad p_5=-240\ , \quad p_6=0\ ,$$

so the elliptic curve $E/\mathbf{Q}(t)$

$$y^2-300xy-240ty=x^3-65tx^2+(-476t^2+t^3)x$$

gives a $\mathbf{Q}$-split example for No. 6. Free generators of $E(\mathbf{Q}(t))\simeq A_5^*$ are given by any 5 points among

$$\{Q_i\}=\{(0, 0), (t, t^2), (4t, 2t^2), (9t, 3t^2), (16t, 4t^2), (100t, -10t^2)\}\ .$$

The 30 minimal vectors (roots) in the narrow MWL $E(Q(t))^0 \simeq A_5$ are given by $\{Q_i - Q_j\ (i<j)\}$.

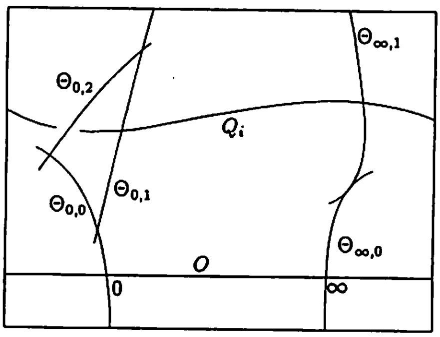

Figure No. 6

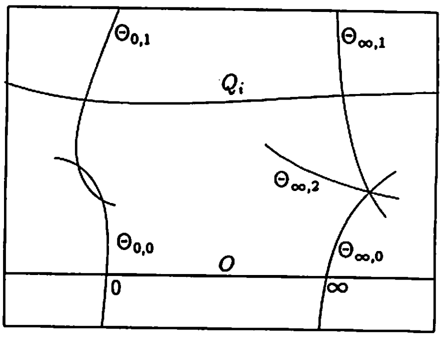

Figure No. 6′

6.2. No. 6 (family of type E_6).

(I) **No. 6′:** $T = A_2 \oplus A_1,\ L = A_5,\ M = A_5^*$

$$y^2 = x^3 + p_4 x^2 + (p_5 t + p_2 t^2)x + (p_6 t^2 + p_3 t^3 + t^4)$$

$$\lambda = (p_2, p_3, p_4, p_5, p_6)\,.$$

This equation is obtained by magic of weights from the weight-table of type E_6 (cf. §3, 3.2).

(II) The discriminant is as follows:

$$\Delta = -16p_4^2(-p_5^2 + 4p_4 p_6)t^2 - 32(2p_5^3 + p_4(\cdots))t^3 + \cdots + -432t^8\,.$$

The singular fibre at $t=\infty$ is of type IV for any λ, as in No. 3. The singular fibres at $t=0$ is either I_2 or III (according to whether $p_4 \neq 0$ or $=0$) under the condition

$$J(\lambda) := -p_5^2 + 4p_4 p_6 \neq 0\,.$$

This can be checked in the same way as before, so it will be omitted. For λ generic, we have again $T = A_2 \oplus A_1$ so that $L = A_5,\ M = A_5^*$.

(III) By the same argument as above, the 12 minimal vectors $P = (x, y) \in E(K) \simeq A_5^*$ have the x-coordinate of the form $x = at$. Then the right hand side of the defining equation in (I) becomes

$$t^2\{t^2 + (a^3 + ap_2 + p_3)t + (a^2 p_4 + ap_5 + p_6)\}\,,$$

which should be a square in $k[t]$.

(IV) Thus we obtain the fundamental algebraic equation:

$$\begin{aligned}\Phi(a, \lambda) &= (a^3+ap_2+p_3)^2-4(a^2p_4+ap_5+p_6)\\ &= a^6+2a^4p_2+2a^3p_3+a^2(p_2^2-4p_4)+a(2p_2p_3-4p_5)+p_3^2-4p_6\\ &= 0\,.\end{aligned}$$

(V) Let $a=u_i$ $(i=1, \cdots, 6)$ be the 6 roots of this equation. Then the relation of roots and coefficients can be rewritten as follows:

$$\begin{cases} p_2 = \varepsilon_2/2 \\ p_3 = -\varepsilon_3/2 \\ p_4 = -(\varepsilon_4-p_2^2)/4 \\ p_5 = (\varepsilon_5+2p_2p_3)/4 \\ p_6 = -(\varepsilon_6-p_3^2)/4\,, \end{cases}$$

where ε_d denotes the d-th elementary symmetric function of u_i $(i=1, \cdots, 6)$ $(u_1+\cdots+u_6=0)$. In this way, we have obtained another excellent family for No. 6.

(VI) The 6 rational points

$$Q_i=\left(u_i t,\ t\left(t+\frac{1}{2}(u_i^3+u_ip_2+p_3)\right)\right) \qquad (i=1, \cdots, 6)$$

generate $E(K)\simeq A_5^*$, where we have $Q_1+\cdots+Q_6=0$ as before. Note that, in this case, we have $u_i=-2sp'_\infty(Q_i)$ as in (No. 3, IV).

(VII) The frame invariant $J(\lambda)$ in (II) now is an invariant of degree 10 with the following expression:

$$J=\frac{1}{2^4}\prod_{i<j<k}(u_i+u_j+u_k)\,,$$

the factors of which correspond to a half of 20 vectors of norm 3/2 in A_5^*. Thus the MWL is nondegenerate if and only if $u_i\neq u_j$ and $u_i+u_j+u_k\neq 0$ for $i<j<k$.

(VIII) *Example.* Take $u_i=2(i-1)$ $(i=1, \cdots, 5)$ and $u_6=-20$, which satisfies the above nondegeneracy condition. Then

$$p_2=-130\,,\quad p_3=1200\,,\quad p_4=6129\,,\quad p_5=-79920\,,\quad p_6=360000\,,$$

so we get the elliptic curve $E/\mathbf{Q}(t)$

$$y^2=x^3-6129x^2+(-79920t-130t^2)x+(360000t^2+1200t^3+t^4)\,,$$

which is a $\mathbf{Q}$-split example for No. 6 using the new family. Free generators of $E(\mathbf{Q}(t))\simeq A_5^*$ are given by the 5 points:

$$(0, t(600+t))\,,\quad (2t, t(474+t))\,,\quad (4t, t(372+t))\,,\quad (6t, t(318+t))\,,\quad (8t, t(336+t))\,.$$

7. No. 7 ($L=D_4\oplus A_1$)

(I) No. 7: $T=A_1^{\oplus 3}, L=D_4\oplus A_1, M=D_4^*\oplus A_1^*$

$$y^2=x^3+x^2(p_6+p_2t)+xt(t-p_4)(t-q_4)+q_2t^2(t-p_4)^2$$

$$\lambda=(p_2, p_4, q_4, p_6, q_2).$$

N.B. This is the first case in the families of Theorem 1 where the elliptic surface has 3 reducible fibres. In addition to the 2 reducible fibres over $t=0, \infty$ as in No. 4, we need one more, say over $t=v$. This $v=p_4$ should serve as an invariant of weight 4 when we employ the weight-table of type E_7. This is a rough idea for getting the above equation.

(II) The discriminant:

$$\Delta=-16t^2(t-p_4)^2\{p_6^2(4p_6q_2-q_4^2)+\cdots+4t^5\}.$$

The singular fibre at $t=\infty$ is of type III for any λ, as in No. 2. Assume $p_4\neq 0$. Then at $t=0$ (or $t=p_4$), the Weierstrass equation reduces to the cubic curve

$$y^2=x^3+p_6x^2 \qquad (\text{or } y^2=x^3+x^2(p_6+p_2p_4))$$

with a singular point at (0, 0) (node or cusp as the case may be). Thus the singular fibre at $t=0$ is either I_2 or III (according to whether $p_6\neq 0$ or $=0$) provided that

$$J_1=q_4^2-4q_2p_6\neq 0.$$

Similarly, the singular fibre at $t=p_4$ is either I_2 or III (according to whether $p_6+p_2p_4\neq 0$ or $=0$) provided that

$$J_2=(p_4-q_4)^2-4q_2(p_6+p_2p_4)\neq 0.$$

For λ generic, there are no other reducible fibres than those three, and so the trivial lattice is $T=A_1^{\oplus 3}$, which implies that $L=D_4\oplus A_1$, $M=D_4^*\oplus A_1^*$ ([OS]). (We note that the orthogonal complement of a root (i.e. of A_1) in D_6 is $D_4\oplus A_1$.)

(III) The lattice $M=D_4^*\oplus A_1^*$ has 2 minimal vectors of minimal norm 1/2 coming from A_1^*, and 24 vectors of norm 1 coming from D_4^*; the latter divides into 3 orbits under $W(D_4)$. The number of roots in L is 26 ($=24+2$).

On the other hand, we have by the formula (3.1)

$$\langle P, P\rangle=2+2(PO)-\begin{cases}\frac{1}{2}\\ 0\end{cases}-\begin{cases}\frac{1}{2}\\ 0\end{cases}-\begin{cases}\frac{1}{2}\\ 0\end{cases}$$

for $P\in M=E(K)$; the 3 terms correspond to the reducible fibres at $t=0, p_4, \infty$, with the value 1/2 if (P) passes through the non-identity component there. In particular, we have $\langle P, P\rangle=1/2$ if and only if $(PO)=0$ and (P) meets all the non-identity components at $t=0, p_4, \infty$. Hence we see that the rational points

$$\pm Q_0=(0,\ \pm u_0 t(t-p_4))\in E(k) \qquad (u_0^2=q_2)$$

give the minimal vectors, since they reduce to the singular points (0, 0) at $t=0, p_4, \infty$.

More generally, the short vectors P meeting the non-identity components at $t=0$, ∞ have the following form:

$$P=(at,\ ct^2+dt)\,.$$

The condition for such a P to satisfy the defining equation is:

$$\begin{cases} c^2 = a+q_2 \\ 2cd = a^3+a^2p_2-a(p_4+q_4)-2p_4q_2 \\ d^2 = a^2p_6+p_4^2q_2+ap_4q_4\,. \end{cases}$$

(IV) Eliminating a, d, we obtain the fundamental algebraic equation, which factorizes as follows:

$$\Phi(c, \lambda)=(c^2-q_2)^2\Psi(c, \lambda)=0\,,$$

where

$$\begin{aligned}\Psi(c, \lambda)=c^8&+c^6(2p_2-4q_2)+c^4(p_2^2-2p_4-6p_2q_2+6q_2^2-2q_4)\\ &+c^2(-2p_2p_4-4p_6-2p_2^2q_2+6p_2q_2^2-4q_2^3-2p_2q_4+4q_2q_4)\\ &+(p_4-p_2q_2+q_2^2-q_4)^2\,.\end{aligned}$$

The first factor corresponds to $\pm Q_0$ mentioned above: $c^2=q_2$, $a=0$. The 8 roots of Ψ define 8 rational points $\pm Q_i$ $(i=1, \cdots, 4)$, corresponding to 8 vectors of norm 1 in D_4^* forming one orbit under $W(D_4)$, which generate an index 2 subgroup (cf. No. 4 or No. 5).

(V) Let $\pm u_i$ $(i=1, \cdots, 4)$ be the 8 roots of this equation, and write down the relation of roots and coefficients. Then, letting ε'_{2d} denote the d-th elementary symmetric function of $u_1^2, \cdots, u_4^2$, we obtain the following:

$$\begin{cases} q_2 = u_0^2 \\ p_2 = -(-4q_2+\varepsilon'_2)/2 \\ p_4 = (p_2^2-4p_2q_2+4q_2^2-\varepsilon'_4+2\varepsilon_4)/4 \\ q_4 = (p_2^2-8p_2q_2+8q_2^2-\varepsilon'_4-2\varepsilon_4)/4 \\ p_6 = (-2p_2p_4-2p_2^2q_2+6p_2q_2^2-4q_2^3-2p_2q_4+4q_2q_4+\varepsilon'_6)/4\,, \end{cases}$$

where we set $\varepsilon_4=u_1\cdots u_4$. This gives an explicit formula of the fundamental invariants of $W(D_4\oplus A_1)=W(D_4)\times W(A_1)$ in terms of the splitting variables $u_0, u_1, \cdots, u_4$, and we conclude that the family in question is an excellent family for No. 7.

(VI) Generators. The 5 rational points

$$Q_i=((u_i^2-q_2)t,\ u_it^2+d_it) \qquad (i=1, \cdots, 4)$$

and Q_0 generate an index 2 subgroup of $E(K) \simeq D_4^* \oplus A_1^*$. Taking one more point

$$Q_5 = ((u_5^2 - q_2)(t - p_4),\ u_5(t - p_4)(t - \cdots)),\quad u_5 = \frac{1}{2}(u_1 + \cdots + u_4),$$

we get generators of the full MWL $E(K)$.

(VII) The frame invariant is given by

$$J(\lambda) = p_4 J_1 J_2 \qquad \text{(cf. II)}.$$

In terms of u_i, p_4, J_1, J_2 are expressed as follows:

$$p_4 = \prod{}' \frac{1}{2}(u_1 + u_2 + u_3 + u_4),$$

$$J_1 = \prod{}' \left(u_0 + \frac{1}{2}(u_1 - u_2 - u_3 - u_4)\right)\left(u_0 - \frac{1}{2}(u_1 - u_2 - u_3 - u_4)\right),$$

$$J_2 = \prod_{i=1}^{4} (u_0 + u_i)(u_0 - u_i),$$

where the product for p_4, J_1 is taken over 4 choices of changing even number of signs of u_2, u_3, u_4 of the given linear form. The other invariant $\delta(\lambda)$ in Theorem 2 is equal to

$$\delta = \left\{u_0 \prod_{1 \le i < j \le 4} (u_i + u_j)(u_i - u_j)\right\}^2.$$

Therefore the MWL is nondegenerate if and only if $u_0 \neq 0$, $u_i \neq \pm u_j$ for any $1 \le i < j \le 4$ and none of the linear factors of J vanish.

(VIII) *Example.* Let $u_0 = 6$, $u_1 = 2$, $u_2 = 4$, $u_3 = 8$, $u_4 = 16$. Then

$$q_2 = 36,\quad p_2 = -98,\quad p_4 = 2025,\quad q_4 = 5825,\quad p_6 = 271350.$$

So the equation of the elliptic curve $E/\mathbf{Q}(t)$ is:

$$y^2 = x^3 + (271350 - 98t)x^2 + t(t - 5825)(t - 2025)x + 36t^2(t - 2025)^2$$

and the generators of $E(\mathbf{Q}(t))$ are given by

$$Q_0 = (0, 6t^2 - 12150t),\quad Q_1 = (-32t, 2t^2 - 6930t),\quad Q_2 = (-20t, 4t^2 - 4500t),$$

$$Q_3 = (28t, 8t^2 - 26280t),\quad Q_4 = (220t, 16t^2 + 126000t),$$

$$Q_5 = (-35(t - 2025),\ (t - 2025)(t + 20475)).$$

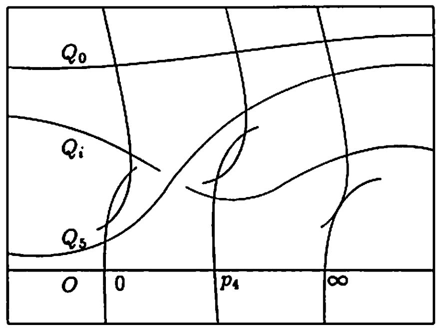

Figure No. 7

8. No. 8, No. 16, No. 27, No. 43 ($L=A_4, \cdots, A_1$)

8.1. No. 8 ($L=A_4$).

(I) **No. 8:** $T=A_4,\ L=A_4,\ M=A_4^*$

$$y^2+p_5xy+p_3t^2y=x^3+x^2(p_4t)+x(p_2t^3)+t^5\ , \quad \lambda=(p_2, p_3, p_4, p_5)$$

Magic of weights works here using the weight-table of type E_8 (cf. 3.2).

(II) The discriminant:

$$\Delta=-t^5\{p_5^4(p_3^2p_4-p_2p_3p_5+p_5^2)+\cdots+432t^5\}$$

The singular fibre at $t=\infty$ is of type II, as in No. 1. The singular fibre at $t=0$ is of type I_5 under the condition:

$$p_5\neq 0\ , \quad J_1=p_3^2p_4-p_2p_3p_5+p_5^2\neq 0\ .$$

This is because then the discriminant has order 5 at $t=0$ and the Weierstrass equation reduces to the cubic curve $y^2+p_5xy=x^3$ with a node at (0, 0). Thus we can write

$$f^{-1}(0)=\Theta_0+\cdots+\Theta_4\ ,$$

where the 5 irreducible components Θ_j are smooth rational curves forming a pentagon, numbered in a cyclic way. (We choose Θ_0 as the identity component as usual.)

For λ generic, there are no other reducible fibres, and the trivial lattice is $T=A_4$ and $L=A_4$, $M=A_4^*$ ([OS]).

(III) The lattice $M=A_4^*$ has 10 minimal vectors of minimal norm 4/5, which divide into 2 orbits under $W(A_4)=\mathscr{S}_5$, and M is generated by any 4 of the 5 vectors in an orbit (cf. (No. 6, III)). The number of roots in $L=A_4$ is 20.

On the othr hand, for $P\in E(K)\simeq A_4^*$, suppose that (P) meets Θ_j, then the formula (3.1) gives

$$\langle P, P\rangle = 2 + 2(PO) - \begin{cases} \dfrac{6}{5} & (j=2, 3) \\ \dfrac{4}{5} & (j=1, 4) \\ 0 & (j=0) . \end{cases}$$

Thus the minimal vectors are exactly those P for which $(PO)=0$ and (P) meets Θ_j for $j=2$ or 3; the latter distinguishes the 2 orbits.

(IV) There are 5 rational points of the form

$$P = \left(\frac{t^2}{u^2}, \frac{t^3}{u^3}\right) \qquad (u = sp_{\infty}(P)) .$$

Indeed, substituting the coordinates into the defining equation, we get the fundamental algebraic equation in this case:

$$\Phi(u, \lambda) = u^5 + p_2 u^3 - p_3 u^2 + p_4 u - p_5 = 0$$

which is the generic algebraic equation of degree 5.

(V) Let $u_1, \cdots, u_5$ be the 5 roots; we have $u_1 + \cdots + u_5 = 0$. Then p_d $(d=2, 3, ,4, 5)$ is equal to the d-th elementary symmetric function of $u_1, \cdots, u_5$. Thus we see that the family in question is an excellent one for No. 8.

(VI) Let

$$Q_i = \left(\frac{t^2}{u_i^2}, \frac{t^3}{u_i^3}\right) \qquad (i = 1, \cdots, 5) .$$

Obviously, each $P=Q_i$ is a short vector (i.e. $(PO)=0$) such that (P) passes through a non-identity component Θ_j $(j>0)$ (note that P reduces to the node $(0, 0)$ at $t=0$). Observe that these 5 points are conjugate under the Galois group over $Q(\lambda)$, i.e. they form a single orbit under $W(A_4)$. Hence $\{Q_i\}$ give 5 minimal vectors in $M = A_4^*$, which generate M. Incidentally, notice that they pass through one and the same component, say Θ_2. For the corresponding sections (Q_i) (in the elliptic surface) are disjoint everywhere, since $Q_i - Q_k$ is a root in $L = A_4$ for any $i \neq k$.

(VII) The frame invariant is given by

$$J(\lambda) = p_5 J_1 \qquad \text{(cf. II)} ,$$

which is expressed as follows in terms of u_i:

$$p_5 = u_1 \cdots u_5 ,$$

$$J_1 = -\prod_{i<j} (u_i + u_j) .$$

So far we have omitted the proof of such identities as above. As an illustration, let us check the last identity. Let $F(u) = \Phi(u, \lambda)$; it has 5 roots u_i. Assume for a moment that $u_i+u_j=0$. Then $F(u)$ and $F(-u)$ have a common zero $u=u_i$. By eliminating u, we obtain a nontrivial relation among p_d, which turns out to be $J_1=0$. Thus J_1 is divisible by the product of u_i+u_j. Comparing the degree, we see that J_1 is a constant multiple of this product. A special case determines the constant to be -1.

The MWL is nondegenerate if and only if $u_i\neq 0$ and $u_i\neq \pm u_j$.

(VIII) *Example.* Let $u_1=1$, $u_2=2$, $u_3=3$, $u_4=4$, $u_5=-10$. Then it is easy to see that the above nondegeneracy condition holds. Since we have

$$p_2=-65\,,\quad p_3=-300\,,\quad p_4=-476\,,\quad p_5=-240\,,$$

the defining equation of the elliptic curve $E/Q(t)$ is:

$$y^2-240xy-300t^2y=x^3-476tx^2-65t^3x+t^5\,.$$

(A direct calculation shows that the discriminant has 5 simple roots (conjugate over Q) at $t\neq 0$ and order 5 zero at $t=0$, which checks also that the MWL is nondegenerate for this example.)

The minimal vectors $\{Q_i\}\in E(Q(t))\simeq A_4^*$ are as follows:

$$(t^2,t^3)\,,\quad \left(\frac{t^2}{4},\frac{t^3}{8}\right),\quad \left(\frac{t^2}{9},\frac{t^3}{27}\right),\quad \left(\frac{t^2}{16},\frac{t^3}{64}\right),\quad \left(\frac{t^2}{100},\frac{-t^3}{1000}\right),$$

and the remaining minimal vectors $\{-Q_i\}$ are given by

$$(t^2,\,-t^3+540t^2)\,,\quad \left(\frac{t^2}{4},\frac{t^3}{8}+360t^2\right),\quad \text{etc.}$$

(Note that $-P$ is not equal to $(x,-y)$ for $P=(x,y)$ in general. Of course, we can rewrite the curve by a simple coordinate change in a more familiar style:

$$y^2=x^3+(14400-476t)x^2+(36000t^2-65t^3)x+22500t^4+t^5\,,$$

but then the generatos $\{Q_i\}$ lose its simplest expression.) Other short vectors such as Q_i+Q_j (of norm 6/5) or Q_i-Q_j (norm 2: roots of A_4) can be computed via the addition theorem. For instance,

$$Q_1+Q_2=\left(\frac{t^2}{9}+196t,\,\frac{t^3}{27}+98t^2+47040t\right)$$

$$Q_2-Q_1=(345600-1324t+t^2,\,-165888000+1153920t-1986t^2+t^3)\,.$$

Observe that $sp_\infty(Q_1+Q_2)=u_1+u_2=3$ or $sp_\infty(Q_2-Q_1)=2-1=1$ is reflected in the highest coefficients. Also we see that (Q_2-Q_1) is disjoint from (O) everywhere ([S2], Lemma 10.9), i.e. (Q_1) and (Q_2) are disjoint and pass the same irreducible component at $t=0$. Let us check directly that this component must be Θ_j with $j=2$ or 3. First we have $\langle Q_i,Q_i\rangle=2-c$, where $c=contr_0(Q_i,Q_i)=4/5$ or 6/5 according as $j=1,4$ or $j=2,3$. Then (3.2) implies

$$\langle Q_1,Q_2\rangle=1+(Q_1O)+(Q_2O)-(Q_1Q_2)-c=1+0+0-0-c=1-c\,.$$

Thus

$$\langle Q_1+Q_2, Q_1+Q_2\rangle=\langle Q_1, Q_1\rangle+\langle Q_2, Q_2\rangle+2\langle Q_1, Q_2\rangle$$
$$=(2-c)+(2-c)+2(1-c)=6-4c.$$

On the other hand, the expression for Q_1+Q_2 shows that it has a norm smaller than 2, since it reduces to $(0, 0)$ at $t=0$. This implies that $c>1$, hence $c=6/5$, proving that $j=2$ or 3 and hence that Q_i are minimal vectors with norm 4/5. (This argument works for arbitrary λ with nondegenerate MWL, while the proof in (VI) is for generic λ. N.B. The generic case, combined with a specialization argument, also proves the general case.)

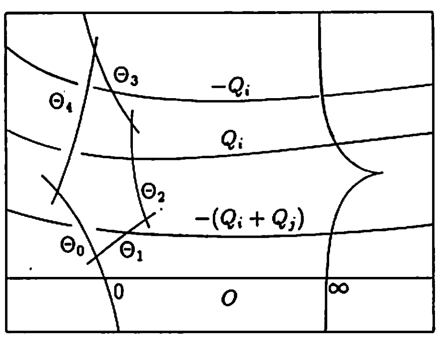

Figure No. 8

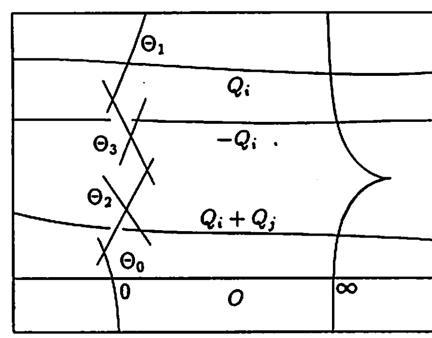

Figure No. 16

8.2. No. 16 ($L=A_3$).

(I) **No. 16:** $T=D_5,\ L=A_3,\ M=A_3^*$

$$y^2+p_3t^2y=x^3+x^2(p_4t)+x(p_2t^3)+t^5\,, \quad \lambda=(p_2, p_3, p_4)\,.$$

This is obtained from No. 8 by letting $p_5=0$. It turns out that magic of weights works in this case too.

(II) The discriminant:

$$\Delta=-t^7\{16p_3^2p_4^3+\cdots+432t^3\}\,.$$

The singular fibre at $t=\infty$ is of type II, as before. The singular fibre at $t=0$ is of type I_1^* under the condition:

$$p_3\neq 0\,, \quad p_4\neq 0\,.$$

This is because then the discriminant has order 7 at $t=0$ and the Weierstrass equation reduces to the cuspidal cubic $y^2=x^3$ (cf. [K], [N], [T]). Thus we have

$$f^{-1}(0)=\Theta_0+\cdots+\Theta_3+2\Theta_4+2\Theta_5\,,$$

where the 6 irreducible components Θ_j intersect as follows:

$$(\Theta_0\Theta_4)=(\Theta_2\Theta_4)=1\,,\quad (\Theta_1\Theta_5)=(\Theta_3\Theta_5)=1\,,\quad (\Theta_4\Theta_5)=1$$

with all other $(\Theta_i\Theta_j)=0$. In this case, the group structure on $f^{-1}(0)^{\#}/\Theta_0^{\#}$ is a cyclic group of order 4, which is indexed by $j=0, 1, 2, 3$ as if $j\in Z/4Z$.

For λ generic, there are no other reducible fibres, and the trivial lattice is $T=D_5$ and $L=A_3$, $M=A_3^*$ ([OS]).

(III) The lattice $M=A_3^*$ has 8 minimal vectors of minimal norm 3/4, forming 2 orbits under $W(A_3)=\mathscr{S}_4$, and M is generated by any 3 of the 4 vectors in an orbit. The number of roots in $L=A_3$ is 12.

On the other hand, for $P\in E(K)\simeq A_3^*$, suppose that (P) meets Θ_j, then by (3.1) (cf. [S2], p. 229), we have

$$\langle P, P\rangle = 2+2(PO)-\begin{cases}\dfrac{5}{4} & (j=1, 3)\\ 1 & (j=2)\\ 0 & (j=0)\,.\end{cases}$$

Thus the minimal vectors are exactly those P for which $(PO)=0$ and (P) meets Θ_j for $j=1$ or 3; the latter distinguishes the 2 orbits.

(IV) Arguing as in (No. 8, III), we obtain the fundamental algebraic equation for No. 16:

$$\Phi(u, \lambda)=u^4+p_2u^2-p_3u+p_4=0\,.$$

Observe that this is deduced from the previous one by setting $p_5=0$ (and dividing by $u\neq 0$).

(V) Let $u_1, \cdots, u_4$ be the roots of the above equation. This is compatible with the notation of (No. 8, III) by letting $u_5=0$ (recall that p_5 is the product of $u_1, \cdots, u_5$). Hence the formula of the fundamental invariants for $W(A_3)$ is the same as before (but with $u_5=0$), and it follows that our family is an excellent one for No. 16.

(VI) The 4 rational points $Q_i=(t^2/u_i^2, t^3/u_i^3)$ $(i=1, \cdots, 4)$ defined as in (No. 8, VI), give a half of minimal vectors, and any 3 among them give free generators of the MWL $E(K)\simeq A_3^*$. (Note that Q_5 "degenerates" to O under the specialization $u_5\to 0$).

(VII) The frame invariant is given by

$$J=p_3p_4\,,\quad p_3=-\prod_{1\le i<j\le 3}(u_i+u_j)\,,\quad p_4=u_1\cdots u_4$$

Hence the MWL is nondegenerate ($\simeq A_3^*$) if and only if we have $u_i\neq 0$ and $u_i\neq \pm u_j$ for all i, j.

(VIII) *Example.* Take $u_1=1$, $u_2=2$, $u_3=3$, $u_4=-6$ (and $u_5=0$). Then

$$y^2-60t^2y=x^3-36tx^2-25t^3x+t^5$$

has the MWL $E(Q(t))\simeq A_3^*$, which is generated by

$$Q_1=(t^2, t^3), \quad Q_2=\left(\frac{t^2}{4}, \frac{t^3}{8}\right), \quad Q_3=\left(\frac{t^2}{9}, \frac{t^3}{27}\right).$$

8.3. No. 27 ($L=A_2$).

(I) **No. 27:** $T=E_6$, $L=A_2$, $M=A_2^*$

$$y^2+p_3t^2y=x^3+p_2t^3x+t^5\,, \quad \lambda=(p_2, p_3)\,.$$

This is obtained from No. 16 by letting $p_4=0$. Again magic of weights works here.

(II) The discriminant:

$$\Delta=-t^8\{27p_3^4+(64p_2^3+216p_3^2)t+432t^2\}\,.$$

The singular fibre at $t=\infty$ is of type II, as before. The singular fibre at $t=0$ is of type IV^* under the condition:

$$p_3\neq 0\,.$$

For the discriminant has order 8 at $t=0$, while the j-invariant has no pole there (cf. [K], [N], [T]). There are 7 irreducible components of which 3 have multiplicity 1:

$$f^{-1}(0)=\Theta_0+\Theta_1+\Theta_2+\cdots,$$

and the algebraic group $f^{-1}(0)^{\#}$ is the product of $\Theta_0^{\#}\simeq G_a$ and a cyclic group of order 3, indexed by $j=0, 1, 2$.

For λ generic, the trivial lattice is $T=E_6$ and $L=A_2$, $M=A_2^*$.

(III) The root lattice $L=A_2$ is one of the most famous lattice: the hexagonal lattice (cf. [CS]). It has 6 roots (norm 2), and its dual lattice $M=A_2^*$ has also 6 minimal vectors (of minimal norm 2/3), but they divide into 2 orbits under $W(A_2)=\mathscr{S}_3$. Any 2 vectors in an orbit give free generators of M.

On the other hand, we have for $P\in E(K)\simeq A_2^*$

$$\langle P, P\rangle=2+2(PO)-\begin{cases}\dfrac{4}{3} & (j=1, 2)\\ 0 & (j=0)\,,\end{cases}$$

if (P) meets Θ_j (cf. (3.1), [S2], p. 229). Thus the minimal vectors are the short vectors meeting Θ_j for $j=1$ or 2, and the latter distinguishes the 2 orbits.

(IV) The fundamental algebraic equation for No. 27:

$$\Phi(u, \lambda)=u^3+p_2u-p_3=0$$

is obtained from (No. 16, IV) by setting $p_4=0$.

(V) Let u_1, u_2, u_3 be the roots of the above cubic equation ($u_1+u_2+u_3=0$). Again this is compatible with the notation of No. 8 and No. 16 by letting $u_4=u_5=0$ (recall that p_4 is the product of $u_1, \cdots, u_4$). It will now be evident that the family in question is excellent for No. 27.

(IV) The 3 rational points $Q_i=(t^2/u_i^2, t^3/u_i^3)$ ($i=1, 2, 3$) defined as in (No. 8, VI), give a half of 6 minimal vectors, and any 2 among them give free generators of the MWL $E(K) \simeq A_2^*$.

(VII) The frame invariant is given by

$$J = p_3 = u_1 u_2 u_3 .$$

Thus the MWL is nondegenerate ($\simeq A_2^*$) if and only if we have $u_i \neq 0$ and $u_i \neq u_j$ for any $i<j$, or $p_3 \neq 0$, $4p_2^3 + 27p_3^2 \neq 0$ in terms of $\lambda = (p_2, p_3)$.

(VIII) *Example.* Take $u_1 = 1$, $u_2 = 2$, $u_3 = -3$. Then

$$y^2 - 6t^2 y = x^3 - 7t^3 x + t^5$$

has the MWL $E(Q(t)) \simeq A_2^*$, which is generated by

$$Q_1 = (t^2, t^3), \quad Q_2 = \left(\frac{t^2}{4}, \frac{t^3}{8}\right), \quad Q_3 = \left(\frac{t^2}{9}, \frac{-t^3}{27}\right) \quad (= -Q_1 - Q_2).$$

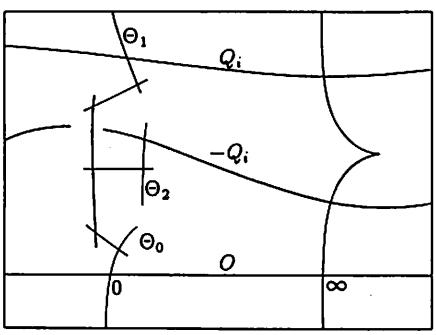

Figure No. 27

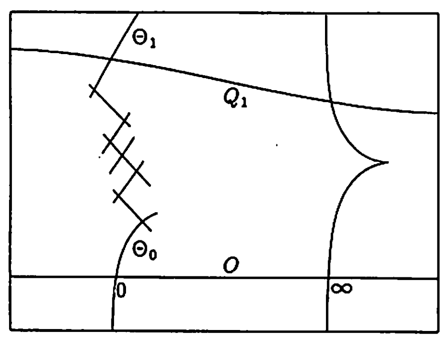

Figure No. 43

8.4. No. 43 ($L = A_1$).

(I) **No. 43:** $T = E_7$, $L = A_1$, $M = A_1^*$

$$y^2 = x^3 + p_2 t^3 x + t^5 , \quad \lambda = p_2 .$$

This is obtained from No. 27 by letting $p_3 = 0$. Again magic of weights works here.

(II) The discriminant is:

$$\Delta = -16t^9(4p_2^3 + 27t) .$$

The singular fibre at $t = \infty$ is of type *II*, as before. The singular fibre at $t = 0$ is of type *III** under the condition:

$$p_2 \neq 0 ,$$

since the discriminant has order 9 at $t = 0$, while the j-invariant has no pole there (cf. [K], [N], [T]). There are 8 irreducible components of which 2 have multiplicity 1:

$$f^{-1}(0)=\Theta_0+\Theta_1+\cdots,$$

and the algebraic group $f^{-1}(0)^{\sharp}$ is the product of $\Theta_0^{\sharp}\simeq G_a$ and a cyclic group of order 2, indexed by $j=0, 1$.

If $p_2\neq 0$, the trivial lattice is $T=E_7$ and $L=A_1$, $M=A_1^*$.

(III) The lattice $M=A_1^*$ has 2 minimal vectors of minimal norm 1/2, which are interchanged by $W(A_1)=\{\pm 1\}$, and M is generated by them. On the other hand, for $P\in E(K)\simeq A_1^*$, we have

$$\langle P, P\rangle=2+2(PO)-\begin{cases}\dfrac{3}{2} & (j=1)\\ 0 & (j=0),\end{cases}$$

if (P) meets Θ_j (cf. [S2], p. 229). Thus P is a minimal vector if and only if $(PO)=0$ and (P) meets Θ_1.

(IV) The fundamental algebraic equation for No. 43 reads:

$$\Phi(u, \lambda)=u^2+p_2=0\,.$$

(V) Let u_1, u_2 be the 2 roots of the above equation ($u_1+u_2=0$). This is compatible with the previous notation of No. 8, No. 16 and No. 27. Obviously the family defines an excellent family for No. 43.

(VI) The rational point $Q_1=(t^2/u_1^2, t^3/u_1^3)$ is a generator of $E(K)\simeq A_1^*$.

(VII) The frame invariant is given by

$$J=p_2=u_1u_2=-u_1^2\,.$$

Thus the MWL is nondegenerate ($\simeq A_1^*$) if and only if we have $u_1\neq 0$, or equivalently, $p_2\neq 0$.

(VIII) *Example.* Take $u_1=1$, $u_2=-1$. Then

$$y^2=x^3-t^3x+t^5$$

has the MWL $E(\boldsymbol{Q}(t))\simeq A_1^*$, which is generated by $Q_1=(t^2, t^3)$. We have $Q_2=-Q_1=(t^2, -t^3)$ and

$$2Q_1=\left(\frac{t^2-6t+1}{4}, \frac{t^3+15t^2-9t+1}{8}\right),$$

the latter being a root of the narrow MWL $E(\boldsymbol{Q}(t))^0\simeq A_1$.

Incidentally the familiar elliptic curve defined over $\boldsymbol{Q}(j)$ with the j-invariant j

$$y^2=x^3-sx+s\,, \qquad s=\frac{27j}{4(j-12^3)}$$

can be transformed to the above example by a simple coordinate change. The obvious rational point $(x, y)=(1, 1)$ corresponds to Q_1 and is a generator of $E(\boldsymbol{Q}(j))$, if j is a variable over $\boldsymbol{Q}$.

REMARK. As shown in the above discussion, the 4 families treated in this section are related by specializing the parameters: $p_5 \to 0$, $p_4 \to 0$ and $p_3 \to 0$, or in terms of u_i, $u_5 \to 0$, $u_4 \to 0$ and $u_3 \to 0$. While the variables "downstairs" are sufficient for describing how the singular fibre at $t=0$ changes, i.e. how the frame is broken (in the terminology of 3.2), the degeneration of MWL can be best described in terms of the variables "upstairs" or the splitting variables u_i.

9. No. 9, No. 10, No. 11 ($L=D_4$, $A_3 \oplus A_1$, $A_2^{\oplus 2}$)

9.1. No. 9 ($L=D_4$).

(I) **No. 9:** $T=D_4$, $L=D_4$, $M=D_4^*$ (cf. [S5], §6)

$$y^2=x^3+x(p_4-t^2)+(q_6+q_4t+q_2t^2), \quad \lambda=(q_2, q_4, p_4, q_6).$$

This case has been treated in [S5], which we review here to compare with other cases and also to make some supplements. The equation is weighted homogeneous of total weight 6; the relevant weight-table (cf. §3, 3.2) should be called of type D_4.

(II) The discriminant is a polynomial of degree 6:

$$\Delta=-2^4\{4(p_4-t^2)^3+27(q_6+q_4t+q_2t^2)^2\},$$

and we have a singular fibre of type I_0^* at $t=\infty$:

$$f^{-1}(\infty)=\Theta_0+\Theta_1+\Theta_2+\Theta_3+2\Theta_4.$$

There are no other reducible fibres in general, so for λ generic, the trivial lattice is $T=D_4$ and $L=D_4$, $M=D_4^*$ (cf. [OS]).

(III) In a standard realization (cf. (No. 4, III)), we have

$$D_4 \subset Z^4 \subset D_4^*.$$

The Weyl group $W(D_4)$ is generated by $\mathscr{S}_4$ (permutations of 4 coordinates) and sign-change at 2 coordinates, and has order $2^3 \cdot 4!$. The lattice $M=D_4^*$ has 24 minimal vectors of minimal norm 1, which divide into 3 orbits under $W(D_4)$. If we denote by $\{e_i\}$ the unit vectors in Z^4, then they are: (i) $\{\pm e_i\}$, (ii) $\{\pm f, f-e_i-e_j\}$ $(i<j)$, (iii) $\{\pm(f-e_i)\}$, where we set $f=\frac{1}{2}(e_1+\cdots+e_4)$. The quotient group D_4^*/D_4 is isomorphic to $Z/2Z \oplus Z/2Z$, with representatives $\{0, e_1, f, f-e_1\}$. In particular, M is generated by $\{e_1, e_2, e_3, f\}$. The roots in $L=D_4$ are $\{\pm e_i \pm e_j\ (i \neq j)\}$.

The minimal vectors $P \in E(K) \simeq D_4^*$ are precisely the short vectors (P) which meet Θ_j $(j>0)$ (use (3.1)); the 3 cases $j=1, 2, 3$ correspond to the 3 orbits under $W(D_4)$. In terms of Weierstrass coordinates, they are given by

$$P=(at+b, dt+e).$$

(IV) The condition for a, b, d, e is easily written down. In particular, we see $a=0$, 1 or -1, which correspond to the 3 cases as above, say $j=1$, 2 or 3 (by renaming Θ_j if necessary). Let $a=0$; then $b=q_2-d^2$, $e=q_4/(2d)$ (if $d \neq 0$), and elimination yields the fundamental algebraic equation for d:

$$\Phi(d, \lambda)=d^8-3q_2d^6+(p_4+3q_2^2)d^4-(q_6+q_2p_4+q_2^3)d^2+\left(\frac{1}{2}q_4\right)^2=0 .$$

(V) Let $\pm u_i$ ($i=1, \cdots, 4$) be the roots of this equation. Then the relation of roots and coefficients gives the formula (cf. [S5], Th. (D_4), p. 678):

$$\begin{cases} q_2 = \varepsilon'_2/3 \\ p_4 = \varepsilon'_4-3q_2^2 \\ q_4 = u_1\cdots u_4 \\ q_6 = \varepsilon'_6-q_2p_4-q_2^3 . \end{cases}$$

Here ε'_{2d} denotes the d-th elementary symmetric function of $u_1^2, \cdots, u_4^2$. This gives an explicit formula of the fundamental invariants of $W(D_4)$, and we see that the family in question is an excellent family for No. 9 (cf. No. 4, No. 5).

(IV) The 4 rational points

$$Q_i=(q_2-u_i^2, u_it+u_ju_ku_l) \qquad (i=1, \cdots, 4) , \quad \{i, j, k, l\}=\{1, 2, 3, 4\}$$

generate the sublattice $\simeq \mathbf{Z}^4$ of $E(K)\simeq D_4^*$. The full MWL is generated by Q_i and one more rational point corresponding to $-f$, which has the form

$$Q_0=(x, y) , \quad \begin{cases} x=-t+q_2+\sum_{i<j} u_iu_j \\ y=\left(\sum_i u_i\right)t-\left\{\sum_{i<j}(u_i^2u_j+u_iu_j^2)+2\sum_{i<j<k}u_iu_ju_k\right\} . \end{cases}$$

(VII) By (II), the frame invariant can be taken as the constant 1. The MWL is nondegenerate if and only if $u_i\neq \pm u_j$ ($i<j$).

(VIII) For an example, see [S5], p. 683.

9.2. No. 10 ($L=A_3\oplus A_1$).

(I) **No. 10:** $T=A_3\oplus A_1$, $L=A_3\oplus A_1$, $M=A_3^*\oplus A_1^*$

$$y^2+p_3xy=x^3+x^2(p_2t)+x(p_4t^2+t^3)+q_2t^4 , \quad \lambda=(p_2, p_3, p_4, q_2) .$$

The equation is obtained by magic of weights using the weight-table of type E_7.

(II) The discriminant is:

$$\Delta=-t^4\{p_3^4(-p_4^2+p_3^2q_2)+\cdots+64t^5\} .$$

We have a singular fibre of type III at $t=\infty$, as in No. 2. The singular fibre at $t=0$ is of type I_4 under the condition:

$$p_3\neq 0 , \quad J_1=-p_4^2+p_3^2q_2\neq 0 .$$

This is verified as in (No. 5, II), noting that the reduced curve at $t=0$ has a node at (0, 0). Let

$$f^{-1}(0)=\Theta_0+\Theta_1+\Theta_2+\Theta_3\,,\quad f^{-1}(\infty)=\Theta'_0+\Theta'_1\,.$$

There are no other reducible fibres in general. Hence, for λ generic, the trivial lattice is $T=A_3\oplus A_1$ and $L=A_3\oplus A_1$, $M=A_3^*\oplus A_1^*$ (cf. [OS]).

(III) The lattice $M=A_3^*\oplus A_1^*$ has 2 minimal vectors of minimal norm 1/2 (coming from the factor A_1^*), say $\pm\bar{u}_0$, and 8 vectors of norm 3/4 (coming from the other factor) which divide into 2 orbits under $W(A_3)$. If we denote by $\bar{u}_i$ $(i=1,\cdots,4)$ an orbit, then $\{\bar{u}_i\,(i=0,\cdots,3)\}$ forms a basis of M (note that $\sum_{i=1}^{4}\bar{u}_i=0$).

The rational points $Q_i\in E(K)$ corresponding to $\bar{u}_i\in M$ can be described as follows. They are short vectors, i.e. $(Q_iO)=0$, and the minimal vectors $\pm Q_0$ meet the components Θ_2 and Θ'_1; the formula (3.1) says $2-1-1/2=1/2$. The 4 points Q_i pass through Θ'_1 and either one of Θ_j $(j=1,3)$, and (3.1) reads $2-3/4-1/2=3/4$. In terms of Weierstrass coordinates, they are given by

$$P=(at,ct^2+dt)\,.$$

(IV) Substituting it into the defining equation, we get the condition for a, c, d:

$$a=c^2-q_2\,,\quad d=0\quad\text{or}\quad d=-ap_3\,,\quad a^3+a^2p_2-acp_3+ap_4=2cd\,.$$

Consider the case $d=0$ ($d=-ap_3$ corresponds to changing $P\to -P$). Then eliminating a, we get the fundamental algebraic equation:

$$\begin{aligned}\Phi(c,\lambda)&=a^3+a^2p_2-acp_3+ap_4\\&=(c^2-q_2)(c^4+c^2(p_2-2q_2)-cp_3+p_4-p_2q_2+q_2^2)=0\,.\end{aligned}$$

(V) Let $c=\pm u_0$ be the roots of the first factor c^2-q_2; then we have $a=0$ and

$$Q_0=(0,u_0t^2)\,,\quad q_2=u_0^2\,.$$

Next let $\pm u_i$ $(i=1,\cdots,4)$ be the roots of the second factor of $\Phi(c,\lambda)$. By the relation of roots and coefficients, we have $u_1+u_2+u_3+u_4=0$ and

$$\begin{cases}\\ \varepsilon_2=p_2-2q_2\\ \varepsilon_3=p_3\\ \varepsilon_4=p_4-p_2q_2+q_2^2\end{cases}\quad\text{i.e.}\quad\begin{cases}q_2=u_0^2\\ p_2=\varepsilon_2+2q_2\\ p_3=\varepsilon_3\\ p_4=\varepsilon_4+p_2q_2-q_2^2\,.\end{cases}$$

Here ε_d is the d-th elementary symmetric function of $u_1,\cdots,u_4$. This gives an explicit formula of the fundamental invariants of the Weyl group $W(A_3\oplus A_1)=W(A_3)\times W(A_1)$, and we see that the family in question is an excellent family for No. 10.

(VI) Generators. Let

$$Q_i=((u_i^2-q_2)t,u_it^2)\qquad(i=1,\cdots,4)\,.$$

Then $Q_1+\cdots+Q_4=0$ and $\{Q_0,Q_1,Q_2,Q_3\}$ generate the full MWL $E(K)\simeq M=A_3^*\oplus A_1^*$.

(VII) By (II), the frame invariant is given by

$$J(\lambda)=p_3J_1=p_3(-p_4^2+p_3^2q_2).$$

In terms of splitting variables u_i, we have

$$p_3=-(u_1+u_2)(u_1+u_3)(u_2+u_3)$$

$$J_1=-\prod_{i=1}^{4}(u_0+u_i)(u_0-u_i).$$

The invariant $\delta(\lambda)$ is given by the product of all roots of $L=A_3\oplus A_1$, expressed in terms of u_i:

$$\delta=(2u_0)^2\prod_{1\le i<j\le 4}(u_i-u_j)^2.$$

Therefore the MWL is nondegenerate if and only if $u_0\neq 0$ and $u_i\neq \pm u_j$ $(0\le i<j\le 4)$.

(VIII) *Example.* Take $u_0=4, u_1=0, u_2=1, u_3=2, u_4=-3$, which satisfy the above condition. Then $q_2=16, p_2=25, p_3=-6, p_4=144$. So the elliptic curve $E/\mathbf{Q}(t)$

$$y^2-6xy=x^3+25tx^2+(144t^2+t^3)x+16t^4$$

has the MWL $E(\mathbf{Q}(t))\simeq M=A_3^*\oplus A_1^*$. The rational points $Q_0, \cdots, Q_4$ are given as follows:

$$(0, 4t^2),\quad (-16t, 0),\quad (-15t, t^2),\quad (-12t, 2t^2),\quad (-7t, -3t^2),$$

of which the first 4 elements form free generators. The duplication formula on E gives

$$2Q_0=\left(\frac{1}{64}(20160-1312t+t^2), \frac{1}{512}(-2419200+137280t-856t^2-t^3)\right),$$

which confirms that the point $2Q_0$ has norm $2^2\cdot 1/2=2$, i.e., it is a root in $A_1\subset L$. (This also checks that (Q_0) passes through Θ_2.) For other $P=2Q_i$, the x-coordinate is a rational function of degree 4 in t, reflecting the fact that their norm $2^2\cdot 3/4=3$ shows $(PO)=1$.

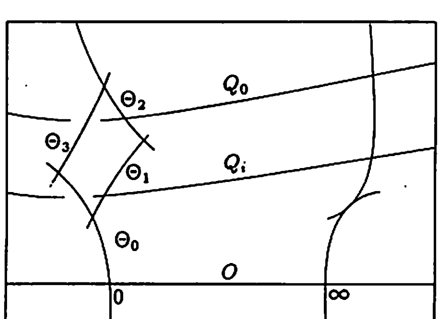

Figure No. 10

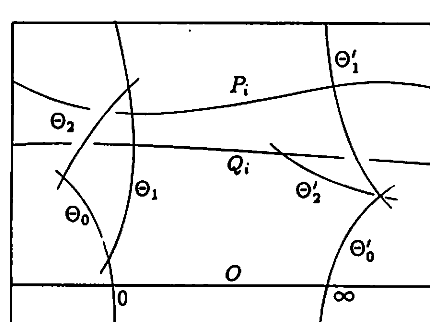

Figure No. 11

9.3. No. 11 ($L=A_2^{\oplus 2}$).

(I) **No. 11:** $T=A_2^{\oplus 2}$, $L=A_2^{\oplus 2}$, $M=A_2^{*\oplus 2}$

$$y^2+p_2xy+p_3ty=x^3+x(q_2t^2)+(q_3t^3+t^4)\,,\quad \lambda=(p_2,p_3,q_2,q_3)\,.$$

This equation is obtained by magic of weights using the weight-table of type E_6.

(II) The discriminant is:

$$\Delta=t^3\{-p_2^3(-p_3^3-p_2^2p_3q_2+p_2^3q_3)+(-27p_3^4+p_2(\cdots))t+\cdots-432t^5\}\,.$$

We have a singular fibre of type IV at $t=\infty$, as in No. 3. The singular fibre at $t=0$ is of type I_3 or IV under the condition:

$$J=-p_3^3-p_2^2p_3q_2+p_2^3q_3\neq 0\,.$$

This is verified as in No. 6, noting that the reduced curve at $t=0$ has a node or a cusp at (0, 0) according as p_2 is zero or not. Let

$$f^{-1}(0)=\Theta_0+\Theta_1+\Theta_2\,,\quad f^{-1}(\infty)=\Theta'_0+\Theta'_1+\Theta'_2\,.$$

For λ generic, there are no other reducible fibres, so the trivial lattice is $T=A_2^{\oplus 2}$, which implies $L=A_2^{\oplus 2}$, $M=A_2^{*\oplus 2}$ (cf. [OS]).

(III) The lattice A_2^* has 6 minimal vectors of norm 2/3, which form 2 orbits under $W(A_2)=\mathscr{S}_3$; 3 vectors in each orbit sum up to 0 and any 2 among them generate this lattice. Hence the lattice $M=A_2^{*\oplus 2}$ has 12 minimal vectors of minimal norm 2/3, which form 4 orbits under $W(L)=W(A_2)\times W(A_2)$ of the form:

$$\{\bar{u}_1,\bar{u}_2,\bar{u}_3\}\,,\quad \{-\bar{u}_1,-\bar{u}_2,-\bar{u}_3\}\,,\quad \{\bar{v}_1,\bar{v}_2,\bar{v}_3\}\,,\quad \{-\bar{v}_1,-\bar{v}_2,-\bar{v}_3\}\,,$$

where $\bar{u}_1+\bar{u}_2+\bar{u}_3=0$, $\bar{v}_1+\bar{v}_2+\bar{v}_3=0$.

By (3.1), $P\in E(K)$ is a minimal vector if and only if $(PO)=0$ and (P) passes through a non-identity component both at $t=0$ and $t=\infty$. In terms of Weierstrass coordinates, we have

$$P=(at,ct^2+dt)\,.$$

As in No. 3, we have $c=\pm 1$ and $d=\pm 2sp'_\infty(P)$.

(IV) It follows from the defining equation that a, c, d should satisfy the condition:

$$c=1 \text{ or } -1\,,\quad d=0 \text{ or } d=-ap_2-p_3\,,\quad a^3+a(q_2-cp_2)-2cd-cp_3+q_3=0\,.$$

The 4 choices of c, d correspond to the 4 orbits mentioned above. The choice $c=1$, $d=0$ (or $c=-1$, $d=-ap_2-p_3$) leads to one of the fundamental equations:

$$\Phi_1(a,\lambda)=a^3+a(-p_2+q_2)-p_3+q_3=0\,,$$

and the choice $c=-1$, $d=0$ (or $c=1$, $d=-ap_2-p_3$) to another:

$$\Phi_2(a,\lambda)=a^3+a(p_2+q_2)+p_3+q_3=0\,.$$

(V) Let $a=u_i$ $(i=1,2,3)$ (resp. v_i $(i=1,2,3)$) be the roots of the first (resp.

second) equation. By the relation of roots and coefficients, we have $u_1+u_2+u_3=0$, $v_1+v_2+v_3=0$ and

$$\begin{cases} p_2=\frac{1}{2}(-\varepsilon_2+\eta_2) \\ q_2=\frac{1}{2}(\varepsilon_2+\eta_2), \end{cases} \qquad \begin{cases} p_3=\frac{1}{2}(\varepsilon_3-\eta_3) \\ q_3=\frac{1}{2}(-\varepsilon_3-\eta_3). \end{cases}$$

Here ε_d is the d-th elementary symmetric function of u_1, u_2, u_3, and η_d is that of v_1, v_2, v_3.

Therefore the two extensions of polynomial rings with Galois group $W(A_2)=\mathscr{S}_3$

$$\boldsymbol{Q}[u_1, u_2]\supset \boldsymbol{Q}[\varepsilon_2, \varepsilon_3], \qquad \boldsymbol{Q}[v_1, v_2]\supset \boldsymbol{Q}[\eta_2, \eta_3]$$

glue into a single extension with Galois group $W(A_2)\times W(A_2)$

$$\boldsymbol{Q}[u_1, u_2, v_1, v_2]\supset \boldsymbol{Q}[\varepsilon_2, \varepsilon_3, \eta_2, \eta_3]=\boldsymbol{Q}[p_2, q_2, p_3, q_3].$$

This proves that the family in question is an excellent family for No. 11.

(VI) Generators. Let

$$P_i=(u_it, t^2), \quad Q_i=(v_it, -t^2) \qquad (i=1, \cdots, 3).$$

Then $\{P_1, P_2, Q_1, Q_2\}$ generate the full MWL $E(K)\simeq M=A_2^{*\oplus 2}$. Note that $P_1+P_2+P_3=0$, since the 3 points are collinear, lying on the line $y=t^2$, and similarly for Q_i.

(VII) The frame invariant is given by J in (II), which is expressed as

$$J(\lambda)=\frac{1}{8}\prod_{i,j=1}^{3}(u_i-v_j)$$

in terms of splitting variables u_i, v_j. The invariant $\delta(\lambda)$ is equal to the product of the discriminants of Φ_1 and Φ_2. Therefore the MWL is nondegenerate if and only if u_1, u_2, u_3, v_1, v_2, v_3 are mutually distinct.

(VIII) *Example.* Take $u_1=0$, $u_2=1$, $v_1=2$, $v_2=3$, which satisfy the above condition. Then we get an elliptic curve $E/\boldsymbol{Q}(t)$

$$y^2-9xy+15ty=x^3-10t^2x+15t^3+t^4,$$

having the MWL $E(\boldsymbol{Q}(t))\simeq M=A_2^{*\oplus 2}$. The generactors P_1, P_2, Q_1, Q_2 are given by

$$(0, t^2), \quad (t, t^2), \quad (2t, -t^2), \quad (3t, -t^2).$$

10. No. 15 and No. 26 ($L=A_2\oplus A_1$, $A_2^{\oplus 2}$)

10.1. No. 15 ($L=A_2\oplus A_1$).

(I) **No. 15:** $T=A_5$, $L=A_2\oplus A_1$, $M=A_2^*\oplus A_1^*$

$$y^2+q_2p_3xy+p_3t^2y=x^3+p_2q_2tx^2+(p_2+q_2)t^3x+t^5, \quad \lambda=(p_2, p_3, q_2).$$

Note that this equation is obtained from No. 8 by specializing the parameters in a way less obvious than the case No. 16. It will be seen below that this arises quite naturally when we use the splitting variables.

(II) The discriminant is:

$$\Delta = t^6\{q_2^3 p_3^4 (q_2^3 + p_3^2 - 2q_2^2 p_2 + q_2 p_2^2) + \cdots - 432t^4\}\,.$$

The singular fibre at $t=\infty$ is the same type II as in No. 8. The singular fibre at $t=0$ is of type I_6 under the condition:

$$q_2 \neq 0\,, \quad p_3 \neq 0\,, \quad J_1 = q_2^3 + p_3^2 - 2q_2^2 p_2 + q_2 p_2^2 \neq 0\,,$$

since Δ has order 6 at $t=0$ and the reduced equation gives a nodal cubic $y^2 + q_2 p_3 xy = x^3$ (with the node $(0,0)$). It has 6 irreducible components Θ_j ($j=0, \cdots, 5$) forming a hexagonal cycle.

For λ generic, the trivial lattice is $T = A_5$, and we have $L = A_2 \oplus A_1$, $M = A_2^* \oplus A_1^*$ (cf. [OS]).

(III) Given $P \in E(K) \simeq M$, suppose that (P) meets Θ_j. By the formula (3.1), we have

$$\langle P, P\rangle = 2 + 2(PO) - \begin{cases} \dfrac{5}{6} & (j=1,5) \\ \dfrac{4}{3} & (j=2,4) \\ \dfrac{3}{2} & (j=3) \\ 0 & (j=0)\,. \end{cases}$$

Hence P has norm $1/2$ (resp. $2/3$ or $7/6$) if and only if $(PO)=0$ and (P) passes through Θ_3 (resp. Θ_2, Θ_4 or Θ_1, Θ_5).

(IV) As in No. 8, there are 5 rational points:

$$P = \left(\frac{t^2}{u^2}, \frac{t^3}{u^3}\right) \qquad (u = sp_\infty(P))\,,$$

where u satisfies the fundamental algebraic equation:

$$\begin{aligned} \Phi(u, \lambda) &= u^5 + (p_2 + q_2)u^3 - p_3 u^2 + p_2 q_2 u - q_2 p_3 \\ &= (u^2 + q_2)(u^3 + p_2 u - p_3) = 0\,. \end{aligned}$$

(V) Let $\pm u_0$ be the roots of $u^2 + q_2 = 0$ and let u_1, u_2, u_3 be the roots of $u^3 + p_2 u - p_3 = 0$ ($u_1 + u_2 + u_3 = 0$). The relation of roots and coefficients gives

$$\begin{cases} q_2 = -u_0^2 \\ p_2 = u_1 u_2 + u_1 u_3 + u_2 u_3 \\ p_3 = u_1 u_2 u_3\,. \end{cases}$$

This represents the fundamental invariants of the Weyl group $W(L)=W(A_2)\times W(A_1)$, and we can conclude that our family is an excellent family for No. 15.

REMARK. At this point, we can explain the degeneration of MWL from No. 8 to No. 15. In the notation of (No. 8, II), we should have $p_5\neq 0$ and $J_1=0$ in order to make the singular fibre at $t=0$ to be of type I_6. In terms of the splitting variables $u_1, \cdots, u_5$ $(u_1+\cdots+u_5=0)$ in (No. 8, V), the invariants p_5, J_1 are expressed as in (No. 8, VII) and so we should have $u_i\neq 0$ for all i and $u_i+u_j=0$ for some $i<j$.

Now let $u_4=u_0$ and $u_5=-u_0$ so that $u_1+u_2+u_3=0$. Then the elementary symmetric functions p'_d of $u_1, \cdots, u_5$, which were denoted by p_d in No. 8, are rewritten as follows:

$$p'_2=p_2+q_2\,,\quad p'_3=p_3\,,\quad p'_4=p_2q_2\,,\quad p'_5=q_2p_3$$

This defines the family for No. 15 under consideration.

(VI) Let

$$Q_i=\left(\frac{t^2}{u_i^2},\frac{t^3}{u_i^3}\right)\qquad (i=0,\cdots,3)\,.$$

Observe that $\{\pm Q_0\}$ and $\{Q_1, Q_2, Q_3\}$ are stable under the Galois group $W(L)$ (for λ generic). It follows that $\pm Q_0$ are the minimal vectors of norm 1/2 and Q_1, Q_2, Q_3 form one orbit of vectors of norm 2/3, and that Q_1, Q_2 and Q_0 generate the MWL $E(K)\simeq M=A_2^*\oplus A_1^*$. In view of (III), (Q_0) meets Θ_3, while (Q_i) $(i=1,2,3)$ meet either Θ_2 or Θ_4.

(VII) The frame invariant is given by

$$J(\lambda)=q_2p_3J_1\qquad (\text{cf. II})\,,$$

where J_1 is expressed as follows in terms of u_i:

$$J_1=-\prod_{i=1}^{3}(u_0-u_i)(u_0+u_i)\,.$$

Thus the MWL is nondegenerate if and only if all $u_0, \cdots, u_3$ are different from 0 and from each other and $u_0\neq -u_i$ for any $i=1,2,3$.

(VIII) *Example.* Let $u_0=4$, $u_1=1$, $u_2=2$, $u_3=-3$. Then the elliptic curve

$$y^2+96xy-6t^2y=x^3+112tx^2-23t^3x+t^5$$

has a rank 3 MWL $E(\mathbf{Q}(t))\simeq M=A_2^*\oplus A_1^*$, which is generated by

$$Q_1=(t^2,t^3)\,,\quad Q_2=\left(\frac{t^2}{4},\frac{t^3}{8}\right),\quad Q_0=\left(\frac{t^2}{16},\frac{t^3}{64}\right).$$

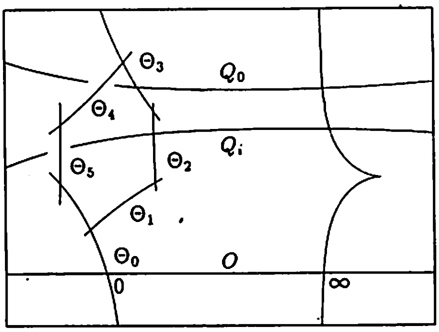

Figure No. 15

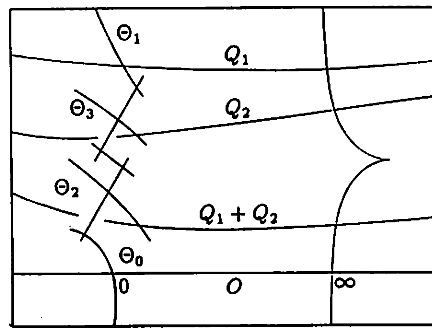

Figure No. 26

10.2. No. 26 ($L=A_1^{\oplus 2}$)

(I) **No. 26:** $T=D_6$, $L=A_1^{\oplus 2}$, $M=A_1^{*\oplus 2}$

$$y^2=x^3+p_2q_2tx^2-(p_2+q_2)t^3x+t^5\,,\quad \lambda=(p_2,q_2)\,.$$

This is obtained from No. 15 by letting $p_3=0$ and changing the sign of p_2, q_2.

(II) The discriminant is:

$$\Delta=16t^8\{p_2^2q_2^2(p_2-q_2)^2+\cdots-27t^2\}\,.$$

The singular fibre at $t=\infty$ is the same type II as in No. 8. The singular fibre at $t=0$ is of type I_2^* provided that

$$p_2\neq 0\,,\quad q_2\neq 0\,,\quad p_2\neq q_2\,.$$

For Δ has order 8 and the j-invariant has a pole of order 2 at $t=0$. It has 7 irreducible components Θ_j ($j=0,\cdots,6$), which we arrange in a similar way to (No. 16, II); in particular, Θ_j ($j=0,\cdots,3$) have multiplicity 1.

For λ generic, the trivial lattice is $T=D_6$, and $L=A_1^{\oplus 2}$, $M=A_1^{*\oplus 2}$ (cf. [OS]).

(III) The lattice M is generated by 2 minimal vectors (of norm 1/2) which are orthogonal to each other. There are 4 minimal vectors and 4 vectors of norm 1. By (3.1), if (P) meets Θ_j, then

$$\langle P,P\rangle=2+2(PO)-\begin{cases}\dfrac{3}{2} & (j=1,3)\\ 1 & (j=2)\\ 0 & (j=0)\,.\end{cases}$$

Hence P has norm 1/2 (resp. 1) if and only if $(PO)=0$ and (P) passes through Θ_1 or Θ_3 (resp. Θ_2).

(IV) In this case, there are 4 rational points of the form:

$$P=\left(\frac{t^2}{u^2}, \frac{t^3}{u^3}\right),$$

and the fundamental algebraic equation is given by

$$\Phi(u, \lambda)=(u^2-p_2)(u^2-q_2)=0\,.$$

(V) Let u_1, u_2 be the splitting variables such that

$$p_2=u_1^2\,, \quad q_2=u_2^2\,.$$

For generic (p_2, q_2), the extension $Q(u_1, u_2)/Q(p_2, q_2)$ is a Galois extension whose Galois group is the Weyl group $W(L)=W(A_1)\times W(A_1)=\{\pm 1\}\times\{\pm 1\}$. It is easy to see that our family is an excellent family for No. 26.

(VI) Let

$$Q_i=\left(\frac{t^2}{u_i^2}, \frac{t^3}{u_i^3}\right) \qquad (i=1, 2)\,.$$

We claim that Q_1, Q_2 are minimal vectors such that $\langle Q_1, Q_2\rangle=0$. Indeed, let σ be the automorphism of $Q(u_1, u_2)$ sending u_1 to $-u_1$ and leaving u_2 fixed. Then we have $Q_1^\sigma=-Q_1$, $Q_2^\sigma=Q_2$. On the other hand, we have $\langle Q_1^\sigma, Q_2^\sigma\rangle=\langle Q_1, Q_2\rangle$ in general (see [S2], Prop. 8.13). This proves $\langle Q_1, Q_2\rangle=0$. Now both Q_1, Q_2 have norm <2 by (3.1) mentioned above. Therefore the only possibility is that they are minimal vectors in $M=A_1^{*\oplus 2}$. (The claim can be also proven by other argument which works even if the extension is trivial; cf. example below).

(VII) The frame invariant is given by

$$J(\lambda)=p_2q_2(p_2-q_2)\,,$$

while $\delta=\{(2u_1)(2u_2)\}^2=2^4p_2q_2$. Hence the MWL is nondegenerate if and only if $u_1\neq 0$, $u_2\neq 0$, $u_1\pm u_2\neq 0$.

(VIII) *Example.* Let $u_1=1$, $u_2=2$. Then we get the elliptic curve

$$y^2=x^3+4tx^2-5t^3x+t^5\,,$$

and the rational points

$$Q_1=(t^2, t^3)\,, \quad Q_2=\left(\frac{t^2}{4}, \frac{t^3}{8}\right)$$

forming an orthogonal basis of $E(Q(t))\simeq M=A_1^{*\oplus 2}$. By the addition theorem, we have

$$Q_1-Q_2=(-4t+t^2, 6t^2-t^3)\,, \quad Q_1+Q_2=\left(\frac{1}{9}(-36t+t^2), \frac{1}{27}(126t^2+t^3)\right);$$

these are also short vectors and, in fact, $\{\pm Q_1\pm Q_2\}$ are all the vectors of norm 1. Note also that the above expression implies that (Q_1) and (Q_2) pass through different irreducible components at $t=0$; by renumbering if necessary, (Q_1) meets Θ_1 and (Q_2)

meets Θ_3, and $\{\pm Q_1 \pm Q_2\}$ meet Θ_2. The 4 roots in $L=A_1^{\oplus 2}$ are given by $\pm 2Q_1$, $\pm 2Q_2$; for instance,

$$2Q_1=\left(\frac{1}{4}(9+2t+t^2), \frac{1}{8}(-27-33t+3t^2+t^3)\right).$$

This shows directly that Q_1 is a minimal vector, since $2Q_1$ has norm ≤ 2 as a short vector so that $\langle Q_1, Q_1\rangle \leq 2/2^2=1/2$. Combined with the above formula for $Q_1 \pm Q_2$ showing that their norm is less than 2, we deduce that $|\langle Q_1, Q_2\rangle| < 1/2$, which implies that $\langle Q_1, Q_2\rangle=0$ by (3.2) and [S2], p. 229.

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