Elliptic fibrations of maximal rank on a supersingular K3 surface *

Tetsuji Shioda

To I.R.Shafarevich for his 90-th birthday

Abstract

We study a class of elliptic K3 surfaces defined by an explicit Weierstrass equation to find elliptic fibrations of maximal rank on K3 surface in positive characteristic. In particular, we show that the supersingular K3 surface of Artin invariant 1 (unique by Ogus) admits at least one elliptic fibration with maximal rank 20 in every characteristic $p > 7$, $p \neq 13$, and further that the number, say $N(p)$, of such elliptic fibrations (up to isomorphisms), is unbounded when $p \to \infty$; in fact, we prove that $\lim_{p\to\infty} N(p)/p^2 \geq (1/12)^2$.

1 Introduction

For a smooth projective K3 surface X defined over an algebraically closed field k, the Picard number $\rho := \mathrm{rkNS}(X)$ has the well-known upper bound:

$$\rho \leq \begin{cases} h^{1,1} = 20 & \text{if } k = \mathbf{C} \\ b_2 = 22 & \text{if } p = \mathrm{char}(k) > 0. \end{cases} \tag{1.1}$$

When X has an elliptic fibration with a section $\Phi : X \to \mathbf{P}^1$, i.e. for an elliptic K3 surface (X, Φ), the Mordell-Weil rank r has accordingly the upper bound:

$$r \leq \begin{cases} 18 & \text{if } k = \mathbf{C} \\ 20 & \text{if } p = \mathrm{char}(k) > 0. \end{cases} \tag{1.2}$$

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In this note, we focus on the case of maximal rank in (1.1) and (1.2).

First consider (1.1). The K3 surfaces over $k = \mathbf{C}$ with maximal Picard number $\rho = 20$ are called *singular* K3 surfaces. The simplest examples are Kummer surfaces of product of two isogenous elliptic curves with complex multiplication, and they have played an important role in the proof of the Torelli theorem for K3 surfaces in the celebrated work of Piateckii-Shapiro and Shafarevich [12]. A complete classification of singular K3 surfaces has been given by Inose-Shioda [6], based on [12] and Kodaira's theory of elliptic surfaces [7], in terms of positive-definite even integral binary quadratic forms on the transcendental lattice of such a surface.

The K3 surfaces in positive characteristic with maximal $\rho = 22$ are called *supersingular* K3 surfaces. By Artin [1], supersingular K3 surfaces form a 9-dimentional moduli space which has the stratification by means of the Artin invariant $\sigma (1 \leq \sigma \leq 10)$ defined by $|\det(\mathrm{NS}(X))| = p^{2\sigma}$. The simplest example of supersingular K3 surfaces are the Kummer surfaces of product of two supersingular elliptic curves, which have $\sigma = 1$. The uniqueness of the supersingular K3 surface with Artin invariant 1 is proven by Ogus [10].

Next, as for (1.2), we need to find a singular or supersingular K3 surface having some elliptic fibration without reducible fibres. The first example of complex elliptic K3 surfaces (X, Φ) with $r = 18$ having explicit defining equation has been found by Kuwata [8] by using Inose's fibration on Kummer surfaces [4, 5] and taking suitable base change to kill the reducible fibres; more generally, he gets most values of rank in the range $r \leq 18$ $(k = \mathbf{C})$ in this way. In our previous papers ([16, 17, 18, 19]), we have studied the sections of these elliptic K3 surfaces by the method of Mordell-Weil lattices [15].

In this note, we extend the above construction to characteristic $p > 0$ to find K3 surfaces with elliptic fibration of maximal rank 20. Thus we consider the elliptic K3 surfaces defined by the Weierstrass equation:

$$F^{(n)}_{\alpha,\beta} : \ y^2 = x^3 - 3\alpha x + (t^n + \frac{1}{t^n} - 2\beta) \tag{1.3}$$

where α, β are arbitrary in k and $n = 1, 2, \ldots, 6$. By Inose [5], $F^{(2)}_{\alpha,\beta}$ is isomorphic to the Kummer surface of the product of two elliptic curves C_1, C_2:

$$F^{(2)}_{\alpha,\beta} \cong (\mathrm{Km}(C_1 \times C_2), \Phi) \tag{1.4}$$

where (α, β) and the absolute invariants j_1, j_2 of C_1, C_2 are related by

$$\alpha^3 = j_1 j_2, \quad \beta^2 = (1 - j_1)(1 - j_2). \tag{1.5}$$

[N.B. Throughout this paper, the absolute invariant j is normalized in the classical way so that $j = 1$ (instead of $j = 12^3$) for $y^2 = x^3 - x$.]

It will be shown among others that the maximal rank $r = 20$ is attained by taking $n = 5$ or 6 and by choosing C_1, C_2 to be non-isomorphic supersingular elliptic curves in the above construction.

The main results are summarized in the next section. The idea of proof in characteristic p will be discussed in §3, together with necessary preparation, and the proof will be given in §4. Examples and remarks are given at the end of the note.

2 Main results for $p > 0$

Let $\mathrm{char}(k) = p > 0$. In the following, we always assume that

$$p > 3 \text{ (for } n \leq 4 \text{ or } n = 6) \quad \text{and} \quad p > 5 \text{ (for } n = 5). \tag{2.1}$$

Theorem 2.1 *For any $n \leq 6$, the rank of the Mordell-Weil lattice $\mathrm{MW}(F^{(n)}_{\alpha,\beta}) = F^{(n)}_{\alpha,\beta}(k(t))$ is given by the following formula:*

$$r^{(n)}_{\alpha,\beta} = h + \mathrm{Min}\{4(\mathrm{n} - 1), 16\} - \begin{cases} 0 & \text{if } j_1 \neq j_2 \\ n & \text{if } j_1 = j_2 \neq 0, 1 \\ 2n & \text{if } j_1 = j_2 = 0 \text{ or } 1 \end{cases} \tag{2.2}$$

where h denotes the rank of the group of homomorphisms $C_1 \to C_2$:

$$h = \mathrm{rk}\ \mathrm{Hom}(C_1, C_2) \in \{0, 1, 2, 4\}. \tag{2.3}$$

As an immediate consequence, we have

Corollary 2.2 *In characteristic $p > 0$, we have:*

$$r^{(n)}_{\alpha,\beta} = 20 \Leftrightarrow h = 4, j_1 \neq j_2, n = 5 \text{ or } 6. \tag{2.4}$$

Note that the case $h = \mathrm{rkHom}(C_1, C_2) = 4$ occurs if and only if both C_1 and C_2 are supersingular elliptic curves in characteristic $p > 0$. Let us denote by $\mathcal{H}_p$ the set of supersingular j-invariants in characteristic p:

$$\mathcal{H}_p := \{j = j(C) | C : \text{supersingular elliptic curve in char. } p\}. \tag{2.5}$$

We introduce a new notation:

$$S_{j_1,j_2}^{(n)} := F_{\alpha,\beta}^{(n)}, \quad M_{j_1,j_2}^{(n)} := \mathrm{MW}(S_{j_1,j_2}^{(n)}) \tag{2.6}$$

where

$$\alpha = \sqrt[3]{j_1 j_2}, \quad \beta = \sqrt{(1-j_1)(1-j_2)} \tag{2.7}$$

(the choice of cube or square roots here is irrelevant).

Theorem 2.3 *Assume that $j_1, j_2 \in \mathcal{H}_p$. Then, for any $n \leq 6$, the elliptic surface $S_{j_1,j_2}^{(n)}$ is a supersingular K3 surface of Artin invariant equal to 1, i.e.*

$$|\det(\mathrm{NS}(S_{j_1,j_2}^{(n)}))| = p^2. \tag{2.8}$$

If $j_1 \neq j_2$ and $n = 5$ or 6, then the Mordell-Weil lattice $\mathrm{MW}(S_{j_1,,j_2}^{(n)})$ has maximal rank 20 and determinant p^2:

$$\det \mathrm{MW}(S_{j_1,j_2}^{(n)}) = p^2. \tag{2.9}$$

Now recall that a supersingular K3 surface of Artin invariant 1 is unique up to isomorphisms in a given characteristic p by Ogus [10].

Corollary 2.4 *For any $j_1, j_2 \in \mathcal{H}_p$ and any $n \leq 6$, $S_{j_1,j_2}^{(n)}$ are isomorphic to each other as K3 surfaces.*

In the other direction, most of these K3 surfaces are not isomorphic as elliptic surfaces. In particular, this observation leads to the following formulation:

Theorem 2.5 *Let $X(p)$ be a fixed supersingular K3 surface of Artin invariant 1 in char p. Then for any $p > 7, p \neq 13$, $X(p)$ admits at least two non-isomorphic elliptic fibrations of maximal rank 20. More precisely, if we denote by $N(p)$ the number of non-isomorphic elliptic fibrations of maximal rank 20 on $X(p)$, then we have*

$$N(p) \geq |\mathcal{H}_p| \cdot (|\mathcal{H}_p| - 1) \tag{2.10}$$

Therefore $N(p)$ is unbounded when $p \to \infty$, and we have

$$\lim_{p\to\infty} \frac{N(p)}{p^2} \geq (\frac{1}{12})^2. \tag{2.11}$$

3 Preliminaries for the proof

First we recall that the proof of the rank formula (2.2) in char 0 is based on the following three facts:

(1) the Picard numbers of two K3 surfaces X, Y are equal if there is a dominant rational map $X \to Y$ (Inose [4], [19, Lem.3.1]);

(2) the standard formula relating the Picard number of an elliptic surface (S, Φ) and the MW-rank:

$$\rho = r + 2 + \sum_v (m_v - 1), \tag{3.1}$$

where the summation runs over $v \in \mathbf{P}^1$ such that $\Phi^{-1}(v)$ is a reducible fibre with $m_v > 1$ irreducible components. (see [15]).

(3) the Picard number of a Kummer surface:

$$\rho(\mathrm{Km}(C_1 \times C_2)) = 16 + \rho(C_1 \times C_2) = 18 + \mathrm{rkHom}(C_1, C_2). \tag{3.2}$$

In char $p > 0$, while (2) and (3) above continue to hold true, the fact (1) is not always true. We have only a weaker statement:

(1*) If there is a dominant map $X \to Y$ between two K3 surfaces, then

$$\rho(X) \leq \rho(Y). \tag{3.3}$$

This follows from the inequality of Lefshetz number $\lambda(X) := b_2(X) - \rho(X)$:

$$\lambda(X) \geq \lambda(Y) \tag{3.4}$$

which holds more generally for any two surfaces with a dominant map $X \to Y$ and which implies for example that a unirational surface is supersingular in the sense that $\rho = b_2$ (cf. [13]).

Thus we proceed as follows. We shall construct suitable sublattices of the Mordell-Weil lattices

$$M^{(n)} := MW(F^{(n)}_{\alpha,\beta}) \tag{3.5}$$

to match the desired rank formula.

We refer to our previous paper [17] for the proof of the first two of the following Lemmas. As is well-known, $\mathrm{Hom}(C_1, C_2)$ has the structure of an even integral lattice by taking the norm of a homomorphism $\varphi : C_1 \to C_2$ to be $2\deg(\varphi)$. (Note the change of notation that the norm of φ was taken to be $\deg(\varphi)$ in [17].)

Given a lattice L, we denote by $L[n]$ the lattice structure on L with the norm (or pairing) multiplied by n. For the root lattices A_2, E_8 etc., we refer to [2].

Lemma 3.1 *([17, Th.1.1]) Assume $j_1 \neq j_2$, Then there is a natural isomorphism*

$$M^{(1)} \simeq \mathrm{Hom}(C_1, C_2), \quad R_\varphi \leftrightarrow \varphi \tag{3.6}$$

such that the height of $R_\varphi \in M^{(1)}$, $\langle R_\varphi, R_\varphi \rangle$, is equal to $2\deg(\varphi)$.

Lemma 3.2 *([17, Th.1.2]) Assume $j_1 \neq j_2$, Then $M^{(2)}$ contains the sublattice*

$$M^{(2)} \supset M^{(1)}[2] \oplus (A_2^*)^{\oplus 2}[2] \tag{3.7}$$

with finite index which is equal to 2^h and where A_2^* is the dual of the root lattice A_2. In particular, we have

$$r^{(2)} = r^{(1)} + 4, \quad r^{(1)} = h. \tag{3.8}$$

Lemma 3.3 *Let $n = 2m$ for $m = 2$ or 3. Assume $j_1 \neq j_2$, Then $M^{(n)}$ contains the sublattice*

$$M^{(n)} \supset M^{(m)}[2] \oplus E_8[2] \tag{3.9}$$

with finite index which is a 2-power. Thus we have

$$r^{(4)} = r^{(2)} + 8, \quad r^{(6)} = r^{(3)} + 8. \tag{3.10}$$

Proof This is based on the well-known fact that, given an elliptic curve E/K and a separable quadratic extension field K'/K, there is an elliptic curve E'/K (called the K'/K-twist of E/K) such that we have the inclusion of Mordell-Weil groups:

$$E(K') \supset E(K) + E'(K) \supset 2E(K'), \tag{3.11}$$

with $E(K) \cap E'(K)$ a finite 2-torsion group. Applying this to the case $E = F^{(m)}_{\alpha,\beta}$ $(m = 2, 3)$, $K = k(t)$ and $K' = k(t')(t'^2 = 2)$, we find that

$$F^{(n)}(k(t)) \simeq F^{(m)}(k(t')) \supset F^{(m)}(k(t)) + E'(K) \tag{3.12}$$

where the twist

$$E' : y^2 = x^3 - 3\alpha t^2 x + t^3(t^m + \frac{1}{t^m} - 2\beta) \tag{3.13}$$

is a rational elliptic surface. As is easily seen, the Mordell-Weil lattice $E'(K)$ is the root lattice E_8 ([9]) since there are no reducible fibres under the assumption that $j_1 \neq j_2$ and $m = 2$ or 3. Noting that the height of $P \in E(K)$ gets multplied by the degree $[K' : K] = 2$ as an element of $E(K')$ ([15]), we have proven the assertion (3.9). (The same argument for the case $n = 2, m = 1$ proves Lemma 3.2.) *q.e.d.*

Lemma 3.4 *Assume that $n = 5$ or 3, and $j_1 \neq j_2$, Then $M^{(n)}$ contains the sublattice*

$$M^{(n)} \supset M^{(1)}[n] \oplus \tilde{L} \tag{3.14}$$

with finite index, where $\tilde{L}$ is a lattice of rank 16 for $n = 5$ (or of rank 8 for $n = 3$). Therefore we have

$$r^{(3)} = r^{(1)} + 8, \quad r^{(5)} = r^{(1)} + 16. \tag{3.15}$$

Proof We prove the case $n = 5$ of the claim (3.14) here, and omit the case $n = 3$, since the same argument works.

Denoting by $E/k(w)$ the elliptic curve:

$$E : y^2 = x^3 - 3\alpha x + w - 2\beta, \tag{3.16}$$

and letting $s = t + 1/t, \quad T = t^n, \quad w = T + 1/T$, we have

$$M^{(n)} = E(k(t)), \quad M^{(1)} = E(k(T)). \tag{3.17}$$

Note (cf. [18]) that $k(t)/k(w)$ is a Galois extension with the Galois group $G = \langle \sigma, \tau \rangle$ where $\sigma : t \to \zeta_n \cdot t, \tau : t \to 1/t$ and ζ_n is a primitive n-th root of unity. By Galois theory, the subfields $k(T)$, $k(s)$ and $k(w)$ of $k(t)$ are respectively the fixed fields of $< \sigma >$, $< \tau >$ and G.

Now let $L = E(k(s))$ be the Mordell-Weil group of $k(s)$-rational points of

$$E/k(s) : y^2 = x^3 - 3\alpha x + (s^5 - 5s^3 + 5s) - 2\beta \tag{3.18}$$

which is a rational elliptic surface. The structure of the Mordell-Weil lattice on L is again the root lattice E_8 under the assumption $j_1 \neq j_2$. Next let

$L' = L^\sigma = E(k(s'))$ be the image of L under σ where $s' = s^\sigma = \zeta t + 1/(\zeta t)$. Then $L' \simeq E_8$ has rank 8, and $\tilde{L} := L + L'$ (as a subgroup of $E(k(t))$) has rank 16, since $L \cap L' = E(k(s) \cap k(s')) = E(k(w)) = \{0\}$. Thus we have the inclusion of lattices:

$$M^{(5)} \supset M^{(1)}[5] + (L + L')[2], \tag{3.19}$$

where the two summands on the right hand side are orthogonal to each other. (For more details and also for the fact that $(L + L')[2] = M^{(5)}_{gen}$, see [19, Lem.7.2, 7.3, 7.4].)

Hence the rank $r^{(5)} = \mathrm{rk} M^{(5)}$ satisfies

$$r^{(5)} \geq r^{(1)} + 16. \tag{3.20}$$

On the other hand, by (3.1) and the general inequality (3.3), applied to the rational map $F^{(5)} \to F^{(1)}$, we have

$$\rho^{(5)} = r^{(5)} + 2 \leq \rho^{(1)} = r^{(1)} + 2 + 8 + 8. \tag{3.21}$$

Thus we conclude

$$r^{(5)} = r^{(1)} + 16, \tag{3.22}$$

which proves Lemma in case $n = 5$ under consideration. *q.e.d.*

4 Proof of Theorems

4.1 Proof of Theorem 2.1

By collecting the results about the ranks $r^{(n)}$ obtained above, i.e. by (3.8), (3.10) and (3.15), we have shown that, under the condition $j_1 \neq j_2$,

$$r^{(n)} = h + \mathrm{Min}\{4(\mathrm{n} - 1), 16\}, \quad \mathrm{h} = \mathrm{rkHom}(\mathrm{C}_1, \mathrm{C}_2) \tag{4.1}$$

for all $n \leq 6$, i.e. the rank formula (2.2) in any char p as in (2.1).

Thus we have proven Theorem 2.1 in case $j_1 \neq j_2$.

Next , to check the excluded case $j_1 = j_2$, we assume $C_1 = C_2 = C$ without loss of generality. Then the elliptic fibration on the Kummer surface $\mathrm{Km}(C_1 \times C_2)$ under consideration, i.e. the elliptic surface $F^{(2)} = F^{(2)}_{\alpha,\beta}$, has extra reducible fibres at $t \neq 0, \infty$, which arise from the graph of automorphisms of the elliptic curve C (see [17, Prop.7.1]). This makes the rank drop by 2, i.e. to $r^{(2)} = h + 2$ if $\mathrm{Aut}(C) = \{\pm 1\}$, and by 4, i.e. to $r^{(2)} = h$ if $|\mathrm{Aut}(C)| > 2$. Then, considering the base change from $F^{(1)}$ to $F^{(2)}$, we easily see that, in case $n = 1$, $r^{(1)} = h - 1$ or $h - 2 = 0$ according as $j(C) \neq 0, 1$ or otherwise. Thus we have verified (2.2) for $n = 1, 2$ in case $j_1 = j_2$.

The rest for $n = 3, 4, 5, 6$ and $j_1 = j_2$ can be verified by modifying the arguments given in Lemmas 3.3 and 3.4, by using the above result for $n = 1, 2$. This proves Theorem 2.1. *q.e.d.*

4.2 Proof of Theorem 2.3

Once the rank formula (2.2) is established, it is immediate that the Picard number of $S^{(n)}$ is independent of $n \leq 6$:

$$\rho(S^{(n)}_{j_1,j_2}) = \rho(F^{(n)}_{\alpha,\beta}) = h + 18 \quad (n \leq 6) \tag{4.2}$$

for any j_1, j_2 or any α, β, by using (3.1).

We assume that $j_1, j_2 \in \mathcal{H}_p$, $j_1 \neq j_2$. Then we have $h = 4$, and the elliptic surfaces $S^{(n)}_{j_1,j_2}$ are supersingular K3 for any $n \leq 6$. Let $\sigma(n)$ be its Artin invariant. We claim:

$$\sigma(n) = 1 \quad \text{for all } n \leq 6. \tag{4.3}$$

This is known to be true for $n = 2$, since $S^{(2)} \simeq \mathrm{Km}(C_1 \times C_2)$ with supersingular elliptic curves C_1, C_2.

We consider the determinant of Modell-Weil lattices. Let

$$d^{(n)} := \det M^{(n)} = \det \mathrm{MW}(S^{(n)}_{j_1,,j_2}), \quad (1 \leq n \leq 6). \tag{4.4}$$

For $n = 1$, Lemma 3.1 implies:

$$d^{(1)} = \det \mathrm{Hom}(C_1 \times C_2) = \det \mathrm{NS}(C_1 \times C_2) = p^2 \tag{4.5}$$

For $n > 2$, we shall see that $d^{(n)}$ is equal to $d^{(1)}$ times some rational number without p-factor, using Lemmas in the previous section.

First, for $n = 2m (m = 2, 3)$, Lemma 3.3 implies that

$$d^{(n)} = \det(M^{(m)}[2]) \cdot \det(E_8[2])/\nu^2 = d^{(m)} \cdot (\text{2-power}) \tag{4.6}$$

where ν is the index in (3.3) and a 2-power. Since $\det E_8 = 1$, both $d^{(4)}/d^{(2)}$ and $d^{(6)}/d^{(3)}$ are 2-powers. Similarly Lemma 3.2 implies that $d^{(2)}/d^{(1)}$ is of the form $2^a 3^b$, since $\det A_2^* = 1/3$.

Next, for $n = 5$ or $n = 3$, Lemma 3.4 implies that

$$d^{(n)} = \det(M^{(1)}[n]) \cdot \det(\tilde{L}[2])/\nu^2 \tag{4.7}$$

where ν is the index in (3.14). By [19], we have

$$\det \tilde{L} = \det M^{(n)}_{gen} = 5^4 (n = 5) \text{ or } 3^4/4^2 (n = 3). \tag{4.8}$$

It follows that $d^{(3)}/d^{(1)}$ and $d^{(5)}/d^{(1)}$ is a rational number of the form $2^a 3^b 5^c/\nu^2$.

Therefore the p-factor of $d^{(n)} (n \neq 5)$ is at most that of $d^{(1)} = p^2$ if $p > 3$, and the p-factor of $d^{(5)}$ is likewise at most p^2 if $p > 5$. We conclude that the p-factor of $d^{(n)}$ is at most p^2 for all $n \leq 6$, which implies the claim (4.3) that $\sigma(n) = 1$ for all n.

Finally, for $n = 5, 6$ where the trivial lattice is unimodular, $d^{(n)}$ is equal to $|\det NS(S^{(n)})|$, which implies $\det M^{(n)} = p^2$. *q.e.d.*

4.3 Proof of Theorem 2.5

We note first:

Lemma 4.1 *The elliptic fibrations of the following surfaces*

$$S^{(5)}_{j_1,j_2}, \ S^{(6)}_{j_1,j_2} \quad (j_1 \neq j_2, j_1, j_2 \in \mathcal{H}_p) \tag{4.9}$$

are mutually non-isomorphic.

(N.B. The suffix j_1, j_2 of $S^{(n)}_{j_1,j_2}$ is meant to be an unordered pair.)

Proof If two elliptic fibrations over $\mathbf{P}^1$ are isomorphic to each other, the configuration type of Kodaira fibres should be the same, and the set of places of singular fibres in $\mathbf{P}^1$ for one fibration should be transformed to that for the other by an automorphism of $\mathbf{P}^1$. It is easy to check that this cannot happen among the fibrations in (4.9), by using the fact that they are induced from $S^{(1)}_{j_1,j_2}$ by the base changes $t \to t^n$ for $n = 5$ or 6. We omit the computation. *q.e.d.*

Now we are ready to prove Theorem 2.5. Each surface in the list (4.9) is a supersingular K3 surface of Artin invariant 1 by Theorem 2.3, and hence it is isomorphic to a chosen $X(p)$ (by Ogus). Then (4.9) can be regarded as a set of elliptic fibrations of maximal rank 20 on $X(p)$, which are non-isomorphic to each other by the above lemma. In this way, we have proven the inequality (2.10). The cardinality $|\mathcal{H}_p|$ of the supersingular j-invariants in char p is given by the famous Deuring-Eichler-Igusa formula (see [3])

$$|\mathcal{H}_p| = \frac{p-1}{12} + \cdots . \tag{4.10}$$

In particular, we have $|\mathcal{H}_p| \geq 2$ iff $p = 11$ or $p > 13$. This completes the proof of Theorem 2.5. *q.e.d.*

4.4 Examples and remarks

Consider the elliptic curves

$$C_1 : y^2 = x^3 - 1 \;\text{ and }\; C_2 : y^2 = x^3 - x \tag{4.11}$$

with j-invariant $j_1 = 0$ and $j_2 = 1$. It is well-known that we have $0 \in \mathcal{H}_p$ if $p \equiv -1(3)$, and $1 \in \mathcal{H}_p$ if $p \equiv -1(4)$. Hence if $p \equiv -1(12)$, then $0, 1 \in \mathcal{H}_p$, and we can apply Theorem 2.3. Therefore the elliptic surfaces

$$S^{(n)}(p) : y^2 = x^3 + t^n + \frac{1}{t^n} \quad (n \leq 6) \tag{4.12}$$

in char p are mutually isomorphic as K3, and $S^{(5)}(p)$ and $S^{(6)}(p)$ give two non-isomorphic elliptic fibrations of rank 20 on the supersingular K3 surface with Artin invariant $\sigma = 1$.

Let $p = 11$ and $n = 5$ or 6. The Mordell-Weil lattice M of $S^{(n)}(p)$ is a positive-definite even integral lattice of rank 20 and det p^2. The minimal norm (height) of M is 4 and the number of minimal sections, i.e. the kissing number, is 12540. The center density of the lattice M is $\delta(M) = 1/11$. See [11] for more details about this lattice.

Remark 1 Recall ([2]) that the densest lattice known in dimension 20 has the center density $\delta(\Lambda_{20}) = 1/8$. Thus, if we *knew* that the number $N(p)$ (of elliptic fibrations of rank 20 in Theorem 2.5) is at least 1 for $p = 7$, the same argument as above would have produced a lattice of rank 20 and with center density $\delta(M) = 1/p = 1/7$, a new record! But, as $|\mathcal{H}_p| = 1$ for $p = 7$, we cannot find $j_1 \neq j_2$ to apply Theorem 2.3.

Remark 2 If characteristic $p = 3$, the method of this paper gives again the surface $y^2 = x^3 + t^n + 1/t^n$ (for $n \neq 3, 6$), which is not an elliptic, but a quasi-elliptic K3 surface and the Mordell-Weil group is a finite group.

Remark 3 In this note, we have considered elliptic fibrations of maximal rank on a supersingular K3 surface only in case the Artin invariant $\sigma = 1$. As to the existence of such in case $\sigma > 1$, see [14, Ex.8].

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Rikkyo University, Department of Mathematics
3-34-1 Nishi-Ikebukuro Toshima-ku, Tokyo 171-8501, Japan
(shioda@rikkyo.ac.jp)