

Elliptic Surfaces and Davenport-Stothers Triples

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Abstract

Two interesting topics, elliptic surfaces and integral points, have been studied so far rather independently, almost without any cross-references. Their close connection will be discussed, centering around the special theme of Davenport-Stothers triples.

1 Introduction

Davenport [4] has proven that, for any non-constant complex polynomials $f(t)$ and $g(t)$ such that $f^3 \neq g^2$, the degree of the difference $f^3 - g^2$ has the lower bound

$$\deg(f^3 - g^2) \geq \frac{1}{2} \deg(f) + 1 \quad (1)$$

(see §3 below). In 1981, Stothers [25] has generalized (1) to what is now known as the *abc*-theorem (or sometimes Mason's theorem, cf. [7], [8]): for any non-constant relatively prime polynomials a, b, c such that $a + b + c = 0$, the degree of a, b, c is bounded above by the number $N_0(abc)$ of distinct zeros of abc minus one, i.e.

$$\max(\deg(a), \deg(b), \deg(c)) \leq N_0(abc) - 1, \quad (2)$$

and he has studied the cases of equality in (1) in detail.

In view of their beautiful work, we propose to call a triple $\{f, g, h\}$ a *Davenport-Stothers triple* (or a *DS-triple*) of order m if it satisfies the following condition:

$$f^3 - g^2 = h, \quad \deg(f) = 2m, \quad \deg(g) = 3m, \quad \deg(h) = m + 1. \quad (3)$$

(In [25], the pair $[f, g]$ satisfying (3) is called a $(2, 3)$ -pair of order m , but we find it useful to treat $\{f, g, h\}$ as a triple.) In this terminology, Stothers has proven among others the existence of DS-triples of order m for every $m \geq 1$ and the finiteness of the number of essentially distinct DS-triples of order m . Moreover he has enumerated this number (let us denote it by $St(m)$ in this paper) as an explicit function of m ([25, Th.4.6]); in particular, it implies that

$$St(m) = 1 \ (m = 1, 2, 3, 4), \ St(5) = 4, \ St(6) = 6, \ St(7) = 19, \dots \quad (4)$$

and that $St(m)$ grows quite rapidly with m , tending to infinity.

These results of Davenport and Stothers answer some conjectures made by Birch and others [2, §1]. A few explicit examples of DS-triples are known for small m (cf. [2], [4], [5], [27]) some of which will be recalled in §5 below.

The DS-triples are closely related to such objects as certain ramified covers with three point ramification of the complex projective line \mathbf{P}^1 , the permutation representations of the fundamental group $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ and certain finite index subgroups of the modular group $PSL(2, \mathbf{Z})$. In fact, the method of Stothers is based on careful analysis of these relationship.

In this paper, we study this topic by introducing very natural geometric objects, i.e. *elliptic surfaces*. As is wellknown, the general theory of elliptic surfaces has been established by Kodaira [6] in 1960's. The classification of singular fibres has also been given by Néron in algebraic context, simplified later by Tate ([11], [26]). More specific feature of elliptic surfaces has been studied by many authors since 70's. Especially there are quite extensive work on the configuration of singular fibres on rational or K3 elliptic surfaces (e.g. [1], [9], [10], [12], [13], etc.). Note that the information of singular fibres is completely determined by the behavior of the absolute invariant J and the discriminant Δ .

We consider two types of elliptic surfaces $S_{f,g}$ and \mathcal{S}_h over \mathbf{P}^1 (with the generic fibre $E_{f,g}$ and \mathcal{E}_h being elliptic curves over $k(t)$) to be defined below in §2. Given a polynomial triple $\{f, g, h\}$ such that $f^3 - g^2 = h$, $P = (f, g)$ gives an *integral point* of \mathcal{E}_h , while the discriminant of $S_{f,g}$ is equal to h up to a constant. (This interplay of $S_{f,g}$ and \mathcal{S}_h is inspired by the proof of the famous Shafarevich theorem ([16]) for the finiteness of elliptic curves over \mathbf{Q} with good reduction outside a given finite set of primes.)

Now any elliptic surface over \mathbf{P}^1 (t -line) with a section is isomorphic to $S_{f,g}$ for some polynomials f, g , whose discriminant Δ is equal to h up to a

constant. Thus the determination of elliptic surfaces with given configuration of singular fibres is closely related to the problem of determining the *integral points* on the elliptic curve \mathcal{E}_h . It seems to us that this view has escaped so far the major attention, and as a consequence, two interesting themes of elliptic surfaces and integral points have been developed independently, almost without any cross-reference between them. It will be natural to build a bridge to connect these two interesting subjects, and it is from such a viewpoint that we deal with a very special topic centering around Davenport-Stothers triples.

As the first result, we can characterize the DS-triples $\{f, g, h\}$ in terms of elliptic surfaces $S_{f,g}$ or \mathcal{S}_h as follows: $S_{f,g}$ has a “maximal” singular fibre, or \mathcal{S}_h has an integral point $P = (f, g)$ of “maximal” possible height (see Theorem 2.1 for more precise statement). Also, without explicit reference to the triple $\{f, g, h\}$, we can put DS-triples in even richer, tighter connection with other interesting subjects than Stothers. Indeed, we shall show that there are seven equivalent data for a fixed order m : (C1) DS-tuple, (C2) $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$ with 3 points ramification, (C3) elliptic surface with a maximal singular fibre, (C4) integral point of maximal possible height, (C5) permutation representation of $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$, (C6) finite index subgroup $\bar{\Gamma}$ of $PSL(2, \mathbf{Z})$, (C7) finite index subgroup Γ of $SL(2, \mathbf{Z})$, all with some additional condition (see Theorem 2.2 in §2). While (C1), (C2), (C5), (C6) already appear in Stothers’ work, the data (C3), (C4) and (C7) are defined in terms of elliptic surfaces.

For elliptic surfaces, this connection together with Stothers’ work imply the existence and finiteness of elliptic surfaces over \mathbf{P}^1 with a “maximal” singular fibre for any m (Theorem 2.3). Note that the existence (and uniqueness) of such has been known so far only for $m \leq 4$ (cf. [13] for $m = 1$, [1] for $m = 2$, [10] for $m = 3$, [9] for $m = 4$; cf. our recent report [24] for $m = 3, 4$). Moreover, for any DS-triple $\{f, g, h\}$, the elliptic surface $S_{f,g}$ turns out to be an elliptic modular surface in the sense of our earlier work [18] (Theorem 2.4), which seems to be worthy of further investigation.

Another topics connected with DS-triples (which will not be discussed in this paper) is Grothendieck’s dessins d’enfants, since the rational function $J = f^3/h$ obtained from any DS-triple $\{f, g, h\}$ is a so-called Belyi-function by (C3). It seems that this aspect of Stothers work has been strangely forgotten in the related fields (cf. [14], [29]) in spite of its highly original nature.

The paper is organized as follows. In §2, we fix general notation and state some of our main results. In §3 we prepare a few lemmas including

a proof of Davenport's inequality (1), for the proof of two characterization theorems (Theorem 2.1 and 2.2); they are proven in §4 and §6 respectively. In §5 in between, we give a few explicit examples of DS-triples, which form a complete set of representatives of essentially distinct DS-triples for $m \leq 5$ (Theorem 5.1). In §7, we make some supplementary remarks: about existence and finiteness (Theorem 2.3), or the relation to the elliptic modular surfaces (Theorem 2.4).

In §8, we go back to algebraic situation and determine all the integral points in \mathcal{E}_h when $\deg(h)$ is small (Theorem 8.2, etc). The DS-triples are characterized among them by the maximality condition (C4). In §9, we prove the triviality of the Mordell-Weil groups of $S_{f,g}$ for $m > 2$ (Theorem 9.1). In both sections, the height formula in the Mordell-Weil lattices [19] will play a crucial role.

In summary, our method is to combine three ideas: Kodaira's theory of elliptic surfaces [6], Shafarevich's idea using integral points ([16], §2.1), and Mordell-Weil Lattices [19].

2 Main results

2.1 General notation

Let k be an algebraically closed field of characteristic zero (later we take $k = \mathbf{C}$ when we consider the Riemann surface associated with an algebraic curve). For a pair of polynomials $f(t)$ and $g(t)$ in $k[t]$ such that $h := f^3 - g^2 \neq 0$, let $E_{f,g}$ denote the elliptic curve over $k(t)$, defined by the equation:

$$E_{f,g} : y^2 = x^3 - 3f(t)x - 2g(t). \quad (5)$$

Its discriminant $\Delta = \Delta(E_{f,g})$ is given by

$$\Delta = 4(-3f)^3 + 27(2g)^2 = -4 \cdot 3^3(f^3 - g^2) = -108 \cdot h, \quad (6)$$

and the absolute invariant $j(E_{f,g})$ is equal to the rational function J :

$$J = \frac{f^3}{h}, \quad J - 1 = \frac{g^2}{h}. \quad (7)$$

Next, for any $h(t) \in k[t], \neq 0$, let

$$\mathcal{E} = \mathcal{E}_h : Y^2 = X^3 - h(t) \quad (8)$$

be an elliptic curve over $k(t)$ with absolute invariant $j(\mathcal{E}) = 0$. This type of elliptic curves are quite special, but very important because they are a sort of *moduli space* for elliptic curves with a given discriminant. It is based on the following simple observation: Assume that $h = f^3 - g^2$. Then $P = (f, g)$ is an *integral point* of \mathcal{E}_h (in the sense that both coordinates belong to $k[t]$), and conversely any integral point of \mathcal{E}_h defines an elliptic curve (5) whose discriminant is equal to h up to constant. The group $\mathcal{E}_h(k(t))$ of $k(t)$ -rational points is finitely generated unless h is a 6-th power in $k[t]$, and it has a natural height pairing which defines the structure of Mordell-Weil lattice on $\mathcal{E}_h(k(t))$ (cf. [19]). The set $\mathcal{E}_h(k[t])$ of integral points forms a finite subset of bounded height (see Lemma 3.3 below). (As mentioned before, Shafarevich used a similar idea in arithmetic case.)

We denote by $S = S_{f,g}$ the elliptic surface over \mathbf{P}^1 defined by the equation (5), i.e. the Kodaira-Néron model of the elliptic curve $E = E_{f,g}$ over $k(t)$. Similarly we denote by $\mathcal{S} = \mathcal{S}_h$ the elliptic surface over \mathbf{P}^1 defined by the equation (8). (Here and in the following, by an elliptic surface, we mean a smooth projective elliptic surface over \mathbf{P}^1 with a section.) The type of singular fibres are defined as in Kodaira [6]. We use the special symbol $I_{5m-1}^{(*)}$ to denote a singular fibre of type I_{5m-1} for m even and of type I_{5m-1}^* for m odd.

2.2 Statement of main results

Fix a positive integer m , and let $f, g, h \in k[t]$, $k = \mathbf{C}$.

Theorem 2.1 *For a triple $\{f, g, h\}$ satisfying $f^3 - g^2 = h \neq 0$, the following conditions are equivalent to each other:*

- (c1) $\{f, g, h\}$ is a Davenport-Stothers triple of order m , i.e. by definition, $\deg(f) = 2m, \deg(g) = 3m, \deg(h) = m + 1$.
- (c2) Let $J = f^3/h$. The rational function $w = J(t)$ defines a ramified covering $J : \mathbf{P}_t^1 \longrightarrow \mathbf{P}_w^1$ of degree $6m$, unramified outside $\{0, 1, \infty\} \subset \mathbf{P}_w^1$, such that the ramification index is 3 at each of $2m$ points in $J^{-1}(0)$, 2 at each of $3m$ points in $J^{-1}(1)$, $5m - 1$ at $t = \infty$ and 1 at any other points.
- (c3) The elliptic surface $S = S_{f,g}$ has Euler number $e(S) = 12 \lfloor \frac{m+1}{2} \rfloor$ and a maximal singular fibre of type $I_{5m-1}^{(*)}$ at $t = \infty$. The other singular fibres are necessarily of type I_1 , and there are $m + 1$ of them.
- (c4) $\deg(h) = m + 1$ and $P = (f, g)$ is an integral point of maximum possible height $\langle P, P \rangle = 2m$ in $\mathcal{E}_h(k[t])$.

The above conditions are defined in terms of algebraic or algebraic-geometric data, depending explicitly on $\{f, g, h\}$. Implicitly, we can formulate further characterization involving some other condition of topological or complex analytic nature. Namely we have:

Theorem 2.2 *The following seven data (C1), ..., (C7) determine and are determined by each other (up to suitable equivalence):*

(C1) *A Davenport-Stothers triple of order m .*

(C2) *$J : \mathbf{P}_t^1 \longrightarrow \mathbf{P}_w^1$, a ramified covering of degree $6m$, with the ramification scheme as described in (c2).*

(C3) *An elliptic surface S over \mathbf{P}^1 with Euler number $e(S) = 12\lceil \frac{m+1}{2} \rceil$ having a maximal singular fibre of type $I_{5m-1}^{(*)}$ at $t = \infty$.*

(C4) *An integral point of maximum possible height $2m$ in $\mathcal{E}_h(k[t])$ (under the condition $\deg(h) = m + 1$).*

(C5) *A permutation representation of the fundamental group*

$$\sigma : \pi_1(\mathbf{P}_w^1 - \{0, 1, \infty\}) \longrightarrow \mathcal{S}_{6m}$$

such that the positively oriented loop around 0 (resp. 1, ∞) is mapped to $\sigma_0 = (3)^{2m}$ (resp. $\sigma_1 = (2)^{3m}$, $\sigma_\infty = (\sigma_0\sigma_1)^{-1} = (5m-1)(1)^{m+1}$), where $(n)^k$ denotes a product of k disjoint cyclic permutations of length n , and such that σ_0, σ_1 generate a transitive subgroup.

(C6) *An isomorphism of the fundamental group of $\mathbf{P}_t^1 - \Sigma$ into the modular group*

$$\bar{\rho} : \pi_1(\mathbf{P}_t^1 - \Sigma) \xrightarrow{\sim} \bar{\Gamma} \hookrightarrow PSL(2, \mathbf{Z})$$

such that $\bar{\Gamma} = \text{Im}(\bar{\rho})$ is a subgroup of index $6m$ in $PSL(2, \mathbf{Z})$, and the quotient of the upper half plane H by $\bar{\Gamma}$ is isomorphic to $\mathbf{P}^1 - \Sigma$ ($\Sigma = J^{-1}(\infty)$). Letting $\psi : H \rightarrow \mathbf{P}^1 - \Sigma$ be the universal covering map, we have $J(\psi(\tau)) = j(\tau)$ where $j : H \rightarrow \mathbf{C}$ is the elliptic modular function, and $t = \psi(\tau)$ is the Hauptmodul of $\bar{\Gamma}$ -modular functions.

(C7) *The monodromy representation of the fundamental group*

$$\rho : \pi_1(\mathbf{P}_t^1 - \Sigma) \longrightarrow SL(2, \mathbf{Z}) \quad (\Sigma = \{\alpha_1, \dots, \alpha_{m+1}, \infty\})$$

such that the positively oriented loop around α_i (resp. ∞) is mapped to an element γ_i conjugate to u (resp. γ_∞) where

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_\infty = (-1)^m \begin{pmatrix} 1 & 5m-1 \\ 0 & 1 \end{pmatrix}.$$

The image Γ of ρ is a torsionfree subgroup of $SL(2, \mathbf{Z})$, which is a lifting of $\bar{\Gamma}$ in (C6).

We note that the equivalence of the data (C1), (C2), (C5) and (C6) are in Stothers [25]. The conditions (C3), (C4) and (C7) are newly formulated in terms of elliptic surfaces. The obvious advantage of introducing elliptic surfaces as in (C3) is the interpretation of the rational function J in (C2) as the absolute invariant, as well as the existence and uniqueness of the natural lifting of the data $\bar{\Gamma} \subset PSL(2, \mathbf{Z})$ in (C6) to $\Gamma \subset SL(2, \mathbf{Z})$ as in (C7).

As for the existence, Stothers' results [25] and Theorem 2.2 imply the following:

Theorem 2.3 *For any $m \geq 1$, there exist at least one and at most finitely many objects in each of the data (C1), ..., (C7).*

In particular, there exist at least one and at most finitely many elliptic surfaces over \mathbf{P}^1 with a singular fibre of type $I_{5m-1}^{()}$ and $m+1$ singular fibres of type I_1 . The number of such (up to isomorphism) is given by Stothers' function $St(m)$.*

The proof of Theorem 2.2 also shows:

Theorem 2.4 *The elliptic surface S with a maximal singular fibre as in (C3) is isomorphic to the elliptic modular surface attached to the torsionfree subgroup Γ of $SL(2, \mathbf{Z})$ in (C7) in the sense of [18].*

3 Some lemmas

Let us prepare some lemmas for the proof of the Theorems 2.1 and 2.2.

Note that Lemma 3.1(i) below is equivalent to Davenport's inequality (1); we follow almost verbatim the original elegant proof of it due to Davenport [4]. With a slight modification, we can prove the assertion (ii) at the same time. (Compare [7] or [25, §1] for different proofs.) For (iii), we use the abc -theorem (2) to illustrate its relevance to DS-triples.

Lemma 3.1 *Let k be a field of characteristic zero. Given a pair of polynomials $f(t), g(t) \in k[t]$ of degree $2m, 3m$, let $h := f^3 - g^2$.*

(i) If $\deg(h) \leq m$, then $h = 0$, i.e. $f^3 = g^2$.

(ii) If $\deg(h) = m+1$ (i.e. if $\{f, g, h\}$ is a DS-triple), then both f and g have only simple zeros and they are coprime.

(iii) In this case, h has only simple zeros too.

Proof We may assume that f, g have highest coefficient 1. We assume that

$$d := \deg(h) = \deg(f^3 - g^2) \leq m + 1. \quad (9)$$

It means that the coefficients of $t^{6m}, t^{6m-1}, \dots, t^{m+2}$ (and t^{m+1} in case $d \leq m$) of f^3 and g^2 coincide. Hence the sum of the ν -th powers of the roots of $f(t)^3$ and $g(t)^2$ coincide for $\nu = 0, 1, \dots, 5m - 2$ (and $5m - 1$ in case $d \leq m$). Thus if

$$f(t) = (t - \xi_1) \cdots (t - \xi_{2m}), \quad g(t) = (t - \eta_1) \cdots (t - \eta_{3m}),$$

we have

$$3(\xi_1^\nu + \cdots + \xi_{2m}^\nu) = 2(\eta_1^\nu + \cdots + \eta_{3m}^\nu) \quad (10)$$

for $\nu = 0, 1, \dots, 5m - 2$ (and $5m - 1$ in case $d \leq m$).

Let $\theta_1, \dots, \theta_r$ be the distinct numbers of the set $\xi_1, \dots, \xi_{2m}, \eta_1, \dots, \eta_{3m}$ so that $r \leq 5m$. Then, on collecting together equal terms, the above equations take the form

$$c_1 \theta_1^\nu + \cdots + c_r \theta_r^\nu = 0 \quad (11)$$

for $\nu = 0, 1, \dots, 5m - 2$ (and $5m - 1$ in case $d \leq m$). If $c_1 = \cdots = c_r = 0$, then the last equation holds for all ν , hence so does (10), and we get $f^3 = g^2$ identically. Assume $f^3 \neq g^2$ so that we have $(c_i) \neq (0)$. We separate the two cases (i) $d \leq m$ and (ii) $d = m + 1$.

In case (i), it follows from (11) that

$$\det(\theta_i^\nu) = 0 \quad (1 \leq i \leq r, 0 \leq \nu \leq r - 1) \quad (12)$$

and this contradicts the fact that $\theta_1, \dots, \theta_r$ are distinct (van der Monde determinant). Hence we must have $f^3 = g^2$.

Next, in case (ii), it suffices to show that $r = 5m$. For it means that $\xi_1, \dots, \xi_{2m}, \eta_1, \dots, \eta_{3m}$ are all distinct, so that f, g have only simple roots and they have no common roots. Assume that $r \neq 5m$. Then we have $r - 1 \leq 5m - 2$ and we again have (12), which gives a contradiction.

To show (iii), let us use the *abc*-theorem (2). We can apply it to $f^3 - g^2 - h = 0$ since f^3, g^2 are relatively prime by (ii). While the maximum degree among f^3, g^2, h is $6m$, the number of distinct zeros of $f^3 g^2 h$ is the same as that of fgh . Thus we have by (2)

$$6m \leq N_0(fgh) - 1 \leq \deg(f) + \deg(g) + N - 1 = 5m + N - 1$$

where $N = N_0(h)$, the number of distinct zeros of h . It follows that $N \geq m + 1 = \deg(h)$, which implies that h has $m + 1$ distinct zeros. *q.e.d.*

Remark 1) For a field of positive characteristic p , the above proof of (i) works as far as $p > 6m$. Hence Davenport's inequality (1) is true if $p > 6m$. 2) The examples such as $(t^2)^3 - (t^3 + 1)^2 = 1$ ($p = 2$) or $(t^2 + 1)^3 - (t^3)^2 = 1$ ($p = 3$) show that Davenport's inequality (1) can fail in char. $p > 0$.

It is an open question to see if Davenport's inequality holds true for any characteristic $p > 3$ or not (cf. [24, §4]).

Lemma 3.2 *Let S be a complex elliptic surface over \mathbf{P}^1 with arithmetic genus $\chi = \chi(S)$. Then the number m_v of irreducible components of a singular fibre at $v \in \mathbf{P}^1$ has the upper bound $10\chi - 1$.*

Proof It follows from surface theory (e.g. [6]) that the Euler number $e(S) = 12\chi$, the second Betti number $b_2(S) = e(S) - 2$ and the geometric genus $p_g(S) = \chi - 1$ for an elliptic surface over \mathbf{P}^1 . Hence the Hodge number $h^{1,1}$ is equal to $b_2 - 2p_g = 10\chi$.

Now the Picard number formula (cf. [18], [19]) says

$$\rho(S) = r + 2 + \sum_v (m_v - 1) \quad (13)$$

where $r (\geq 0)$ is the Mordell-Weil rank. Then the Lefschetz-Hodge inequality $\rho \leq h^{1,1}$ implies the desired bound $m_v \leq 10\chi - 1$ for any v . [N.B. If $m_v = 10\chi - 1$ for some v , then we infer from (13) that $m_w = 1$ for all $w \neq v$ and that $r = 0$ and $\rho = h^{1,1}$ hold.] *q.e.d.*

Lemma 3.3 *Suppose that $h(t)$ is a polynomial of degree $m + 1$. For any integral point $P = (f, g)$ in $\mathcal{E}_h(k[t])$, its height $\langle P, P \rangle$ is bounded above by $2m$. This bound is attained if and only if $\{f, g, h\}$ is a DS-triple of order m .*

Proof For any rational point $P = (f, g) \in \mathcal{E}_h(k(t))$, its height (in the sense of Mordell-Weil lattice [19]) is given by

$$\langle P, P \rangle = 2\chi(\mathcal{S}) + 2(PO) - \sum_v \text{contr}_v(P) \quad (14)$$

where (PO) is the intersection number of the section (P) and the zero-section (O) in the surface $\mathcal{S} = \mathcal{S}_h$ (cf. [19, Theorem 8.6], [20, §2]). The last term is the sum of local contribution at reducible singular fibres (e.g. at $t = \infty$).

Suppose that P is an integral point, i.e. $f, g \in k[t]$ are polynomials and $\deg f = 2n$. Then the sections $(P), (O)$ can intersect only at a point of \mathcal{S} over $t = \infty$, and we have

$$(PO) = \max\{n - \chi(\mathcal{S}), 0\} \quad (15)$$

which can be seen easily by writing down P in terms of the coordinates at $t = \infty$: $\bar{t} = 1/t, \bar{X} = X/t^{2\chi}, \bar{Y} = Y/t^{3\chi}$. Hence we have

$$\langle P, P \rangle \leq \max\{2\chi(\mathcal{S}), 2n\}.$$

Now $\chi = \chi(\mathcal{S})$ is computed as follows. Writing $h = h_1 h_2^6$ with h_1 free from sixth power, we have $\chi = \deg(h_1)/6$ or $[\deg(h_1)/6] + 1$ according as $\deg(h_1)$ is divisible by 6 or not. Thus $\chi < \deg(h)/6 + 1$. On the other hand, we have $\deg(f) = 2n \leq 2m$ by Davenport's inequality (1). Therefore we have $\langle P, P \rangle \leq 2m$.

Note that the above argument shows that $\langle P, P \rangle < 2m$ in case $n < m$. Hence equality $\langle P, P \rangle = 2m$ holds only if $\{f, g, h\}$ is a DS-triple of order m . Conversely if $\{f, g, h\}$ is a DS-triple of order m , h has simple zeros only (Lemma 3.1(iii)), and hence the elliptic surface \mathcal{S} has irreducible singular fibres (of type *II*) at $t \neq \infty$ and a reducible fibre only at $t = \infty$. Thus the local contribution $\text{contr}_v(P)$ can be nonzero only at $v = \infty$. But the section (P) intersects with (O) since $(PO) > 0$ and this intersection occurs at $t = \infty$. Hence (P) passes through the identity component of the singular fibre over $t = \infty$, which gives $\text{contr}_v(P) = 0$. Hence the height formula gives $\langle P, P \rangle = 2\chi(\mathcal{S}) + 2(PO) = 2m$. *q.e.d.*

4 Proof of Theorem 2.1

4.1 (c1) \Leftrightarrow (c2)

Assume (c1). By Lemma 3.1, f, g, h are pairwise coprime and have only simple zeros. So the rational function $J = f^3/h$ defines a covering $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$ of degree $6m$. Since J has a pole of order $6m - (m + 1) = 5m - 1$ at $t = \infty$, the ramification index at $t = \infty$ is equal to $5m - 1$.

Let us check the other ramification. By (7) and Lemma 3.1, J has the ramification index 3 at every point in $J^{-1}(0)$, and 2 at every point in $J^{-1}(1)$. Then the Riemann-Hurwitz formula for the covering J gives the relation

$$-2 = 6m(-2) + (5m - 1 - 1) + (3 - 1) \cdot 2m + (2 - 1) \cdot 3m + V \quad (16)$$

where $V \geq 0$ is the sum of possible contribution from all the other ramification points. It follows that $V = 0$. In other words, the covering J is unramified outside $0, 1, \infty$ and also at $J^{-1}(\infty) - \{\infty\}$. This proves (c1) \Rightarrow (c2).

The converse is obvious.

4.2 (c1) \Leftrightarrow (c3)

Assume (c1). Then, by (6), the discriminant Δ of $E = E_{f,g}$ is equal to h up to a constant, which has degree $m + 1$. The singular fibres of the elliptic surface $S = S_{f,g}$ lie over the zeros of Δ and at $t = \infty$. Since h has only simple zeros, we have $m + 1$ singular fibres of type I_1 .

On the other hand, the absolute invariant of E is given by $J = f^3/h$, which has a pole of order $5m - 1$ at $t = \infty$. Hence the type of the singular fibre at $t = \infty$ is either I_{5m-1} or I_{5m-1}^* . We see below that it depends on the parity of m .

The Euler number $e(S)$ of the elliptic surface is given by

$$e(S) = 12\chi(S) = m + 1 + \begin{cases} 5m - 1 & \text{for } I_{5m-1} \\ 5m + 4 & \text{for } I_{5m-1}^* \end{cases} \quad (17)$$

where $\chi = \chi(S)$ is the arithmetic genus of S (see [6, Theorem 12.2], χ being denoted $p_a + 1$ there). Thus we have $\chi = m/2$ or $(m + 1)/2$ according to whether m is even or odd, i.e. $\chi = \lfloor \frac{m+1}{2} \rfloor$, and the type of the singular fibre at $t = \infty$ is I_{5m-1} if m is even and I_{5m-1}^* if m is odd. (Recall that we denote this type by $I_{5m-1}^{(*)}$ in this paper.) By Lemma 3.2, this is a maximal singular fibre, since $10\chi - 1 = 5m - 1$ or $5m + 4$ according to the parity of m . This proves (c1) \Rightarrow (c3).

Conversely, assume (c3). Then the discriminant Δ has $m + 1$ simple zeros at $t \neq \infty$, which implies $\deg h = m + 1$. Since the j -invariant $J = f^3/h$ has a pole of order $5m - 1$ at $t = \infty$, $\deg f$ is equal to $2m$. Hence $\{f, g, h\}$ is a DS-triple of order m . This proves (c3) \Rightarrow (c1).

4.3 (c1) \Leftrightarrow (c4)

This is proven in Lemma 3.3, §3.

This completes the proof of Theorem 2.1.

5 Examples of DS-triples

Before going further, we give a few explicit examples of DS-triples of small order m . Two triples are regarded as essentially the same if one is obtained from the other via (i) the change of variable $t \rightarrow at + b$ ($a \neq 0$), (ii) replacing $\{f, g, h\}$ by $\{c^2f, c^3g, c^6h\}$ for some $c \neq 0$, or combination of these operations.

The first five examples are essentially the same as those in [2] ($m = 3, 5$) and [5] ($m = 1, 2, 4$). Three other examples for $m = 5$ are computed by us a few years ago; we label the four examples for $m = 5$ as B, C_+, C_- or A in this order for the sake of later use.

- **m=1**

$$f = t^2 - 1, \quad g = t^3 - \frac{3}{2}t, \quad h = \frac{3}{4}t^2 - 1$$

- **m=2**

$$f = t^4 - 4t, \quad g = t^6 - 6t^3 + 6, \quad h = 8t^3 - 36$$

- **m=3** (Birch [2])

$$f = t^6 + 4t^4 + 10t^2 + 6, \quad g = t^9 + 6t^7 + 21t^5 + 35t^3 + \frac{63}{2}t,$$

$$h = 27t^4 + \frac{351}{4}t^2 + 216$$

- **m=4** (Hall [5])

$$f = t^8 + 6t^7 + 21t^6 + 50t^5 + 86t^4 + 114t^3 + 109t^2 + 74t + 28,$$

$$g = t^{12} + 9t^{11} + 45t^{10} + 156t^9 + 408t^8 + 846t^7 + 1416t^6$$

$$+ 1932t^5 + 2136t^4 + 1873t^3 + \frac{2517}{2}t^2 + \frac{1167}{2}t + \frac{299}{2},$$

$$h = -\frac{27}{4}(4t^5 + 15t^4 + 38t^3 + 61t^2 + 62t + 59)$$

- **m=5** B (Birch [2])

$$f = t(t^9 + 12t^6 + 60t^3 + 96),$$

$$g = t^{15} + 18t^{12} + 144t^9 + 576t^6 + 1080t^3 + 432,$$

$$h = -1728(3t^6 + 28t^3 + 108).$$

- **m=5** C_+

$$\begin{aligned}
f &= t^{10} + 26t^8 + 7(34 + 3\sqrt{-3})t^6 + 24(35 + 18\sqrt{-3})t^4 \\
&\quad + \frac{3}{2}(371 + 1509\sqrt{-3})t^2 + 3(-775 + 543\sqrt{-3}), \\
g &= t(t^{14} + 39t^{12} + \frac{3}{2}(407 + 21\sqrt{-3})t^{10} + \frac{5}{2}(1921 + 423\sqrt{-3})t^8 \\
&\quad + 18(1028 + 717\sqrt{-3})t^6 + 54(253 + 1260\sqrt{-3})t^4 \\
&\quad + \frac{1}{4}(-616509 + 535437\sqrt{-3})t^2 \\
&\quad + \frac{1}{4}(-1524069 + 136485\sqrt{-3})), \\
h &= \frac{27}{8}(17755915 - 17284173\sqrt{-3}) \cdot \\
&\quad (t^6 + 18t^4 + (87 + 18\sqrt{-3})t^2 + (16 + 240\sqrt{-3})).
\end{aligned}$$

- **m=5** C_- [conjugate : change the sign of $\sqrt{-3}$ in the above.]

- **m=5** A

$$\begin{aligned}
f &= t^{10} + \frac{65}{3}t^8 + \frac{45}{2}t^7 + \frac{6895}{36}t^6 + \frac{3829}{12}t^5 + \frac{165175}{144}t^4 \\
&\quad + \frac{51605}{36}t^3 + \frac{1678945}{432}t^2 + \frac{449155}{144}t + \frac{8849347}{2592}, \\
g &= t^{15} + \frac{65}{2}t^{13} + \frac{135}{4}t^{12} + \frac{1390}{3}t^{11} + \frac{3377}{4}t^{10} + \frac{118450}{27}t^9 \\
&\quad + \frac{68695}{8}t^8 + \frac{1074955}{36}t^7 + \frac{3658145}{72}t^6 + \frac{1161107}{9}t^5 + \frac{6662225}{36}t^4 \\
&\quad + \frac{13571900915}{41472}t^3 + \frac{5019815}{16}t^2 + \frac{31879878445}{82944}t + \frac{21951424793}{165888}, \\
h &= (5^{25}/2^{22}3^{12})(432t^6 + 6480t^4 + 7560t^3 + 35820t^2 + 54972t + 166675).
\end{aligned}$$

Theorem 5.1 For $m \leq 5$, the above examples give a complete set of representatives of essentially distinct DS-triples of order m .

Proof This follows from Stothers' enumeration (4).

q.e.d.

Remark Purely algebraic proof is possible for small m . In fact, this must have been known to the above mentioned authors by direct algebraic computation at least for $m \leq 4$, although the uniqueness is not mentioned. On

the other hand, we shall indicate more conceptual algebraic proof based on the theory of Mordell-Weil lattices (§8).

6 Proof of Theorem 2.2

Before the proof, let us make clear what we mean by “suitable equivalence” of the various data. For (C1), we identify two essentially same DS-triples, as defined at the beginning of §5. For (C2), isomorphism of the coverings $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$ in the usual sense (as algebraic curves, or equivalently, as Riemann surfaces). For (C3), isomorphism of the elliptic surfaces over \mathbf{P}^1 (with ∞ fixed). For (C4), this is the same as for (C1). As for (C5), (C6), (C7), equivalence of representations, i.e. up to conjugation in the target group.

6.1

With these convention, it is easy to check the one-one correspondence between the data (C1), ..., (C4) using Theorem 2.1. For example, the correspondence of the data (C1) and (C2) is given by $\{f, g, h\} \mapsto J = f^3/h$, which is well-defined and easily seen to be bijective (modulo “equivalence explained above”). Similarly $\{f, g, h\} \mapsto S_{f,g}$ gives the bijective correspondence of (C1) and (C3). Observe that the correspondence (C3) \Rightarrow (C2) is directly given by the natural map $S \mapsto J$, where J is the absolute invariant, i.e. it is the meromorphic (actually rational) function on the base curve, evaluating the absolute invariant of the fibre elliptic curves of a given elliptic surface. [In Kodaira’s terminology [6], such a J is called the “functional invariant” of an elliptic surface.]

6.2

Now let us recall (C2) \Rightarrow (C5), which is a standard argument (cf. [25], [29]). Given a ramified covering $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$ of degree $6m$, unramified over $U = \mathbf{P}_w^1 - \{0, 1, \infty\}$, we consider the “sheet change” in the corresponding covering Riemann surfaces. By choosing a base point $b_0 \in U$ and identifying $J^{-1}(b_0)$ with $6m$ letters, we get the permutation representation $\sigma : \pi_1(U, b_0) \rightarrow \mathcal{S}_{6m}$ of the fundamental group into the symmetric group of degree $6m$. The ramification data of (c2) implies the description of $\sigma_0, \sigma_1, \sigma_\infty$

in (C5). The transitivity of the image is required for the covering to be connected.

Conversely, any topological data (C5) can be realized by a covering of Riemann spheres (hence by that of projective lines \mathbf{P}^1) as in (C2), by the Riemann's existence theorem. Hence we have (C2) \Leftrightarrow (C5).

6.3

Next we discuss the correspondence (C2) \Rightarrow (C6). (The following argument is perhaps more direct than Stothers' [25], since he considers first more general abc -situations and then specialize to DS-situations. We can avoid this detour here.) Given a ramified covering $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$ of degree $6m$, unramified over $U = \mathbf{P}_w^1 - \{0, 1, \infty\}$, we consider its restriction $J' : \mathbf{P}_t^1 - \Sigma \rightarrow \mathbf{C} = \mathbf{P}_w^1 - \{\infty\}$ ($\Sigma = J^{-1}(\infty)$). We compare it with the elliptic modular function $j : H \rightarrow \mathbf{C}$, inducing the isomorphism $H/PSL(2, \mathbf{Z}) \cong \mathbf{C}$; j has an infinite degree but has locally the same ramification behavior as J' around $w = 0$ and $w = 1$. Letting J'' and j'' denote the further restriction of J' and j over $\mathbf{C} - \{0, 1\}$, both unramified, let us consider the composite $\tilde{\psi} = J''^{-1} \circ j''$. This is a multi-valued holomorphic function on $H - j^{-1}(\{0, 1\})$, and the common manner of ramification implies that $\tilde{\psi}$ can be extended to a holomorphic function on the whole upper-half plane H and it is locally biholomorphic everywhere (i.e. unramified). Since H is simply-connected, this splits into the $6m$ branches of single-valued holomorphic functions $\psi_i : H \rightarrow \mathbf{P}_t^1 - \Sigma$, which is unramified and which is uniquely determined by the choice of $b_1 \in J^{-1}(b_0)$.

Choose b_1 and a branch $\psi = \psi_1 : H \rightarrow \mathbf{P}_t^1 - \Sigma$. It follows from the above that the holomorphic map $j : H \rightarrow \mathbf{C}$ is decomposed as $j = J \circ \psi$:

$$j : H \xrightarrow{\psi} \mathbf{P}_t^1 - \Sigma \xrightarrow{J} \mathbf{C} \cong H/PSL(2, \mathbf{Z}),$$

with ψ unramified, and that H is the universal covering of $\mathbf{P}_t^1 - \Sigma$. Let

$$\bar{\Gamma} = \{\gamma \in PSL(2, \mathbf{Z}) \mid \psi(\gamma \circ \tau) = \psi(\tau), \forall \tau \in H\}.$$

Then $\bar{\Gamma}$ is a torsion-free subgroup of index $6m$ in the modular group $PSL(2, \mathbf{Z})$, isomorphic to the fundamental group $\pi_1(\mathbf{P}_t^1 - \Sigma)$, such that $H/\bar{\Gamma} \cong \mathbf{P}_t^1 - \Sigma$. This establishes (C2) \Rightarrow (C6).

Conversely, given a subgroup $\bar{\Gamma}$ of $PSL(2, \mathbf{Z})$ as in (C6), we recover $J : \mathbf{P}_t^1 \rightarrow \mathbf{P}_w^1$, as an extension of $H/\bar{\Gamma} \rightarrow H/PSL(2, \mathbf{Z}) \cong \mathbf{C}$. Thus we see (C2) \Leftrightarrow (C6).

6.4

Finally we consider (C3) \Rightarrow (C7). Given an elliptic surface $\Phi : S \rightarrow \mathbf{P}^1$, with the singular fibres lying over $\Sigma \subset \mathbf{P}^1$, the smooth elliptic fibration over $\mathbf{P}^1 - \Sigma$ gives rise to a locally constant sheaf \mathcal{G} of rank 2 on $\mathbf{P}^1 - \Sigma$ (called the “homological invariant” in [6]) and the associated monodromy representation:

$$\rho : \pi_1(\mathbf{P}^1 - \Sigma) \longrightarrow SL(2, \mathbf{Z}).$$

The compatibility of the homological invariant with the functional invariant J of an elliptic surface (the data in (C2)) [6, §7-8] shows that ρ gives a lifting of $\bar{\rho}$, i.e. $\bar{\rho} = \nu \circ \rho$ where $\nu : SL(2, \mathbf{Z}) \rightarrow PSL(2, \mathbf{Z})$ is the natural map. It follows that $\Gamma = \text{Im}(\rho)$ is a torsionfree subgroup of $SL(2, \mathbf{Z})$, and $-1 \notin \Gamma$.

By Kodaira [6, §9], the local monodromy $\rho(l_\alpha)$ at $\alpha \in \Sigma$ has the following normal form if $\Phi^{-1}(\alpha)$ is a singular fibre of type I_b (resp. I_b^*):

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \text{resp.} \quad - \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Thus for the singular fibre of type I_1 or $I_{5m-1}^{(*)}$ for our elliptic surface S , the local monodromy is given as in (C7).

Conversely, given the data as in (C7), we recover naturally $\bar{\Gamma}$, J and \mathcal{G} , compatible with ρ . By Kodaira [6, §8], there exists a unique elliptic surface with the given functional and homological invariants and with a section. (This is “the basic member” of the family $\mathcal{F}(J, \mathcal{G})$ of elliptic surfaces with given invariants J, \mathcal{G} .) Thus we have (C3) \Leftrightarrow (C7).

This completes the proof of Theorem 2.2.

7 Some comments

7.1 Existence and finiteness

Theorem 2.3 is a direct consequence of Theorem 2.2, if we assume the results of Stothers [25]. In order to make this paper self-contained at least for the existence for each m , we prove the following lemma, which shows the existence for (C5) and hence for other data. Also the finiteness for each m follows easily from (C5).

Lemma 7.1 *For any m , there are at least a pair of $\sigma_0, \sigma_1 \in S_{6m}$ satisfying*

$$\sigma_0 = (3)^{2m}, \sigma_1 = (2)^{3m}, \sigma_0\sigma_1 = (5m-1)(1)^{m+1}$$

and generating a transitive subgroup of S_{6m} .

Proof We prove this by induction on m . For $m = 1$, take

$$\sigma_0 = (123)(456), \sigma_1 = (14)(23)(56), \sigma_0\sigma_1 = (1542)(3)(6).$$

Assume that we have constructed $\sigma_0, \sigma_1 \in S_{6m}$ satisfying the required condition as follows:

$$\begin{cases} \sigma_0 &= (123)(456) \cdots (6m-2, 6m-1, 6m) \\ \sigma_1 &= (14)(23) \cdots (6m-1, 6m) \\ \sigma_0\sigma_1 &= (15 \cdots, 6m-1, 6m-2, \cdots)(3) \cdots (6m). \end{cases}$$

Then we define $\sigma'_0, \sigma'_1 \in S_{6(m+1)}$ as follows: first let

$$\sigma'_0 = \sigma_0 \cdot (6m+1, 6m+2, 6m+3)(6m+4, 6m+5, 6m+6).$$

Next we define σ'_1 from σ_1 by replacing the transposition $(6m-1, 6m)$ by the product $(6m-1, 6m+1)(6m, 6m+4)$. Then it is immediate to check that

$$\sigma'_0\sigma'_1 = (1 \cdots 6m-1, 6m+2, 6m+1, 6m, 6m+5, 6m+4, 6m-2, \cdots)$$

is a $(5m+4)$ -cycle, and that σ'_0, σ'_1 generate a transitive subgroup of $S_{6(m+1)}$. *q.e.d.*

Remark Miranda-Persson [9] classified semi-stable configurations of singular fibres on elliptic K3 surfaces. They also use a similar method as above (exhibiting certain permutation representations) to show the existence. In particular, the first member in their list ([9, Th.3.1]) is an elliptic K3 surface with five I_1 and one I_{19} , which corresponds to the DS-triple of order $m = 4$. As remarked in [24], the method in the present paper gives not only the existence, but also the defining equation and uniqueness for such.

7.2 Elliptic modular surfaces

Next let us prove Theorem 2.4 stating that the elliptic surfaces in (C3) are elliptic modular surfaces attached Γ defined in (C7). Actually this is essentially done in the last part of the previous section, if the reader is familiar with this notion.

For the sake of completeness, we recall ([18, §4]) the definition of the elliptic modular surface attached to a torsionfree subgroup Γ of finite index in $SL(2, \mathbf{Z})$. The quotient $C' = H/\Gamma$ of the upper-half plane H by Γ becomes a compact Riemann surface, say C , by adjoining a finite number of cusps Σ . The fundamental group of $C' = C - \Sigma$ can be identified with $\Gamma \hookrightarrow SL(2, \mathbf{Z})$, which defines a locally constant sheaf \mathcal{G} of rank 2 on C' , while the natural map $J' : H/\Gamma \rightarrow H/PSL(2, \mathbf{Z}) \cong \mathbf{C}$ is extended to a meromorphic function J on C . Then there is an elliptic surface S over C with the functional invariant J and the homological invariant \mathcal{G} and with a global section (the basic member of the family $\mathcal{F}(J, \mathcal{G})$). The *elliptic modular surface* attached to Γ is, by definition, this elliptic surface which is uniquely determined up to isomorphism.

Therefore the argument in §6.4 says exactly that the elliptic surface S in (C3) is the elliptic modular surface attached to the group Γ in the corresponding (C7).

To be more explicit, it can be defined as follows. Let $\tilde{\Gamma} = \Gamma \times \mathbf{Z}^2$ act on the product space $H \times \mathbf{C}$ via $(\gamma, n_1, n_2) : (\tau, z) \mapsto (\tau', z')$, where $\tau' = (a\tau + b)/(c\tau + d)$, $z' = z/(c\tau + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\tilde{\Gamma}$ has a structure of group so that this becomes a group action. The quotient $S' = (H \times \mathbf{C})/\tilde{\Gamma}$, together with the natural projection $\Phi' : S' \rightarrow H/\Gamma$, defines a smooth (open) elliptic surface over C' with a zero-section. We obtain a smooth elliptic surface $\Phi : S \rightarrow C$, by filling in a singular fibre of type I_b (resp. I_b^*) over a cusp of the first (resp. second) kind with cusp-width b .

In the case under consideration, say for $S = S_{f,g}$ with a DS-triple $\{f, g, h\}$ of order m , the singular fibres are one of type I_{5m-1}^* and $m+1$ of type I_1 . So the subgroup Γ of index $12m$ in $SL(2, \mathbf{Z})$ has a cusp with cusp-width $5m-1$ (of the first or second kind according to the parity of m), and $m+1$ cusps of the first kind with cusp-width 1.

For $m = 1$, S is a rational elliptic surface with singular fibres I_1, I_1, I_4^* , and the corresponding Γ has been determined by Schmickler-Hirzebruch [13] in her study of elliptic surfaces over \mathbf{P}^1 with three singular fibres, who showed that it is a congruence subgroup isomorphic to $\Gamma_0(4)/\pm 1$.

For $m = 2$, S is a rational elliptic surface with singular fibres I_1, I_1, I_1, I_9 , and Γ has been determined by Beauville [1] to be a certain congruence subgroup of level 9.

For $m > 2$, on the contrary, our Γ *cannot be congruence subgroups*. This follows from Sebbar’s work [15] classifying torsionfree congruence subgroups of genus 0 of $SL(2, \mathbf{Z})$, since the set of cusp-widths such as $1, \dots, 1, 5m - 1$ cannot appear in his list for $m > 2$.

It should be interesting to study these elliptic modular surfaces $S = S_{f,g}$ for $m > 2$ attached to non-congruence subgroups Γ . For example, for $m = 3$ and 4, S is a “singular” K3 surface in the sense $\rho = h^{1,1} = 20$ (cf. [24]), and we hope to discuss this aspect for them in some other occasion.

8 Integral points of \mathcal{E}_h

Let us consider the integral points of the elliptic curve $\mathcal{E}_h : Y^2 = X^3 - h$, assuming that h has only simple zeros and $\deg(h) = m + 1$.

8.1 The Mordell-Weil lattices

Proposition 8.1 *Assume $m \leq 5$. Then the structure of the Mordell-Weil lattice on $\mathcal{E}_h(K)$ ($K = k(t)$) depends only on m , and it is the dual lattice A_2^*, D_4^*, E_6^* of the root lattice A_2, D_4, E_6 for $m = 1, 2, 3$ and the root lattice E_8 for $m = 4$ and 5.*

Proof For $m \leq 5$, the associated elliptic surface \mathcal{S}_h is a rational elliptic surface. For such, the Mordell-Weil lattice is determined by the method of [19, §10] and [12], outlined as follows. When h has only simple zeros, the trivial sublattice is $T = U \oplus V$, with U hyperbolic and $V = E_6, D_4, A_2 (m = 1, 2, 3)$ or $\{0\} (m = 4, 5)$, as easily seen from the singular fibre at ∞ , which is independent of h (cf. [20, §3] where the case $h = t^{m+1} + 1$ is treated). Taking the orthogonal complement of T in $N = NS(\mathcal{S}_h) \cong U \oplus E_8^-$, we obtain the narrow Mordell-Weil lattice $A_2, D_4, E_6 (m = 1, 2, 3)$ and $E_8 (m = 4, 5)$, and the full Mordell-Weil lattice as its dual lattice. *q.e.d.*

The following table collects some information about the lattices in question: $L = A_2^*, D_4^*, E_6^*$ and E_8 (cf. [3]). We denote by μ the minimal norm of L , and by N_l the number of elements of L with norm l . In particular, $\tau = N_\mu$ is the number of the minimal vectors in L , known as the kissing number of L in the terminology of sphere packings, while N_2 is the number of the “roots” in the root lattice A_2, D_4, E_6 or E_8 . The blank box indicates

that any rational point of $\mathcal{E}_h(K) \cong L$ with this height (=norm) cannot be an integral point in $\mathcal{E}_h(k[t])$ because of the Davenport inequality (1).

m	1	2	3	4	5
L	A_2^*	D_4^*	E_6^*	E_8	
μ	$2/3$	1	$4/3$	2	
N_μ	6	24	54	240	
N_2	6	24	72		
N_4		24	270	2160	
N_6			720	6720	
N_8				17520	
N_{10}				30240	

Theorem 8.2 For $m \leq 5$, all the rational points $P \in \mathcal{E}_h(k(t))$ with height ≤ 2 are integral points. More precisely, we have the following results, in which a, b, c, d, e, g, h are constants:

- (i) For $m = 1$, there exist exactly 6 integral points of the form $P = (b, dt + e)$,
- (ii) for $m = 2$, exactly 24 of the form $P = (at + b, dt + e)$, and
- (iii) for $m = 3$, exactly 54 of the form $P = (at + b, ct^2 + dt + e)$.
- (iv) For any $m \leq 5$, there exist exactly $N_2 = 6, 24, 72$ or 240 points of the form $P = (gt^2 + at + b, ht^3 + ct^2 + dt + e)$.

For h generic (with $\deg(h) = m + 1$), there are no more integral points with height > 2 .

Proof For the first part, we refer to [19, §10] and [12]. As for the last assertion, it is obvious for $m = 1$. We can characterize for $m = 2$ and 3 the cases of h which have integral points with height > 2 (see below). For $m = 4$, see [23, §3]. Actually, the method there works for any $m \leq 5$. *q.e.d.*

8.2 Extra integral points for $m = 2$

Let us study the question of “extra” integral points. By coordinate change of t , we can normalize h to be monic and without degree m -term.

Proposition 8.3 Let $m = 2$ and $h = t^3 + q_1t + q_0(4q_1^3 + 27q_0^2 \neq 0)$. Then $\mathcal{E}_h(k(t))$ has an integral point of height 4 if and only if $q_1 = 0$, i.e. $h = t^3 + q_0$. In this case, $P = (t, e)(e = \sqrt{-q_0})$ is a point of height 1 and $Q = 2P$ is an integral point of height 4.

Proof If $\langle P, P \rangle = 1$ and $Q = 2P$, then $\langle Q, Q \rangle = 4\langle P, P \rangle = 4$. Since $N_1 = 24 = N_2$ for $L = D_4^*$ by the table, the map $P \rightarrow 2P$ is a bijection from the set of points of height 1 to that of points of height 4.

Now we have $P = (at + b, dt + e)$ by Theorem 8.2 (ii), and the duplication formula on the elliptic curve shows that the X -coordinate of $Q = 2P$ is equal to

$$-2(at + b) + \left\{ \frac{3}{2} \cdot \frac{(at + b)^2}{dt + e} \right\}^2.$$

This becomes a polynomial, i.e. Q is an integral point, if and only if $d = 0$. If this is the case, writing down the relation $P = (at + b, e) \in \mathcal{E}_h$, we have the identity in t : $e^2 = (at + b)^3 - (t^3 + q_1t + q_0)$. Comparing the coefficients of t^n , we have $a^3 = 1, b = 0, e^2 = -q_0$ and $q_1 = 0$. This implies that $h = t^3 - e^2$ and $P = (t, e) \in \mathcal{E}_h$. Then $2P = (-2t - 9t^4/4q_0, \dots)$ is an integral point of height 4. *q.e.d.*

The above proposition gives a characterization of DS-triples of order $m = 2$, together with the uniqueness (up to coordinate change) and the method of construction. For example, taking $h = 8t^3 - 36$, it is easy to find a point $P = (2t, 6) \in \mathcal{E}_h(k(t))$. Then $Q = 2P = (f, g)$ gives $f = t^4 - 4t, g = t^6 - 6t^3 + 6$; thus we obtain the DS-triple $\{f, g, h\}$ of order 2 cited in §5.

8.3 Extra integral points for $m = 3$

For $m = 3$, a similar idea works to determine the cases of extra integral points, of height 4 or 6 (the latter being the case of DS-triple of order $m = 3$) in the framework of Mordell-Weil lattices. Summarizing the results of Shinki and Yanagida ([17], [28]), we can state the following:

Proposition 8.4 *Let $m = 3$ and assume $h = t^4 + q_2t^2 + q_1t + q_0$ has no multiple zeros, i.e. the discriminant δ of h is non-zero ($\delta = 256q_0^3 - 27q_1^4 + 144q_0q_1^2q_2 - 128q_0^2q_2^2 - 4q_1^2q_2^3 + 16q_0q_2^4 \neq 0$). Let $\lambda = q_2^2/q_0$ and $H = 1827904q_0^3 - 185193q_1^4 + 782496q_0q_1^2q_2 + 1397968q_0^2q_2^2 - 359464q_1^2q_2^3 + 6860q_0q_2^4 - 117649q_2^6$.*

Then $\mathcal{E}_h(k(t))$ has an integral point of height ≥ 4 only in the following cases:

(i) $q_1 \neq 0$ and $H = 0$. In this case, there are exactly 6 integral points of height 4.

(ii) $q_1 = 0$. In this case, there are 6 integral points of height 4 which are invariant under $t \mapsto -t$, and no more except for the cases (iii) and (iv).

(iii) $q_1 = 0$ and $\lambda = -676/343$. In this case, there are 6 more integral points of height 4.

(iv) $q_1 = 0$ and $\lambda = 169/128$. There are exactly 6 integral points of height 6. This corresponds to the case for the DS-triples of order $m = 3$.

Thus the number of integral points in $\mathcal{E}_h(k(t))$ is equal to 126 for general h , 132 in the cases (i) and (ii), and 138 in the cases (iii) and (iv).

8.4 Remark for $m = 4$

For $m = 4$, the same method should work, but we have not verified every detail. On the other hand, we have checked the following facts for the DS-triple $\{f, g, h\}$ of order $m = 4$ by Hall (cited in §5). While $\mathcal{E}_h(k(t)) \cong E_8$ has rank 8, the subgroup $\mathcal{E}_h(\mathbf{Q}(t))$ of $\mathbf{Q}(t)$ -rational points has rank 2. It contains, besides the integral point $P = (f, g)$ of maximal height 8, another integral point of height 2 (a “root”) $Q_1 = (t^2 - 2t + 4, t^3 + \dots)$. These two are linearly independent because under the specialization map $s = sp_\infty$, they are mapped to 0 and 1. Moreover, by using two other $\mathbf{Q}(\sqrt{-3})(t)$ -rational roots, the integral point $P = (f, g)$ can be expressed as

$$-P = Q_1 + Q_2 - Q_3,$$

where we set $Q_2 = (-3\omega(t^2 + 2t + 3), 3\sqrt{-3}t^3 + \dots)$ and $Q_3 = (-3\omega'(t^2 + 2t + 3), 3\sqrt{-3}t^3 + \dots)$, ω, ω' being the cube roots of 1. The computation above is based on the theory of algebraic equations arising from Mordell-Weil lattices, as developed in [21] and [22].

9 Mordell-Weil groups of $S_{f,g}$

For any elliptic modular surface S , it is known ([18]) that $r = 0$ and $\rho = h^{1,1}$. For our $S = S_{f,g}$ with a maximal singular fibre, this follows also from the formula (13); see the proof of Lemma 3.2. The following theorem 9.1 gives a more precise result that for such an S , the Mordell-Weil group is trivial for $m > 2$. The proof is based on the theory of Mordell-Weil lattices ([19]). It is worth noting that, even for dealing with torsion in Mordell-Weil groups, the height formula can play an indispensable role.

Theorem 9.1 (i) *The Mordell-Weil group of $S = S_{f,g}$ (or $E_{f,g}$) in (C3) is trivial for any $m > 2$. For $m = 1$ or 2, it is isomorphic to the cyclic group*

of order 2 or 3, respectively. (ii) For $m > 2$, the Néron-Severi lattice of S is isomorphic to the trivial lattice $U \oplus V$, where U is a rank 2 hyperbolic lattice and V is the (negative-definite) root lattice of type A_{5m-2}^- or D_{5m+3}^- depending on the parity of m .

Proof Let $S = S_{f,g}$ be an elliptic surface with a maximal singular fibre of type $I_{5m-1}^{(*)}$ and let $E = E_{f,g}$, $K = k(t)$. Let $N = NS(S)$ be the Néron-Severi lattice and $T = U \oplus V$ the trivial sublattice, where U is a rank 2 hyperbolic lattice generated by the zero-section (O) and a fibre, and V is the sublattice generated by the irreducible components of $I_{5m-1}^{(*)}$ -fibre at $v = \infty$. We have $V^- = A_{5m-2}^-$ or D_{5m+3}^- for m even or odd, so $\det T$ is equal to $5m-1$ or 4 according to the parity of m . Now the Mordell-Weil group $E(K)$ is isomorphic to N/T by [19, Th.1.3]. Letting $\nu = [N : T]$ be the index of T in N , we have $\det N = \det T/\nu^2$, and $E(K)$ is an abelian group of order ν . Thus, for the proof of Theorem 9.1, it suffices to prove the following:

$$(i) \ m = 1 \Rightarrow \nu = 2, (ii) \ m = 2 \Rightarrow \nu = 3, (iii) \ m > 2 \Rightarrow \nu = 1.$$

For $m = 1$ or 2, S is a rational elliptic surface and $\det N = 1$. In case $m = 1$, we have $\det T = \det D_8 = 4$. Hence $\nu = 2$. In case $m = 2$, we have $\det T = \det A_8 = 9$. Hence $\nu = 3$. Thus we have shown (i) and (ii).

To show (iii), assume $m > 2$ and $\nu > 1$ (and we derive a contradiction). Take $P \in E(K)$, $P \neq O$. The height formula (14) (applied to E instead of \mathcal{E}) reads

$$\langle P, P \rangle = 2\chi(S) + 2(PO) - \text{contr}_\infty(P),$$

where we have $\langle P, P \rangle = 0$ and $(PO) = 0$ since P is a torsion point.

Suppose first that m is odd. Then $2\chi(S) = m + 1$, while $\text{contr}_\infty(P)$ is either 1 or $1 + (5m-1)/4$ (by (8.16) of [19]). But $m+1 = 1$ or $= 1 + (5m-1)/4$ cannot hold for $m > 2$, a contradiction.

Next suppose that m is even. Then $2\chi(S) = m$, while $\text{contr}_\infty(P) = i(5m-1-i)/(5m-1)$ if the section (P) passes through the i -component of the singular fibre of type I_{5m-1} ($0 \leq i < 5m-1$) (*ibid*); set $i(P) = i$. The height formula is now rewritten as $m = i(5m-1-i)/(5m-1)$. Rewriting it again, we have

$$(2i - (5m-1))^2 = (5m-1)(m-1).$$

Hence both $5m-1$ and $m-1$ must be square integers; let $5m-1 = u^2$, $m-1 = v^2$ with $u, v > 0$. We have then

$$2i = 5m-1 \pm uv.$$

Now we may assume that $0 < i < (5m - 1)/2$ (replacing P by $-P$ if necessary). Let $Q = 2P \neq O$, and apply the same argument as above to Q . Note here that $i(Q) = 2i$, because $P \mapsto i(P)$ is a group homomorphism from $E(K)$ to the “component group” $\mathbf{Z}/(5m - 1)\mathbf{Z}$.

Therefore we must have

$$2 \cdot 2i = 5m - 1 + uv, \quad 2i = 5m - 1 - uv.$$

Eliminating i and then u, v , we see easily that the only solution to the above is $m = 2$. Thus we have arrived at a contradiction.

This completes the proof.

(Added Dec. 1, 2004) This paper has been completed more than a year ago. In the meantime, a very interesting book [30] has appeared, and its chapter two “Dessins d’Enfants” deals with closely related subjects with the present paper. In particular, it gives a proper credit to Stothers’ work and remarks (on p. 128) about the determination of Davenport-Stothers triples of order $m = 5$ that “... the task is not easy”. As the reader sees, this question has been treated in §5 of the present paper.

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