主論文

THESIS

Braneworld Cosmological Perturbations

ブレーン宇宙摂動論

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January, 2006

Abstract

We explore aspects of cosmological perturbations in braneworld scenarios. In order to correctly evaluate perturbations on a four-dimensional brane, we need to solve higher dimensional bulk perturbations, which reduces to the problem of solving partial differential equations with appropriate boundary conditions. Hence it is not as easy as in the four-dimensional standard cosmological scenario, and the lack of understanding about the behavior of the brane cosmological perturbations has constrained so far the predictability of this revolutionary picture of the universe. In the Randall-Sundrum-type setup, the five-dimensional anti-de Sitter bulk with a maximally symmetric brane is the special case where gravitational wave (tensor) perturbations are exactly solvable. Using this as a stepping stone, we further develop various formulations toward understanding the generation and evolution of cosmological perturbations in the braneworld. We start with an analytic evaluation of leading order corrections to the evolution of cosmological tensor perturbations at low energies. We explicitly show, by a perturbative expansion scheme, that the corrections are indeed suppressed by ℓ^2 and $\ell^2 \ln \ell$, where ℓ is the bulk curvature scale. Next, seeking for the possible high energy effects particular to the braneworld scenario, we consider the primordial spectrum of tensor perturbations generated quantum-mechanically from inflation on the brane. To investigate the effect of nontrivial motion of the brane, first we use the toy model called the junction model and then develop a numerical scheme based on the Wronskian formulation, and show that the primordial tensor spectrum in the braneworld can be reproduced with quite good accuracy by a simple map from the result of the conventional four-dimensional calculation. We find that during inflation the vacuum fluctuations in the initial Kaluza-Klein gravitons contribute to the final amplitude of the zero mode at a significant level on small scales. Then, using the same numerical formulation, we compute the late time spectrum of gravitational waves, evolving through the radiation-dominated stage after their generation from inflation. We show that in this case the effect of the initial Kaluza-Klein vacuum fluctuations is subdominant, and therefore the damping due to the Kaluza-Klein mode generation and the enhancement due to the modification of the background Friedmann equation are in fact the two dominant effects, but they almost cancel each other, leading to the same spectral tilt as the standard four-dimensional result. Finally, we introduce a $(5+m)$-dimensional vacuum description of five-dimensional bulk inflaton models with exponential potentials that makes an analysis of cosmological perturbations simple and transparent, and derive separable master equations for all types of perturbations in this class of models.

Contents

Chapter 1

Introduction

Cosmological inflation [39, 131, 96, 69, 94] explains how the large-scale structure observed today emerges by stretching the quantum fluctuations in inflaton and possibly other fields in the universe. Small inhomogeneities in the energy density exit the horizon, and at a later stage they come back again inside the horizon, being responsible for the Cosmic Microwave Background (CMB) temperature anisotropies of the order of 1 part in 10^5, which were indeed found by the Cosmic Background Explorer (COBE) in 1992 [7], and have been observed with better resolution and sensitivity by the Wilkinson Microwave Anisotropy Probe (WMAP) [8, 70, 136, 121, 147]. Inflation also predicts the gravitational wave background arising due to the quantum fluctuations in the graviton field. The detection of the gravitational waves of inflationary origin is a challenging future task for the Laser Interferometer Space Antenna (LISA) [148] or the Deci-hertz Interferometer Gravitational Wave Observatory (DECIGO) [132]. These cosmological fluctuations have rich information on the early universe, and hence provide us with powerful tools to probe fundamental physics. Then, *what if there exist extra dimensions?*

Motivated by string theory [122], there has been a growing attention to braneworld scenarios, in which our observable universe is realized as a "brane" embedded in a higher dimensional "bulk" spacetime (for pedagogical reviews, see e.g., Refs [129, 89, 102]). In this revolutionary picture, Standard Model particles are assumed to be trapped on the brane while gravity is free to access the bulk. This is the key idea of braneworlds, allowing for rather "large" extra dimensions as Newton's law is confirmed to hold down only to $\mathcal{O}(0.1\text{ mm})$ by the current table-top experiments [99, 22]. In the naïve construction proposed by Arkani-Hamed, Dimopoulos, and Dvali (ADD) [2, 3], the braneworld idea is used to address the hierarchy problem. If d extra dimensions are compactified with radius L, the gravitational Lagrangian reads $\mathcal{L} \sim M_{4+d}^{2+d}\;{}^{(4+d)}R \sim M_{4+d}^{2+d} L^d\;{}^{(4)}R$, and hence the Planck scale $M_{\rm Pl}$ is related to the $(4+d)$-dimensional fundamental scale via the volume of extra dimensions as

$$M_{\rm Pl}^2 \sim M_{4+d}^{2+d} L^d. \tag{1.1}$$

Taking, for example, $d = 2$ and $L \sim 0.1$ mm, we have the low fundamental scale, $M_{4+d} \sim 1$ TeV, which is of the order of the electroweak scale.

Among various models, the Randall-Sundrum model [123, 124] is of particular interest because it includes nontrivial gravitational dynamics despite rather a simple construction. In the Randall-Sundrum type 2 model [124] with a single brane embedded in an anti-de Sitter (AdS) bulk, although the fifth dimension extends infinitely, the warped structure of the bulk geometry results in the recovery of four-dimensional general relativity on the brane at scales larger than the bulk curvature scale ℓ or at low energies [31]. In order to reveal five-dimensional effects particular to the braneworld scenario, we have to focus on the scales smaller than ℓ. It is natural to consider inflationary scenarios in the braneworld context, and cosmological perturbations from inflation will be quite useful for the purpose of probing such small scales. Therefore, it is worthwhile clarifying the behavior of perturbations in the brane universe.

Is it indeed observationally relevant to investigate cosmological dynamics on scales smaller than the size of the extra dimension? To see this point, let us take a brief look at the possibility of a direct detection of a gravitational wave background generated by some dynamics of the extra dimension with a typical length scale ℓ [47]. Suppose that the extra dimensional dynamics has left its trace when the Hubble radius was comparable to the typical size of the extra dimension: $H_*^{-1} \sim \ell$. The characteristic frequency of the gravitational waves redshifted to the present day is $f_0 = H_* a_*/a_0$. Using the relation $a_* T_* = a_0 T_0$ and the Friedmann equation $H_*^2 \sim T_*^4/M_{\rm Pl}^2$, we have the estimate

$$f_0 \sim 10^{-4}\ {\rm Hz} \times \left(\frac{\ell}{1\ {\rm mm}}\right)^{-1/2}. \tag{1.2}$$

(We assumed here for simplicity that the standard Friedmann equation holds at energy scales as high as $\sim T_*$.) Extra dimensions between $\ell \sim 1$ and 10^{-6} mm can produce backgrounds peaked in the LISA band (10^{-1} to 10^{-4} Hz); ground-based observations such as the Laser Interferometer Gravitational-Wave Observatory (LIGO) [149] (up to ∼1000 Hz) can detect activity from extra dimensions as small as $\ell \sim 10^{-14}$ mm [47].

While the cosmological perturbation theory in the conventional four-dimensional universe has been rather established [67, 110, 4, 94], calculating cosmological perturbations in the braneworld still remains to be a difficult problem; we are far from taming it. In this thesis, we explore aspects of cosmological perturbations in the braneworld. After an introductory review of basic, well understood facts about cosmology and perturbations in the braneworld, we first evaluate leading order corrections to the cosmological evolution of gravitational waves at low energies, confirming that it is indeed small. Then, seeking for possible high energy effects, we move on to study the primordial and late time tensor spectra by invoking a newly developed numerical method as well as the so called "junction model." Finally, as an application of the simple braneworld setup where the cosmological perturbations are exactly solvable, we discuss a braneworld model with bulk scalar fields and its perturbations.

The plan of this thesis is as follows:

Chapter 2 We overview homogeneous and isotropic cosmology on a brane.

Chapter 3 We summarize basic facts on cosmological (tensor) perturbations in the Randall-Sundrum braneworld.

Chapter 4 We analytically derive leading order corrections to the cosmological evolution of tensor perturbations in the Randall-Sundrum braneworld at low energies. The original contribution is based on
T. Kobayashi and T. Tanaka, "Leading order corrections to the cosmological evolution of tensor perturbations in braneworld," JCAP **0410**, 015 (2004) [63].

Chapter 5 We introduce the junction model to investigate the effect of nontrivial motion of the brane on the primordial tensor spectrum. The original contribution is based on
T. Kobayashi, H. Kudoh and T. Tanaka, "Primordial gravitational waves in inflationary braneworld," Phys. Rev. D **68**, 044025 (2003) [61].

Chapter 6 We formulate a numerical method to calculate the primordial spectrum of gravitational waves, extending the result of the previous chapter. The original contribution is based on
T. Kobayashi and T. Tanaka, "Quantum-mechanical generation of gravitational waves in braneworld," Phys. Rev. D **71**, 124028 (2005) [65].

Chapter 7 Using the numerical formulation in the previous chapter, we clarify the late time power spectrum of gravitational waves, evolving through the radiation-dominated epoch after their generation during inflation. The original contribution is based on
T. Kobayashi and T. Tanaka, "The spectrum of gravitational waves in Randall-Sundrum braneworld cosmology," hep-th/0511186 [66].

Chapter 8 In the context of the bulk inflaton model, we present a new and powerful technic to generate background cosmological solutions and to compute perturbations. The original contribution is based on
T. Kobayashi and T. Tanaka, "Bulk inflaton shadows of vacuum gravity," Phys. Rev. D **69**, 064037 (2004) [62].

Chapter 9 We draw the conclusions of this thesis.

We have tried to make each chapter to be organized in a self-contained manner so that the reader should be able to follow the content of one chapter independently from another with prior reading of Chapters 2 and 3.

Chapter 2

Overview of braneworld cosmology

In this chapter, we outline the Randall-Sundrum braneworld scenario and homogeneous, isotropic cosmology on the brane.

2.1 Kaluza-Klein picture

To begin with, we will briefly discuss the idea of the Kaluza-Klein scenario. Let us consider the simplest setup with one extra spatial dimension y in addition to the usual four-dimensional (flat) spacetime x^μ. Let us assume that the extra dimension is a homogeneous circle of radius L so that y runs from 0 to $2\pi L$. A complete set of solutions to a five-dimensional Klein-Gordon equation (a wave equation of a free massless particle), $\Box\phi(x^\mu, y) = 0$, is given by

$$\phi(p^\mu, n) = e^{ip^\mu x_\mu} \cdot e^{iny/L}, \tag{2.1}$$

where $n = 0, \pm1, \pm2, \cdots$. The four-momentum p^μ and the eigenvalue n of the extra momentum obey

$$p^\mu p_\mu + \frac{n^2}{L^2} = 0. \tag{2.2}$$

Inhomogeneous modes with $n \neq 0$ are called Kaluza-Klein (KK) modes, and each mode can be seen as a particle with mass

$$m_n = \frac{|n|}{L}, \tag{2.3}$$

from the four-dimensional point of view. The KK modes can be excited at the energy scale $\sim 1/L$, and so in this scenario the size of the extra dimension must be microscopic ($L \lesssim 10^{-17}$ cm) because no KK partners of ordinary particles such as photons have been observed.

2.2 Randall-Sundrum braneworld

The Randall-Sundrum (RS) braneworld models [123, 124] give us a completely different mechanism to realize an effectively four-dimensional world in a five-dimensional spacetime. A basic ingredient of braneworld scenarios is that Standard Model particles and fields are localized on

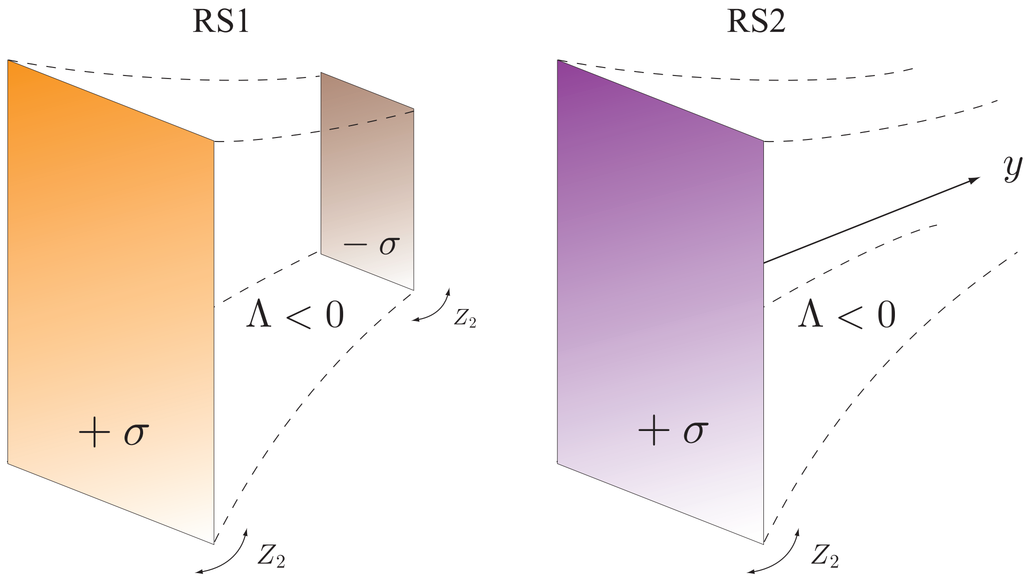

Figure 2.1: Schematic of the Randall-Sundrum braneworld.

a four-dimensional object (the "brane") while gravity can propagate in the higher dimensional spacetime (the "bulk"). In the RS braneworld gravity is localized at the four-dimensional brane not due to compactification, but due to the curvature of the bulk spacetime, and for this reason the mechanism is called "warped compactification".

The RS braneworld is described by the following action:

$$S = \frac{M_5^3}{2}\int d^5x\sqrt{-g}\,(R - 2\Lambda_5) - \sum_i \int d^4x\sqrt{-q_i}\sigma_i, \tag{2.4}$$

where M_5 is the fundamental scale of gravity,

$$\Lambda_5 = -\frac{6}{\ell^2} \tag{2.5}$$

is the bulk cosmological constant, and σ_i is the tension of the i-th brane. Since the cosmological constant Λ_5 is negative, the bulk is basically given by an anti-de Sitter (AdS) spacetime. There are two types of models of the Randall-Sundrum braneworld, and they are classified as follows:

RS1 Let us assume the five-dimensional metric of the form

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{2.6}$$

In the so called RS1 model [123], two branes are placed at $y = 0$ and $y = L$, and the Z_2-symmetry is imposed as $y \leftrightarrow -y$ and $y + L \leftrightarrow -y + L$. The five-dimensional Einstein equations read

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{\ell^2}, \tag{2.7}$$

$$3\left(\frac{a'}{a}\right)' = -\frac{1}{M_5^3}\left[\sigma_1\delta(y) + \sigma_2\delta(y-L)\right], \tag{2.8}$$

where a prime stands for a derivative with respect to y. From these two equations we can see that the branes have equal and opposite tensions,

$$\sigma_1 = -\sigma_2 = \frac{6M_5^3}{\ell} =: \sigma, \tag{2.9}$$

and the "warp factor" is given by

$$a(y) = e^{-|y|/\ell}. \tag{2.10}$$

The effective Planck scale $M_{\rm Pl}$ on the negative tension brane (second brane) is [123]

$$M_{\rm Pl}^2 = \ell M_5^3 \left(e^{2L/\ell} - 1\right), \tag{2.11}$$

and so if we set the separation of the branes to be $L \sim 35\ell$, we have $M_5 \sim \ell^{-1} \sim \mathcal{O}(\text{TeV})$. This implies that the RS1 model helps to solve the hierarchy problem, which is one of the original motivation of the braneworld scenario.

One should note that the effective Planck scale on the positive tension brane is

$$M_{\rm Pl}^2 = \ell M_5^3 \left(1 - e^{-2L/\ell}\right). \tag{2.12}$$

RS2 The RS2 is a model with a single brane in an AdS bulk, and obtained by taking the negative tension brane to infinity, $L \to \infty$ [124]. Then, from Eq. (2.12) we can see that the Planck scale on the brane is

$$M_{\rm Pl}^2 = \ell M_5^3. \tag{2.13}$$

Although the extra dimension extends infinity, the warped structure of the bulk geometry results in its finite contribution to the five-dimensional volume:

$$\int d^5x\sqrt{-g} = \frac{\ell}{2}\int d^4x,$$

which means that the effective size of the extra dimension is ℓ. Gravitational perturbations around the Minkowski brane in the RS2 model are extensively investigated in [31], and it is shown that for $r \gg \ell$ the Newtonian potential on the brane is given by

$$V(r) \simeq -\frac{Gmm'}{r}\left(1 + \frac{2\ell^2}{3r^2}\right), \tag{2.14}$$

where the second term is the small correction to four-dimensional gravity arising due to the contribution of the massive KK modes of gravitons[1]. At this moment, table-top experiments confirm that Newton's law holds at scales down to $\mathcal{O}(0.1 \text{ mm})$ [99, 22]. Therefore we have

$$\ell \lesssim 0.1 \text{ mm}. \tag{2.15}$$

This in turn gives limits on the other parameters:

$$\sigma \gtrsim (1 \text{ TeV})^4, \quad M_5 \gtrsim 10^5 \text{ TeV}. \tag{2.16}$$

In what follows, we will mainly concentrate on the RS2 model.

For later purposes, we rewrite the metric (2.6) in a conformally flat form by introducing the Poincaré coordinates as

$$ds^2 = \frac{\ell^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \tag{2.17}$$

with $z = \ell e^{y/\ell}$. In these coordinates, the brane is located at $z = \ell$.

[1]The recovery of Einstein gravity beyond the linear order is confirmed in Refs. [34, 82, 144].

2.3 Cosmology on the brane

2.3.1 Friedmann-Robertson-Walker brane

Our primary interests are in cosmological aspects of the braneworld, and in this section we shall consider a homogeneous and isotropic, Friedmann-Robertson-Walker (FRW) cosmological model. In the RS setup, cosmological expansion on the brane is mimicked by its motion through the AdS bulk. To see this, let us consider the five-dimensional Schwarzschild-AdS solution whose metric is given by [71, 51]

$$g_{AB}dx^A dx^B = -f(r)dT^2 + \frac{1}{f(r)}dr^2 + r^2\gamma_{ij}dx^i dx^j, \tag{2.18}$$

where

$$f(r) := k + \frac{r^2}{\ell^2} - \frac{\mathcal{C}}{r^2}, \tag{2.19}$$

and γ_{ij} is the metric of the maximally symmetric three-dimensional space. We allow for a black hole in the bulk, and $\mathcal{C}$ is its mass parameter. If $\mathcal{C}$ vanishes, the bulk spacetime is exactly AdS. In that case, by a coordinate transformation $z = \ell/r$ and appropriate rescaling of the time coordinate we have the familiar form of the metric (2.17) when $k = 0$. As will be seen, nonvanishing $\mathcal{C}$ turns out to bring an interesting consequence on cosmology.

Suppose that the brane is moving in this *static* bulk and its trajectory is parameterized by $T = T(\tau)$ and $r = r(\tau)$. The induced metric on the brane is

$$q_{\mu\nu}dx^\mu dx^\nu = -\left(f\dot{T}^2 - f^{-1}\dot{r}^2\right)d\tau^2 + r^2(\tau)\gamma_{ij}dx^i dx^j, \tag{2.20}$$

where a dot denotes a derivative with respect to τ. Now it is clear that the function $r(\tau)$ plays a role of the scale factor: $r(\tau) = a(\tau)$. The tangent to the brane is

$$u^A = \left(\dot{T}, \dot{r}\right), \quad u_A = \left(-f\dot{T}, f^{-1}\dot{r}\right), \tag{2.21}$$

which is normalized as $u^A u_A = -1$ so that τ is the proper time on the brane. The unit normal to the brane is

$$n_A = \left(\dot{r}, -\dot{T}\right), \quad n^A = \left(-f^{-1}\dot{r}, -f\dot{T}\right). \tag{2.22}$$

From this we can calculate the extrinsic curvature $K_{\mu\nu} := q_\mu{}^A q_\nu{}^B \nabla_A n_B$ on the brane:

$$K_{ij} = -f\dot{T}r^{-1}g_{ij}, \tag{2.23}$$

$$K_{\tau\tau} = u^A u^B \nabla_A n_B = \dot{r}^{-1}\frac{d}{d\tau}\left(f\dot{T}\right). \tag{2.24}$$

We assume that the energy-momentum tensor of the localized matter is given by that of a perfect fluid,

$$T^\mu_\nu = \mathrm{diag}(-\rho - \sigma,\ p - \sigma,\ p - \sigma,\ p - \sigma), \tag{2.25}$$

where we include the tension of the brane σ. Then the junction conditions [52] imply that

$$K^i{}_j = -\frac{1}{6M_5^3}(\rho+\sigma)\delta^i{}_j, \tag{2.26}$$

$$K_{\tau\tau} = -\frac{1}{6M_5^3}(2\rho+3p-\sigma). \tag{2.27}$$

From the spatial component K_{ij} we have

$$H^2 := \left(\frac{\dot{r}}{r}\right)^2 = \frac{1}{(6M_5^3)^2}(\rho+\sigma)^2 - \frac{1}{\ell^2} - \frac{k}{r^2} + \frac{\mathcal{C}}{r^4}. \tag{2.28}$$

After substituting the RS relations $M_{\rm Pl}^2 = \ell M_5^3$ and $\sigma = 6M_5^3/\ell$, we finally obtain the *modified Friedmann equation* on the brane [9, 10, 111, 71, 51]

$$H^2 = \frac{\rho}{3M_{\rm Pl}^2}\left(1+\frac{\rho}{2\sigma}\right) - \frac{k}{a^2} + \frac{\mathcal{C}}{a^4}. \tag{2.29}$$

When $\rho \ll \sigma$ and $\mathcal{C} = 0$, the conventional Friedmann equation is recovered on the brane. The modification to the standard cosmology arises from the quadratic term in ρ relevant at energy scales higher than σ, and the so called "dark radiation" $\mathcal{C}/a^4$ which behaves as an extra radiation component.

From the equation for $K_{\tau\tau}$, it can be seen that the standard conservation law holds on the brane:

$$\dot{\rho} + 3H(\rho+p) = 0. \tag{2.30}$$

For an equation of state $p = w\rho$ with w constant, we have $\rho \propto a^{-3(1+w)}$. Then, assuming that $k = 0$ and $\mathcal{C} = 0$, the modified Friedmann equation (2.29) can be solved to give

$$a = a_* \left(\frac{\tau^2 + c\tau}{\tau_*^2 + c\tau_*}\right)^{1/3(1+w)}, \quad \frac{\rho}{\sigma} = \frac{2\ell^2}{9(1+w)^2}\frac{1}{\tau^2+c\tau}, \tag{2.31}$$

where $c := 2\ell/3(1+w)$. More specifically, in the case of a radiation fluid ($w = 1/3$) we obtain $a \sim \tau^{1/2}$ at low energies ($\rho \ll \sigma$) and $a \sim \tau^{1/4}$ at high energies ($\rho \gg \sigma$).

As an alternative approach, we may also use Gaussian normal coordinates [9, 10, 111] and write the general line element that has the cosmological symmetry as

$$ds^2 = -N^2(\tau,y)d\tau^2 + A^2(\tau,y)\gamma_{ij}dx^i dx^j + dy^2. \tag{2.32}$$

In the Gaussian normal coordinates, the brane does not move but is located at a fixed coordinate position which can be set $y = 0$ without loss of generality. (The equivalence of the two description can be shown by finding the explicit coordinate transformation that links the Gaussian normal and Schwarzschild-AdS coordinates [112].) The five-dimensional Einstein equations together with the junction conditions give the explicit form of the metric functions as [10],

$$\begin{aligned} A(\tau,y) \;=\; & a(\tau)\left\{\frac{1}{2}+\frac{1}{2}\left(1+\frac{\rho}{\sigma}\right)^2+\frac{\ell^2\mathcal{C}}{a(\tau)^4}\right. \\ & \left.+\left[\frac{1}{2}-\frac{1}{2}\left(1+\frac{\rho}{\sigma}\right)^2-\frac{\ell^2\mathcal{C}}{a(\tau)^4}\right]\cosh(2y/\ell)-\left(1+\frac{\rho}{\sigma}\right)\sinh(2y/\ell)\right\}^{1/2}, \end{aligned} \tag{2.33}$$

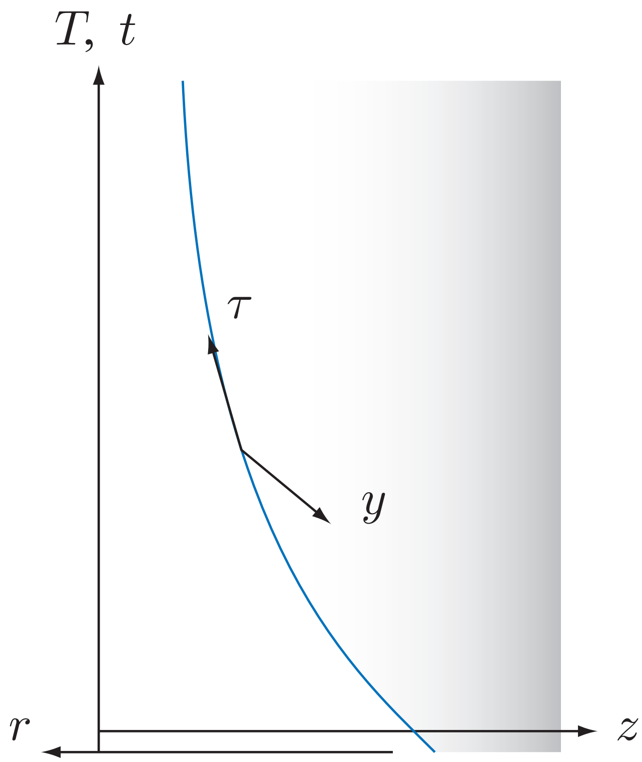

Figure 2.2: Gaussian normal coordinates, Poincaré and Schwarzschild static coordinates. Solid curve represents the world surface of a cosmological brane.

and

$$N(\tau, y) = \frac{\dot{A}(\tau, y)}{\dot{a}(\tau)}. \tag{2.34}$$

In the above, $a(\tau)$ is the scale factor on the brane and $\mathcal{C}$ is equivalent to the black hole mass or the dark radiation parameter in the previous discussion. In deriving Eq. (2.34) we used $N(\tau, 0) = 1$. Of course, by invoking the junction conditions, one can show that the scale factor obeys the modified Friedmann equation (2.29) in this coordinate system as well.

If the dark radiation vanishes, Eq. (2.33) can be simplified to

$$A(\tau, y) = a(\tau)\left[\cosh(y/\ell) - \left(1 + \frac{\rho}{\sigma}\right)\sinh(y/\ell)\right], \tag{2.35}$$

and the explicit form of $N(\tau, y)$ also becomes simple:

$$N(\tau, y) = \frac{A(\tau, y)}{a(\tau)} + \frac{3(\rho + p)}{\sigma}\sinh(y/\ell), \tag{2.36}$$

where we used the conservation law. These two expressions are frequently used in the literature. For a de Sitter brane ($\rho = -p = \text{const.}$), N depends only on y and $A(\tau, y)$ becomes a separable function: $A(\tau, y) = a(\tau)N(y)$, where

$$N(y) = \cosh(y/\ell) - \sqrt{1 + \ell^2 H^2}\,\sinh(y/\ell). \tag{2.37}$$

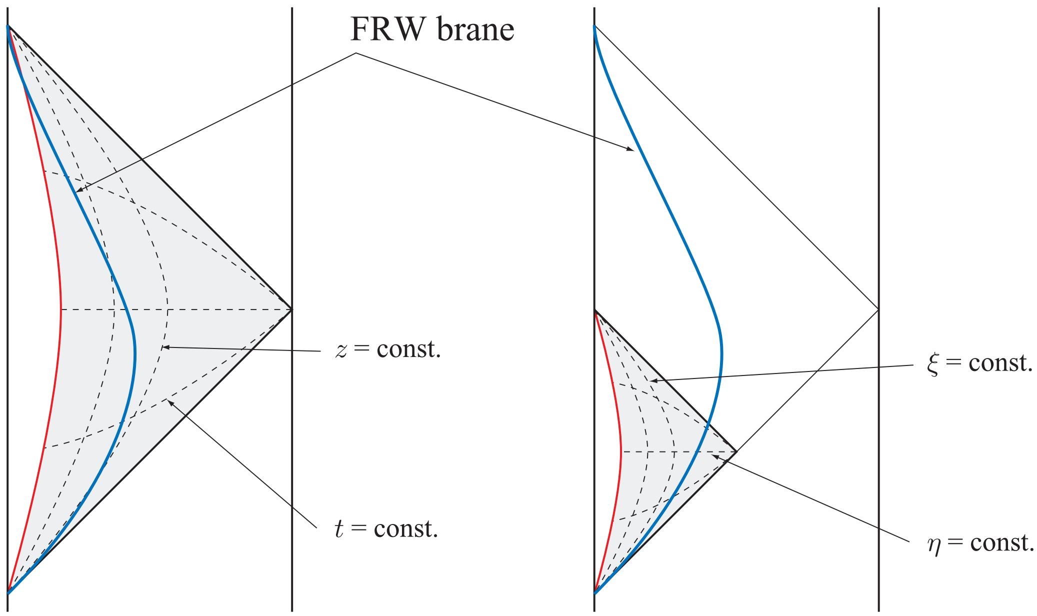

Figure 2.3: Global structures for the Randall-Sundrum model. The Poincaré coordinates (t, z) cover the shaded region in the left figure. For the de Sitter braneworld, (η, ξ) coordinates cover the shaded region in the right figure. Typical trajectories of the FRW brane are shown by thick solid lines.

In this case, by a farther coordinate transformation

$$-\eta = e^{-H\tau}/H, \tag{2.38}$$

$$\sinh\xi = \ell H/N(y), \tag{2.39}$$

we obtain the following bulk metric:

$$ds^2 = \frac{\ell^2}{\sinh^2\xi}\left[\frac{1}{\eta^2}(-d\eta^2 + \delta_{ij}dx^i dx^j) + d\xi^2\right], \tag{2.40}$$

which will be used throughout this thesis. Note that η is the conformal time on the brane. Denoting the brane position in these coordinates as $\xi = \xi_b$, we find from Eq. (2.39) that $\ell H = \sinh\xi_b$.

Let us mention the limitation of the Gaussian normal coordinates. The geodesics orthogonal to the brane eventually will cross, at which the Gaussian normal coordinates are ill defined. This coordinate singularity is at $y = y_c$ defined by

$$\coth(y_c/\ell) = 1 + \frac{\rho}{\sigma}, \tag{2.41}$$

where $A(\tau, y)$ vanishes [see Eq. (2.35)]. Thus, this coordinate system seems not so appropriate for the high energy regime ($\rho \gg \sigma$) because the coordinate singularity is very close to the brane. There is another potential singularity y_h defined by

$$\coth(y_h/\ell) = 1 - (2 + 3w)\frac{\rho}{\sigma}, \tag{2.42}$$

where $N(\tau, y)$ vanishes [see Eq. (2.36)]. This singularity is relevant for $w < -2/3$, and for a de Sitter brane we have $y_c = y_h$.

2.3.2 Dark radiation

We are now in a position to give a short comment on the consequences of the “dark radiation” term in the modified Friedmann equation (2.29). This term corresponds to the projected Weyl tensor $E_{\mu\nu}$ from the brane (four-dimensional) point of view (see Appendix B): $\mathcal{C}/a^4 = E_\tau{}^\tau/3$. The radiation-like behavior can be understood by seeing that $E_{\mu\nu}$ is traceless.

The amount of the dark radiation can be constrained by nucleosynthesis [9]. At the time of nucleosynthesis, the universe should be in the low energy regime ($\rho \ll \sigma$) in order not to spoil the standard cosmology picture, and is dominated by the radiation energy density, which is given by

$$\rho_r|_{\rm N} = g_* \frac{\pi^2}{30} T_{\rm N}^4, \tag{2.43}$$

where g_* is the effective number of relativistic degrees of freedom at that time. In the Standard Model, $g_* = 10.75$. The observed abundances of light elements strongly constrains any deviation Δg_* from that value typically as $\Delta g_* < 2$. Thus, the energy density of the dark radiation $\rho_{\rm DR}(= 3M_{\rm Pl}^2\mathcal{C}/a^4)$ should satisfy the bound

$$\frac{\rho_{\rm DR}}{\rho_r} < 0.16. \tag{2.44}$$

Note that this ratio is invariant because $\rho_{\rm DR}$ has the same dependence on the scale factor as ρ_r. For a more detailed analysis on the observational constraints on the amount of the dark radiation, see Ref. [48].

It is likely that in the early universe gravitational waves escape into the bulk and the parameter $\mathcal{C}$ varies in time due to this process. For example, the high energy collision of particles on the brane results in the production of KK gravitons, which directly leads to the generation of dark radiation [42, 88, 90, 91]. Such a process was studied by using a toy model based on the AdS-Vaidya solution [88, 91] and by resorting to a numerical calculation [90] (see also Refs. [138, 33, 142, 53, 64]). In the most detailed study on this issue by Langlois *et al.* [90], their estimate on the amount of dark radiation generated during the cosmological evolution of the brane universe points to a value close to the current observational bound.

Chapter 3

Basic material on gravitational wave perturbations in braneworld

In the previous chapter we have briefly overviewed homogeneous and isotropic cosmology on a brane. The rest of this thesis treats the generation and evolution of cosmological perturbations in the braneworld [5, 6, 11, 14, 15, 17, 24, 25, 26, 27, 30, 36, 37, 38, 41, 45, 46, 49, 50, 61, 63, 65, 66, 68, 72, 73, 74, 76, 78, 79, 81, 83, 84, 85, 86, 92, 93, 100, 107, 113, 114, 115, 126, 127, 139, 141, 146]. Before going to the main discussions of the thesis, in this chapter we will give some basic facts about cosmological perturbations in the Randall-Sundrum braneworld, focusing on gravitational wave (tensor) perturbations [5, 6, 17, 27, 30, 36, 41, 45, 46, 49, 50, 61, 63, 65, 66, 78, 84, 139].

In the static Poincaré chart, tensor perturbations are given by

$$ds^2 = \frac{\ell^2}{z^2}\left\{-dt^2 + \left[\delta_{ij} + h_{ij}(t,z,x^i)\right]dx^i dx^j + dz^2\right\}. \tag{3.1}$$

(Here and from now on we assume a spatially flat Friedmann-Robertson-Walker brane as a background.) We decompose the perturbations into the spatial Fourier modes and consider one Fourier mode at a time, so that

$$h_{ij} = \sum_{A=+,\times} \frac{\sqrt{2}}{(2\pi M_5)^{3/2}} \int d^3k \phi_{\mathbf{k}}^{(A)}(t,z) e^{i\mathbf{k}\cdot\mathbf{x}} e_{ij}^{(A)}, \tag{3.2}$$

where $e_{ij}^{(A)}$ is a transverse and traceless tensor, and hereafter we suppress the index $\mathbf{k}$. We will also omit the index A for the two polarization states. The meaning of the prefactor $\sqrt{2}/M_5^{3/2}$ will become clear later when considering quantization of perturbations. The equation of motion for ϕ is a Klein-Gordon-type equation:

$$\left(\partial_t^2 + k^2 - \partial_z^2 + \frac{3}{z}\partial_z\right)\phi = 0, \tag{3.3}$$

and it follows from the junction conditions that the boundary condition is given by

$$n^A \partial_A \phi\big|_{\rm brane} = 0, \tag{3.4}$$

where n^A is a unit normal to the brane. In Eq. (3.4) we neglected a possible anisotropic stress term arising from perturbations in the brane energy-momentum tensor. Since Eq. (3.3) is the Bessel equation, its *general solution* can be easily obtained in an analytic form. This is the bonus of working in the Poincaré coordinates. However, the cosmological brane moves in the static bulk, and hence Eq. (3.4) reduces to a moving boundary condition:

$$\left(\ell H\partial_t - \sqrt{1+\ell^2H^2}\partial_z\right)\phi\Big|_{z=z(t)} = 0, \tag{3.5}$$

which is difficult to handle in general.

On the other hand, in the Gaussian normal coordinates the perturbed metric is given by

$$ds^2 = -N^2(\tau,y)d\tau^2 + A^2(\tau,y)\left(\delta_{ij}+h_{ij}\right)dx^i dx^j + dy^2, \tag{3.6}$$

and the tensor perturbation obeys

$$\frac{1}{N^2}\left[\ddot{\phi} + \left(3\frac{\dot{A}}{A} - \frac{\dot{N}}{N}\right)\dot{\phi}\right] + \frac{k^2}{A^2} - \phi'' - \left(3\frac{A'}{A} + \frac{N'}{N}\right)\phi' = 0, \tag{3.7}$$

where the explicit form of $A(\tau,y)$ and $N(\tau,y)$ is found in the previous chapter, and the overdot (prime) denotes a derivative with respect to τ (y). Since the brane is located at a fixed coordinate position, the boundary condition takes a simple form

$$\partial_y\phi(\tau,0) = 0. \tag{3.8}$$

In the Gaussian normal coordinates, however, Eq. (3.7) is not separable for generic functions $A(\tau,y)$ and $N(\tau,y)$. Thus, one will suffer from technical difficulties in any coordinate system.

Fortunately, in the exceptional cases of the maximally symmetric brane, one can choose a coordinate system where the perturbation equation has a separable form and at the same time a simple boundary condition is imposed at a fixed coordinate position. This can be easily seen by noting that we have $A(\tau,y) = a(\tau)N(y)$ for a de Sitter brane, as is argued in the previous chapter. Substituting this into Eq. (3.7), we obtain a manifestly separable equation of motion, subject to the Neumann boundary condition at the brane. The de Sitter braneworld includes the case of a Minkowski brane as a limiting case $H \to 0$. The behavior of gravitational wave perturbations are well understood in these two special cases; let us summarize the basic known results.

Minkowski brane The Minkowski brane sits at $z = \ell =$ const., and hence it is convenient to use the Poincaré chart in this case. The zero mode solution is simply given by

$$\varphi_0(t) = Ae^{-ikt}, \tag{3.9}$$

and the Kaluza-Klein mode solutions are written as

$$\varphi_m(t,z) = Cz^2\left[J_2(mz) - BY_2(mz)\right]e^{-i\omega t}, \tag{3.10}$$

where $\omega = \sqrt{k^2+m^2}$, and the boundary condition at $z = \ell$ requires

$$B = \frac{J_1(m\ell)}{Y_1(m\ell)}. \tag{3.11}$$

At this stage the overall normalization is not so important and we leave it undetermined until Chapter 7.

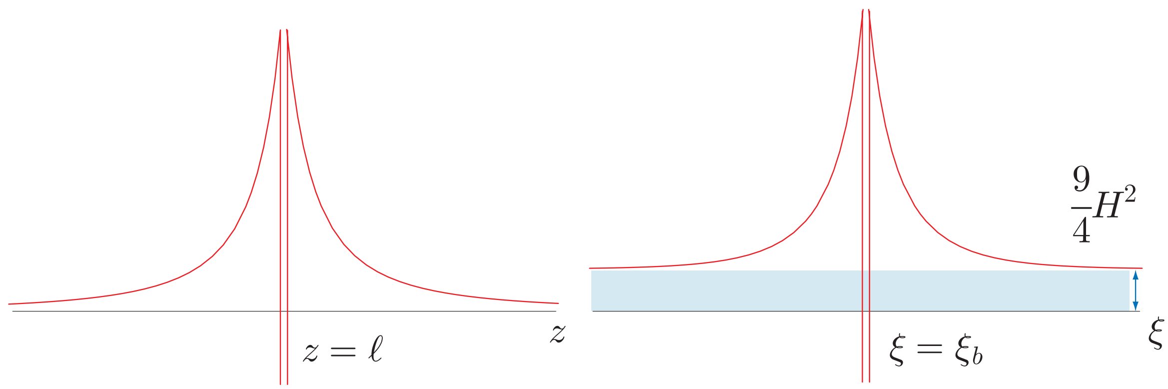

Figure 3.1: "Volcano" potential for gravitons around the Minkowski brane (left) and the de Sitter brane (right).

Since the extra dimension is not compact in the RS2 model, the mass spectrum is continuous and m^2 can take any non-negative value. This can be understood by introducing the canonical variable $\hat{\psi} := z^{-3/2}\varphi_m$. In terms of $\hat{\psi}$, the Klein-Gordon equation can be rewritten in the form of Schrödinger equation,

$$\left[-\frac{\partial^2}{\partial z^2} + V(z)\right]\hat{\psi} = m^2\hat{\psi}, \tag{3.12}$$

$$V(z) = \frac{15}{4z^2} - \frac{3}{\ell}\delta(z-\ell), \tag{3.13}$$

where the boundary condition is incorporated as the delta-function in the potential. From the left figure of Fig. 3.1, it is easy to find that there is a single bound state (zero mode) as well as a gapless continuum of KK modes.

The motion of the cosmological brane in the low energy regime is *non-relativistic*, in the sense that $|dz/dt| \ll 1$, and so such a brane may be approximated by a Minkowski brane [5, 6, 27, 139, 63]. Having this fact in mind, in the next chapter we shall study the evolution of gravitational waves at low energies perturbatively taking into account small deviation from a Minkowski brane [63].

De Sitter brane Even though realistic inflation should not be exactly de Sitter, a de Sitter braneworld [84] is important in that it gives the simplest picture of how cosmological perturbations are generated and evolve during inflation in the early universe[1].

The perturbation equation in the Gaussian normal coordinates reduces to

$$\frac{1}{N^2}\left(\ddot{\phi} + 3H\dot{\phi} + \frac{k^2}{a^2}\phi\right) = \phi'' + 4\frac{N'}{N}\phi', \tag{3.14}$$

where

$$a(\tau) = a_0 e^{H(\tau-\tau_0)}, \tag{3.15}$$

$$N(y) = \cosh(y/\ell) - \sqrt{1+\ell^2H^2}\sinh(y/\ell), \tag{3.16}$$

[1] Appendix A will be helpful to the readers who are not familer with the standard four-dimensional material about the generation of cosmological perturbations from inflation.

with constant H. Equation (3.14) admits a zero mode satisfying

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{k^2}{a^2}\phi_0 = 0, \tag{3.17}$$

as well as massive KK modes,

$$\phi_m(\tau, y) = \psi_m(\tau) \cdot \chi_m(y), \tag{3.18}$$

where $\psi(\tau)$ and $\chi(y)$ obey

$$\ddot{\psi}_m + 3H\dot{\psi}_m + \left(\frac{k^2}{a^2} + m^2\right)\psi_m = 0, \tag{3.19}$$

$$\chi_m'' + 4\frac{N'}{N}\chi_m' + \frac{m^2}{N^2}\chi_m = 0. \tag{3.20}$$

Equation (3.17) is nothing but the Klein-Gordon equation for a massless scalar field in the Friedmann universe, while Eq. (3.19) is equivalent to the equation of motion for a massive scalar field.

Throughout this thesis we work mainly in the coordinate system defined by Eqs. (2.38) and (2.39) when discussing de Sitter inflation on the brane. In terms of the conformal time η, the zero mode solution is given by

$$\phi_0(\eta) = \mathcal{N} \cdot \frac{1}{\sqrt{2k\ell}}\left(\eta - \frac{i}{k}\right)e^{-ik\eta}, \tag{3.21}$$

where $\mathcal{N}$ is a normalization constant to be determined. Note that this coincides with the standard mode function associated with the Bunch-Davies vacuum in de Sitter space [12, 16] up to the normalization factor. In (η, ξ) coordinates the equation for the massive modes are rewritten as

$$\left(\frac{\partial^2}{\partial\eta^2} - \frac{2}{\eta}\frac{\partial}{\partial\eta} + k^2 + \frac{\nu^2 + 9/4}{\eta^2}\right)\psi_\nu(\eta) = 0, \tag{3.22}$$

$$\left(\sinh^3\xi\frac{\partial}{\partial\xi}\frac{1}{\sinh^3\xi}\frac{\partial}{\partial\xi} + \nu^2 + \frac{9}{4}\right)\chi_\nu(\xi) = 0, \tag{3.23}$$

where

$$m^2 = \left(\nu^2 + \frac{9}{4}\right)H^2. \tag{3.24}$$

The mode solutions are found for any non-negative values of ν^2 as

$$\psi_\nu(\eta) = C_3(-\eta)^{3/2}H_{i\nu}^{(1)}(-k\eta), \tag{3.25}$$

$$\chi_\nu(\xi) = C_1\sinh^2\xi\left[P^{-2}_{-1/2+i\nu}(\cosh\xi) - C_2Q^{-2}_{-1/2+i\nu}(\cosh\xi)\right], \tag{3.26}$$

where C_3 and C_1 are normalization constants and C_2 is determined from the boundary condition as

$$C_2 = \frac{P^{-1}_{-1/2+i\nu}(\cosh\xi_b)}{Q^{-1}_{-1/2+i\nu}(\cosh\xi_b)}, \tag{3.27}$$

with ξ_b the position of the brane. The result $m^2 = (\nu^2 + 9/4)H^2 \geq 9H^2/4$ indicates that there is a mass gap between the zero mode and the continuum of massive modes [32][2]. This is in contrast to the case of the Minkowski brane. Since massive fields with $m^2 \geq 9H^2/4$ decay at super-horizon scales during inflaton, it is expected that the zero mode will dominate *on the brane*.

Again using the canonical variable $\hat{\psi} := N^{3/2}\chi_m$, the equation of motion can be rewritten as

$$\left[-\frac{\partial^2}{\partial\xi^2} + V(\xi)\right]\hat{\psi} = \frac{m^2}{H^2}\hat{\psi}, \tag{3.28}$$

$$V(\xi) = \frac{9}{4} + \frac{15}{4\sinh^2\xi} - 3\frac{\sqrt{1+\ell^2H^2}}{\ell H}\delta(\xi - \xi_b), \tag{3.29}$$

where the boundary condition is incorporated as the delta-function well. Noting that $V(\infty) = 9/4$, we clearly see that only massive modes with $m^2/H^2 \geq 9/4$ can exist other than the zero mode (Fig. 3.1).

So far we have left undetermined the overall normalization of the mode functions. To give appropriate normalization constants, we need to go to quantum theory. Now the reason for the presence of the factor $\sqrt{2}/M_5^{3/2}$ in Eq. (3.2) is clear; with this choice the effective action for ϕ derived from the Einstein-Hilbert action is equivalent to the action of a canonically normalized scalar field, whose kinetic term is given by $(1/2)\partial_A\phi\partial^A\phi$. Therefore, we can quantize the graviton field ϕ in a standard textbook manner [12]. Treating the field ϕ as an operator, it can be expanded in terms of the annihilation and creation operators as

$$\hat{\phi}_{\mathbf{k}}(\eta,\xi) = \hat{a}_0\phi_0 + \hat{a}_0^\dagger\phi_0^* + \int d\nu\left(\hat{a}_\nu\phi_\nu + \hat{a}_\nu^\dagger\phi_\nu^*\right), \tag{3.30}$$

where $\hat{a}_n$ and $\hat{a}_n^\dagger$ is respectively the annihilation and creation operators of a zero mode for $n = 0$ and of a KK mode for $n = \nu$. These operators satisfy the usual commutation relations. In quantum field theory, mode functions form an orthonormal system with respect to the Wronskian (the scalar product) $(\phi_1 \cdot \phi_2)$. In the present case the Wronskian is naturally defined as [36, 61]

$$(\phi_1 \cdot \phi_2) := -2i\int_{\xi_b}^{\infty} d\xi \frac{\ell^3}{\eta^2\sinh^3\xi}\left(\phi_1\partial_\eta\phi_2^* - \phi_2^*\partial_\eta\phi_1\right), \tag{3.31}$$

and the Wronskian conditions are given by

$$(\phi_0 \cdot \phi_0) = -(\phi_0^* \cdot \phi_0^*) = 1, \quad (\phi_\nu \cdot \phi_{\nu'}) = -(\phi_\nu^* \cdot \phi_{\nu'}^*) = \delta(\nu - \nu'),$$
$$(\phi_0 \cdot \phi_\nu) = (\phi_0^* \cdot \phi_\nu^*) = (\phi_n \cdot \phi_{n'}^*) = 0, \qquad \text{for} \quad n, n' = 0, \nu.$$

For the zero mode we have

$$\mathcal{N} = C(\ell H), \tag{3.32}$$

where

$$\begin{aligned} C(\ell H) &= \left[2\sinh^2\xi_b\int_{\xi_b}^{\infty} d\xi\frac{1}{\sinh^3\xi}\right]^{-1/2} \\ &= \left[\sqrt{1+\ell^2H^2} + \ell^2H^2\ln\left(\frac{\ell H}{1+\sqrt{1+\ell^2H^2}}\right)\right]^{-1/2}. \end{aligned} \tag{3.33}$$

[2]This result has been generalized in the case of de Sitter brane(s) with a bulk scalar field, showing that the universal lower bound on the mass gap is $\sqrt{3/2}H$ [30].

For $x \ll 1$ we have $C(x) \approx 1$, while we obtain $C(x) \approx \sqrt{3x/2}$ for $x \gg 1$. For the KK modes we separately impose

$$-\frac{i\ell^3}{\eta^2}\left(\psi_\nu \partial_\eta \psi_\nu^* - \psi_\nu^* \partial_\eta \psi_\nu\right) = 1, \tag{3.34}$$

$$2\int_{\xi_b}^{\infty} \frac{d\xi}{\sinh^3 \xi} \chi_{\nu'}^* \chi_\nu = \delta(\nu - \nu'). \tag{3.35}$$

We see from the condition (3.34) that the normalized mode function in the time direction is given by

$$\psi_\nu(\eta) = \frac{\sqrt{\pi}}{2} \ell^{-3/2} e^{-\pi\nu/2} (-\eta)^{3/2} H_{i\nu}^{(1)}(-k\eta). \tag{3.36}$$

The condition (3.35) implies that the weighted mode function $(\sinh \xi)^{-2/3} \chi_\nu(\xi)$ behaves as (a linear combination of) the standard plane wave mode $e^{\pm i\nu\xi}/\sqrt{2\pi}$ asymptotically far from the brane ($\xi \to \infty$). More specifically, it is required that at infinity

$$(\sinh \xi)^{-2/3} \chi_\nu(\xi) \sim \frac{1}{\sqrt{2\pi}} \left(A_+ e^{-i\nu\xi} + A_- e^{i\nu\xi}\right), \tag{3.37}$$

with $|A_+|^2 + |A_-|^2 = 1$. From this and the asymptotic behavior of the associated Legendre functions,

$$P_{-1/2+i\nu}^{-2}(\cosh \xi) \sim \frac{1}{\sqrt{\pi}} \left[\frac{\Gamma(i\nu)}{\Gamma(5/2 + i\nu)} e^{(-1/2+i\nu)\xi} + \frac{\Gamma(-i\nu)}{\Gamma(5/2 - i\nu)} e^{(-1/2-i\nu)\xi}\right], \tag{3.38}$$

$$Q_{-1/2+i\nu}^{-2}(\cosh \xi) \sim \sqrt{\pi} \frac{\Gamma(i\nu - 3/2)}{\Gamma(1 + i\nu)} e^{(-1/2-i\nu)\xi}, \tag{3.39}$$

the normalization constant of the mode in the extra direction is determined as

$$C_1 = \left[\left|\frac{\Gamma(i\nu)}{\Gamma(5/2 + i\nu)}\right|^2 + \left|\frac{\Gamma(-i\nu)}{\Gamma(5/2 - i\nu)} - \pi C_2 \frac{\Gamma(i\nu - 3/2)}{\Gamma(1 + i\nu)}\right|^2\right]^{-1/2}. \tag{3.40}$$

Now let us evaluate the amplitude of quantum fluctuations. The amplitude of perturbations is often discussed in terms of the power spectrum defined by

$$\langle 0|\hat{\phi}_{\mathbf{k}} \hat{\phi}_{\mathbf{k}'}|0\rangle = \frac{(2\pi)^3}{4\pi k^3} \delta^{(3)}\left(\mathbf{k} + \mathbf{k}'\right) \mathcal{P}_\phi(k). \tag{3.41}$$

In the present case $\hat{\phi}_{\mathbf{k}}$ should be understood as the zero mode part of the graviton field, and hence the power spectrum of four-dimensional gravitational waves in the braneworld is given by

$$\mathcal{P}(k) = \frac{k^3}{2\pi^2} \cdot \frac{2}{M_5^3} |\phi_0|^2, \tag{3.42}$$

where the fundamental mass M_5 appeared due to the normalization. Quantum fluctuations in the zero mode are stretched beyond the horizon radius by rapid expansion during inflation, and in the super-horizon regime we have $|\phi_0|^2 \approx C^2(\ell H) H^2 / 2k^3 \ell$. Thus,

$$\mathcal{P}(k) = \frac{2C^2(\ell H)}{M_{\rm Pl}^2} \left(\frac{H}{2\pi}\right)^2, \tag{3.43}$$

where we used $M_{\rm Pl}^2 = \ell M_5^3$. At low energies, we have $C(\ell H) \approx 1$, and we recover the standard four-dimensional result. For high-energy inflation, we have

$$\mathcal{P} \approx \frac{3\ell H}{M_{\rm Pl}^2} \left(\frac{H}{2\pi}\right)^2, \tag{3.44}$$

which is much greater than the standard result in four dimensions. The CMB observations constrain the amplitude of the gravitational waves as $\mathcal{P}^{1/2} < 10^{-5}$, so that

$$\ell H \ll \left(\frac{M_{\rm Pl}}{M_5}\right)^2. \tag{3.45}$$

From this we see that even though the gravitational wave amplitude is enhanced, inflation at energy scales as high as $H \gg \ell^{-1}$ is still allowed. In any case, exact de Sitter inflation on the brane predicts the flat spectrum of gravitational waves, in common with four-dimensional general relativity.

Chapter 4

Leading order corrections to the evolution of tensor perturbations

The second Randall-Sundrum model (RS2) [124] with one brane in an anti de Sitter (AdS) bulk is particularly interesting in the point that, despite an infinite extra dimension, four dimensional general relativity (4D GR) can be recovered at low energies/long distances on the brane. *What is leading order corrections to the conventional gravitational theory?* It was shown that the Newtonian potential in the RS2 braneworld, including the correction due to the bulk gravitational effects with a precise numerical factor [31, 35], is given by

$$V(r) \simeq -\frac{Gmm'}{r}\left(1+\frac{2\ell^2}{3r^2}\right). \tag{4.1}$$

Here ℓ is the curvature length of the AdS space and is experimentally constrained to be $\ell \lesssim 0.1$ mm [99, 22]. It is natural to ask next what are leading order corrections to cosmological perturbations.

Great efforts have been paid for the problem of calculating cosmological perturbations in braneworld scenarios. In order to correctly evaluate perturbations on the brane, we need to solve bulk perturbations, which reduces to a problem of solving partial differential equations with appropriate boundary conditions. Hence it is not as easy as in the standard four dimensional cosmology. A pure de Sitter braneworld [84] explained in the previous chapter, and its variations such as "junction" models [36, 61], which will be discussed in the next chapter, and a special dilatonic braneworld [75, 77, 62], which will be the topic of Chapter 8, are the rare examples where the bulk perturbations are analytically solved. In more generic cases, one has to resort to a numerical calculation [45, 46, 49, 50], or some approximate methods [76, 27, 5, 78]. There is another difficulty concerned with a physical, fundamental aspect of the problem; we do not know how to specify appropriate initial conditions for perturbations with bulk degrees of freedom.

In this chapter, we investigate cosmological tensor perturbations in the RS2 braneworld, and evaluate leading order corrections to the 4D GR result analytically. For this purpose, we make use of the reduction scheme to a four dimensional effective equation which iteratively takes into account the effects of the bulk gravitational fields [139]. As a first step, we concentrate on perturbations which are initially on super-horizon scales.

The chapter is organized as follows. In Sec. 4.1 we define the setup of the problem discussed in the present chapter more precisely. In Sec. 4.2 we briefly review the derivation of the effective equation of motion for tensor perturbations. In Sec. 4.3, using the long wavelength approximation we develop a method for evaluating leading order corrections, and evaluate the corrections considering slow-roll inflation and the radiation stage which follows after inflation. The shortcoming of the long wavelength approximation is cured by introducing another method suitable for analyzing the radiation phase in Sec. 4.4. Section 4.5 is devoted to summary and discussion.

4.1 Setup of the problem and background cosmological model

In this chapter we investigate tensor perturbations in a Friedmann-Robertson-Walker (FRW) braneworld model with an warped (infinite) extra dimension [124, 9, 10, 111, 71, 51]. To determine the spectrum of tensor perturbations observed at late time, we need to solve the perturbation equations with appropriate initial conditions. However, it is not an easy task to specify what initial conditions we should use in the context of braneworld cosmology[1]. At an earlier stage of the evolution of the universe the physical wavelength of perturbation modes is much shorter than the bulk curvature scale. Then, the correction to the four dimensional standard evolution will not remain small. Therefore, in this regime it is not appropriate to discuss the dynamics of perturbation modes under the assumption that a modification to the four dimensional standard one is small. Hence our strategy in this chapter is to isolate the issue of initial conditions from the rest. In this chapter we discuss a relatively easy part, i.e., the late time evolution after the effect of the fifth dimension becomes perturbative.

Our discussion will be restricted to the perturbation modes whose wavelength is initially much longer than the Hubble horizon scale. We analytically derive a formula for small corrections to the solution for the tensor perturbation equation due to the presence of the fifth dimension. We mainly focus on the behavior of a growing solution, since a growing solution will be more important than a decaying one in general. Later we briefly mention the decaying solution in Sec. 4.5.

The background spacetime that we consider is composed of a five dimensional AdS bulk, whose metric is given in Poincaré coordinates as

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + \delta_{ij}dx^i dx^j\right),$$

with a FRW brane at $z = z(t)$. Here ℓ is the curvature length of AdS space. In the original RS2 model [124], a brane is placed at a position of fixed z, and Minkowski geometry is realized on the brane. The induced metric on $z = z(t)$ is

$$\begin{aligned} ds^2 &= a^2(t)\left[-(1-\dot{z}^2)dt^2 + \delta_{ij}dx^i dx^j\right], \\ a(t) &= \ell/z(t), \end{aligned} \tag{4.2}$$

where the overdot denotes ∂_t. From Eq. (4.2), we see that the conformal time on the brane is given by $d\eta = \sqrt{1-\dot{z}^2}dt$. The brane motion is related to the energy density of matter localized on the brane by the modified Friedmann equation [9, 10, 111, 71, 51] as

$$H^2 = \frac{8\pi G}{3}\left(\rho + \frac{\rho^2}{2\sigma}\right), \tag{4.3}$$

[1] The issue of initial conditions for perturbations will be addressed in a later chapter.

and

$$H := \frac{\partial_\eta a}{a^2} = \frac{-\ell^{-1}\dot{z}}{\sqrt{1-\dot{z}^2}}, \tag{4.4}$$

where $\sigma = 3/4\pi G\ell^2$ is a tension of the brane. The quadratic term in ρ in the Friedmann equation modifies the standard cosmological expansion law. At low energies ($\ell H \ll 1$), the brane motion is non-relativistic ($\dot{z} \ll 1$), and hence the conformal time on the brane η almost coincides with the bulk conformal time t.

As a background FRW brane model, we consider slow-roll inflation at low energies followed by a radiation dominant phase. During the slow-roll inflation the wavelength of perturbations stays outside the Hubble scale, and it re-enters the Hubble horizon only during the radiation era.

We fix a model of slow-roll inflation simply by providing a function

$$\epsilon(\eta) := 1 - \frac{\partial_\eta \mathcal{H}}{\mathcal{H}^2}, \tag{4.5}$$

where $\mathcal{H} := \partial_\eta a/a = aH$. During slow-roll inflation, ϵ is supposed to be small. In the $\epsilon \to 0$ limit, we recover the de Sitter case

$$a = \frac{a_0\eta_0}{2\eta_0 - \eta}, \qquad \mathcal{H} = \frac{1}{2\eta_0 - \eta}, \qquad (\eta < \eta_0), \tag{4.6}$$

where we have chosen the normalization of the scale factor and the origin of the η-coordinate so that $a = a_0$ and $\mathcal{H} = 1/\eta_0$ at $\eta = \eta_0$.

In the above alternatively we could specify the potential of the inflaton field localized on the brane. Then, for a given inflaton potential, the evolution of the background scale factor would differ from the conventional four dimensional one. However, as we will show below, the correction to the tensor perturbations coming from the slow-roll inflation phase is always small irrespective of the details of the evolution of the scale factor. Hence, we simply give time-dependence of the slow-roll parameter to specify a model.

We assume that slow-roll inflation terminates at around $\eta = \eta_0$, and a radiation era follows. During the radiation era the energy density decreases like $\rho = \rho_0(a_0/a)^4$ as usual. Then, solving Eq. (4.3) at low energies, we find that the scale factor behaves like

$$a(\eta) = \overset{(0)}{a}(\eta) + \delta a(\eta) + \mathcal{O}(\ell^4), \qquad (\eta > \eta_0), \tag{4.7}$$

with

$$\overset{(0)}{a}(\eta) = a_0\frac{\eta}{\eta_0}, \qquad \delta a(\eta) = a_0\ell^2 H_0^2\left[\frac{1}{6} - \frac{1}{24}\left(\frac{\eta_0}{\eta}\right)^3 - \frac{1}{8}\left(\frac{\eta}{\eta_0}\right)\right], \tag{4.8}$$

and $H_0 = (a_0\eta_0)^{-1}$. Correspondingly, we have

$$\mathcal{H}(\eta) = \overset{(0)}{\mathcal{H}}(\eta) + \delta\mathcal{H}(\eta) + \mathcal{O}(\ell^4), \tag{4.9}$$

with

$$\overset{(0)}{\mathcal{H}}(\eta) = \frac{1}{\eta}, \qquad \delta\mathcal{H}(\eta) = \frac{\ell^2}{6a^2\eta_0^3}\left[\left(\frac{\eta_0}{\eta}\right)^3 - 1\right], \qquad (\eta > \eta_0). \tag{4.10}$$

It is convenient to extend the definition of the slow-roll parameter ϵ to the later epoch by

$$\epsilon(\eta) := 1 - \frac{\partial_\eta \overset{(0)}{\mathcal{H}}}{\overset{(0)}{\mathcal{H}}{}^2}. \tag{4.11}$$

4.2 Tensor perturbations on a FRW brane

We use the method to reduce the five-dimensional equation for tensor perturbations in a Friedmann braneworld at low energies to a four-dimensional effective equation of motion, derived in Ref. [139]. Here we summarize the derivation.

Tensor perturbations on a Friedmann brane are given by

$$ds^2 = \frac{\ell^2}{z^2}\left[dz^2 - dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]. \tag{4.12}$$

We expand the perturbations by using $Y^k_{ij}(\boldsymbol{x})$, a transverse traceless tensor harmonics with comoving wave number k, as $h_{ij} = \sum_k Y^k_{ij}(\boldsymbol{x})\Phi_k(t,z)$. Then the equation of motion for the tensor perturbations in the bulk is given by

$$\left(-\partial_z^2 + \frac{3}{z}\partial_z + \partial_t^2 + k^2\right)\Phi_k = 0, \tag{4.13}$$

Hereafter we will discuss each Fourier mode separately, and we will abbreviate the subscript k. The general solution of this equation is

$$\begin{aligned}\Phi &= \int d\omega \tilde{\Psi}(\omega) e^{-i\omega t} 2(pz)^2 K_2(pz) \\ &= \int d\omega \tilde{\Psi}(\omega) e^{-i\omega t}\left\{1 - \frac{(pz)^2}{4} + \frac{(pz)^4}{16}\left[\frac{3}{4} - \gamma - \ln\left(\frac{pz}{2}\right)\right] + \cdots\right\},\end{aligned} \tag{4.14}$$

where $p^2 = -\omega^2 + k^2$, and γ is Euler's constant. We have chosen the branch cut of the modified Bessel function K_2 so that there is no incoming wave from past null infinity in the bulk. The coefficients $\tilde{\Psi}(\omega)$ are to be determined by the boundary condition on the brane $n^\mu \partial_\mu \Phi|_{z=z(t)} = 0$, where n^μ is an unit normal to the brane, or, equivalently, by the effective Einstein equations on the brane [134],

$${}^{(4)}G_{\mu\nu} = 8\pi G\, T_{\mu\nu} + (8\pi G_5)^2 \pi_{\mu\nu} - E_{\mu\nu}. \tag{4.15}$$

(The derivation of this equation is summarized in Appendix. B.) Here $\pi_{\mu\nu}$ is quadratic in the energy momentum tensor $T_{\mu\nu}$, and the first two terms on the right hand side are totally represented by the variables which reside on the brane. A projected Weyl tensor $E_{\mu\nu} := C_{\alpha\mu\beta\nu} n^\alpha n^\beta$ represents the effects of the bulk gravitational fields, giving rise to corrections to four dimensional Einstein gravity in a fairly nontrivial way. In the present case, this can be written explicitly, and the effective four-dimensional equation reduces to

$$\left(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + k^2\right)\phi = -2E, \tag{4.16}$$

$$-2E = \left[(\ell H)^2(\partial_t^2 + \partial_z^2) - 2\ell H\sqrt{1 + (\ell H)^2}\,\partial_t\partial_z + \partial_z^2 - \frac{1}{z}\partial_z\right]\Phi\Bigg|_{z=z(t)}, \tag{4.17}$$

where $\phi(t) := \Phi(t, z(t))$ is the perturbation evaluated on the brane. At low energies ($\ell H \ll 1$), we can use the approximation like

$$\partial_t^2\Phi \simeq \partial_\eta^2\Phi, \quad \partial_t\partial_z\Phi \simeq -\frac{\ell}{2a}\partial_\eta(\partial_\eta^2+k^2)\Phi, \quad \partial_z^2\Phi \simeq -\frac{1}{2}\int d\omega\tilde{\Psi}e^{-i\omega\eta}p^2 \simeq -\frac{1}{2}(\partial_\eta^2+k^2)\Phi,$$

neglecting higher order corrections of $\mathcal{O}(\ell^4)$. Then, using the lower order equation to eliminate the higher derivative terms in $-2E$, we finally obtain our basic equation:

$$\left(\partial_\eta^2 + 2\overset{(0)}{\mathcal{H}}\partial_\eta + k^2\right)\phi \simeq \frac{\ell^2}{\overset{(0)}{a}{}^2}\left(\mathcal{S}_0[\phi]+\mathcal{S}_1[\phi]+\mathcal{S}_2[\phi]\right), \tag{4.18}$$

where

$$\mathcal{S}_0[\phi] := -\frac{2\overset{(0)}{a}{}^2}{\ell^2}\delta\mathcal{H}\phi', \tag{4.19}$$

$$\mathcal{S}_1[\phi] := (3\overset{(0)}{\mathcal{H}}{}^3 - 2\overset{(0)}{\mathcal{H}}\overset{(0)}{\mathcal{H}}{}')\phi' + k^2\overset{(0)}{\mathcal{H}}{}^2\phi, \tag{4.20}$$

$$\mathcal{S}_2[\phi] := -\frac{1}{2}\int d\omega p^4\tilde{\phi}e^{-i\omega\eta}\left[\ln\left(\frac{p\ell}{2a}\right)+\gamma\right], \tag{4.21}$$

and the prime denotes ∂_η. The first term arises due to the non-standard cosmological expansion included in $\mathcal{H}$, while the last two terms are the corrections from the bulk effects $E_{\mu\nu}$. We can see that all terms are suppressed by ℓ^2 (or $\ell^2\ln\ell$) [2]. $\mathcal{S}_2$ is essentially nonlocal because of the presence of the log term. A time-domain expression for the quantity $p^4\phi$ which appears in $\mathcal{S}_2$ can be rewritten, by eliminating the higher derivative terms in a similar way as before, as

$$\hat{p}^4\phi \simeq -2\left[\left(\overset{(0)}{\mathcal{H}}{}'' - 6\overset{(0)}{\mathcal{H}}\overset{(0)}{\mathcal{H}}{}' + 4\overset{(0)}{\mathcal{H}}{}^3\right)\partial_\eta + 2k^2\left(\overset{(0)}{\mathcal{H}}{}^2 - \overset{(0)}{\mathcal{H}}{}'\right)\right]\phi, \tag{4.22}$$

where we have introduced a derivative operator $\hat{p}^2 = \partial_\eta^2 + k^2$.

4.3 New long wavelength approximation

Let us begin with the case that the wavelength of perturbations is much longer than the Hubble scale. The long wavelength approximation not only allows us an easy treatment of perturbations, but also turns out to be sufficient to derive all the major corrections except for that coming from $\mathcal{S}_0$. We would like to stress that what we call the long wavelength approximation here is different from a low energy expansion scheme or a gradient expansion scheme in the literature [57, 58, 135].

4.3.1 General iteration scheme

The equation we are considering takes the form of

$$\left(\partial_\eta^2 + 2\overset{(0)}{\mathcal{H}}\partial_\eta + k^2\right)\phi = \frac{\ell^2}{\overset{(0)}{a}{}^2}\mathcal{S}[\phi]. \tag{4.23}$$

[2] Hereafter, we will refer to both terms suppressed by ℓ^2 and by $\ell^2\ln\ell$ as $\mathcal{O}(\ell^2)$.

This can be solved iteratively taking ℓ^2 as a small parameter. For this purpose, we write

$$\phi(\eta) = \overset{(0)}{\phi}(\eta) e^{F(\eta)} = \overset{(0)}{\phi}(\eta) \exp\left[\int^{\eta} f(\eta')d\eta'\right]. \tag{4.24}$$

A zeroth order solution $\overset{(0)}{\phi}$ by definition satisfies the equation obtained by setting the right hand side of Eq. (4.23) to be zero. Then f obeys

$$\partial_\eta f + 2\left(\frac{\partial_\eta \overset{(0)}{\phi}}{\overset{(0)}{\phi}} + \overset{(0)}{\mathcal{H}}\right) f = \frac{\ell^2}{\overset{(0)}{a}{}^2 \overset{(0)}{\phi}} \mathcal{S}[\overset{(0)}{\phi}], \tag{4.25}$$

which can be integrated immediately to give

$$f(\eta) = \frac{\ell^2}{\overset{(0)}{a}{}^2 \overset{(0)}{\phi}{}^2} \int^{\eta} d\eta' \overset{(0)}{\phi} \mathcal{S}[\overset{(0)}{\phi}]. \tag{4.26}$$

Integrating this expression, we obtain the first order correction F. In the present case, $\mathcal{S}$ consists of three parts, $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$. We hereafter denote the corrections f and F coming from $\mathcal{S}_i$ as f_i and F_i, respectively.

4.3.2 Long wavelength expansion of the zeroth order solution

In the long wavelength approximation, we can write the zeroth order solution explicitly as

$$\overset{(0)}{\phi} \simeq A_k\left[1 - k^2 \int^{\eta} d\eta' I(\eta')\right], \tag{4.27}$$

$$\partial_\eta \overset{(0)}{\phi} \simeq -A_k k^2 I(\eta), \tag{4.28}$$

where A_k is an amplitude of the perturbation which can depend on k, and

$$I(\eta) := \overset{(0)}{a}{}^{-2}(\eta) \int_{-\infty}^{\eta} \overset{(0)}{a}{}^2(\eta')d\eta'. \tag{4.29}$$

Here we simply keep the terms up to $\mathcal{O}(k^2)$.

We have already introduced a parameter $\epsilon(\eta)$ in Eq. (4.11), which parameterizes the cosmological expansion. In the inflationary universe, it reduces to a slow-roll parameter and is assumed to be small. Slow-roll inflation assumes that time differentiation of ϵ is $\mathcal{O}(\epsilon^2)$. Hence, we treat ϵ as a constant during slow-roll inflation. The transition from inflation to the radiation stage occurs at around $\eta = \eta_0$, and the behavior of ϵ there depends on the details of the reheating process. During the radiation stage, ϵ keeps again a constant value, $\epsilon = 2$, because the scale factor behaves like $\overset{(0)}{a} \propto \eta$. In general ϵ reduces to a constant for the scale factor $\overset{(0)}{a}$ proportional to a power of the conformal time.

Using ϵ and the zeroth order solution in the long wavelength approximation, we can write

$\mathcal{S}_1[\overset{(0)}{\phi}]$ and $\hat{p}^4\overset{(0)}{\phi}$ as

$$\begin{aligned} \mathcal{S}_1[\overset{(0)}{\phi}] &= \overset{(0)}{\mathcal{H}}{}^2\left[(1+2\epsilon)\overset{(0)}{\mathcal{H}}\partial_\eta\overset{(0)}{\phi}+k^2\overset{(0)}{\phi}\right] \\ &\simeq A_k k^2\overset{(0)}{\mathcal{H}}{}^2\left[1-\overset{(0)}{\mathcal{H}}I-2\epsilon\overset{(0)}{\mathcal{H}}I\right], \end{aligned} \tag{4.30}$$

$$\begin{aligned} \hat{p}^4\overset{(0)}{\phi} &\simeq 2\overset{(0)}{\mathcal{H}}{}^2\left\{\left[\epsilon'-2\epsilon(1+\epsilon)\overset{(0)}{\mathcal{H}}\right]\partial_\eta\overset{(0)}{\phi}-2\epsilon k^2\overset{(0)}{\phi}\right\} \\ &\simeq -2A_k k^2\overset{(0)}{\mathcal{H}}{}^2\left\{\epsilon' I-2\epsilon\left[(1+\epsilon)\overset{(0)}{\mathcal{H}}I-1\right]\right\}, \end{aligned} \tag{4.31}$$

which are generic expressions not relying on any specific time dependence of the scale factor.

Equation (4.31) can be further rewritten into a more suggestive form. First, integrating the identity $\partial_\eta(\overset{(0)}{a}{}^2/\overset{(0)}{\mathcal{H}})=(1+\epsilon)\overset{(0)}{a}{}^2$, it can be shown that

$$1-\overset{(0)}{\mathcal{H}}I=\frac{\overset{(0)}{\mathcal{H}}}{\overset{(0)}{a}{}^2}\int_{-\infty}^{\eta}\epsilon\,\overset{(0)}{a}{}^2d\eta'.$$

Then, integration by parts leads

$$(1+\epsilon)\overset{(0)}{\mathcal{H}}I-1=\frac{\overset{(0)}{\mathcal{H}}}{\overset{(0)}{a}{}^2}\int_{-\infty}^{\eta}\epsilon'\,\overset{(0)}{a}{}^2I d\eta', \tag{4.32}$$

and thus we have

$$\hat{p}^4\overset{(0)}{\phi}\simeq -2A_k k^2\overset{(0)}{\mathcal{H}}{}^2\left(\epsilon' I-2\epsilon\frac{\overset{(0)}{\mathcal{H}}}{\overset{(0)}{a}{}^2}\int_{-\infty}^{\eta}\epsilon'\,\overset{(0)}{a}{}^2I d\eta'\right). \tag{4.33}$$

Since ϵ' is of second order in the slow-roll parameters during slow-roll inflation, we can safely neglect this term until the end of inflation. Moreover, for sufficiently smooth ϵ this term is expected to be small, and $\epsilon'=0$ for the scale factor proportional to a power of the conformal time. Only at the time when the equation of state of the universe changes abruptly, i.e., at a sudden transition from inflation to the radiation stage, ϵ' possibly becomes significantly large like a delta function. After such a violent transition, the integral in Eq. (4.33) leaves a constant value and thus this second term decreases in proportion to the damping factor $\overset{(0)}{\mathcal{H}}/\overset{(0)}{a}{}^2$.

4.3.3 Slow-roll inflation

Now let us consider slow-roll inflation on the brane. In this case $\mathcal{S}_0$ vanishes, since $\delta\mathcal{H}=0$ by construction. In the previous section we have also shown that $\hat{p}^4\overset{(0)}{\phi}$ vanishes at first order in the slow-roll parameters during inflation. Consequently, the only relevant term in the present situation is $\mathcal{S}_1$.

Equation (4.30) together with Eq. (4.32) tells us that

$$\mathcal{S}_1\simeq -A_k k^2\overset{(0)}{\mathcal{H}}{}^2\epsilon, \tag{4.34}$$

and thus

$$f(\eta) \;\simeq\; -\frac{k^2\ell^2}{\overset{(0)}{a}{}^2}\int^{\eta} d\eta' \epsilon \overset{(0)}{\mathcal{H}}{}^2. \tag{4.35}$$

Using the relations $\overset{(0)}{\mathcal{H}} \simeq \overset{(0)}{a}/a_0\eta_0$ and $d\eta = a_0\eta_0\, d\overset{(0)}{a}/\overset{(0)}{a}{}^2$, and neglecting the time variation of the slow-roll parameter, we obtain

$$f(\eta) \simeq \frac{k^2\ell^2}{\eta_0 \overset{(0)}{a}{}^2}\epsilon\left(\frac{a_i}{a_0} - \frac{a}{a_0}\right), \tag{4.36}$$

where $a_i = a(\eta_i)$ and η_i is the lower boundary of integration. Integrating this expression, we have

$$F(\eta) \simeq -\frac{k^2\ell^2}{a_i^2}\epsilon\left[\frac{1}{3}\left(\frac{a_i}{\overset{(0)}{a}}\right)^3 - \frac{1}{2}\left(\frac{a_i}{\overset{(0)}{a}}\right)^2\right]. \tag{4.37}$$

We chose the integration constant so that F vanishes in the limit of $a(\eta) \to \infty$. This means that we have renormalized A_k so that it represents the amplitude of fluctuations that we see at late times if slow-roll inflation lasts forever.

Although our present long wavelength approximation is not valid when the wavelength $a_i k^{-1}$ becomes comparable or shorter than the Hubble scale H, we can still arrange a_i to be sufficiently small so that we can neglect the first term in Eq. (4.37). For such a_i, we have an expression independent of a_i,

$$F(\eta) \approx \frac{k^2\ell^2}{2\overset{(0)}{a}{}^2}\epsilon, \tag{4.38}$$

and we do not need to care about the choice of the “initial time” η_i. We see that the correction arising during inflation is very tiny, suppressed by the slow-roll parameter ϵ in addition to the factor $k^2\ell^2/\overset{(0)}{a}{}^2$. For pure de Sitter inflation there is no correction, as is expected.

4.3.4 Transition to the radiation stage

In this section, we investigate the effects of the transition from the inflation stage to the radiation stage. First, just for the illustrative purpose, we plot the behavior of $\mathcal{S}_1$ and $\mathcal{S}_2$ for the modes well outside the horizon in the neighborhood of the transition time η_0 in Fig. 4.1 and Fig 4.2. These plots are for ϵ given by

$$\epsilon(\eta) = \tanh[(\eta - \eta_0)/s] + 1, \tag{4.39}$$

where the parameter s controls the smoothness of the transition. From the figures it can be seen that the corrections from $E_{\mu\nu}$ become significant only around the transition time.

From now on we shall consider the limiting case where ϵ is given by $\epsilon(\eta) = 2\theta(\eta - \eta_0)$. Here we neglected the tiny effect of the non vanishing slow-roll parameter during inflation. For this instantaneous transition model, the scale factor and the comoving Hubble parameter for $\eta < \eta_0$ are given by Eq. (4.6), while for $\eta > \eta_0$ they are given by $\overset{(0)}{a}$ and $\overset{(0)}{\mathcal{H}}$ in Eqs. (4.8) and (4.10).

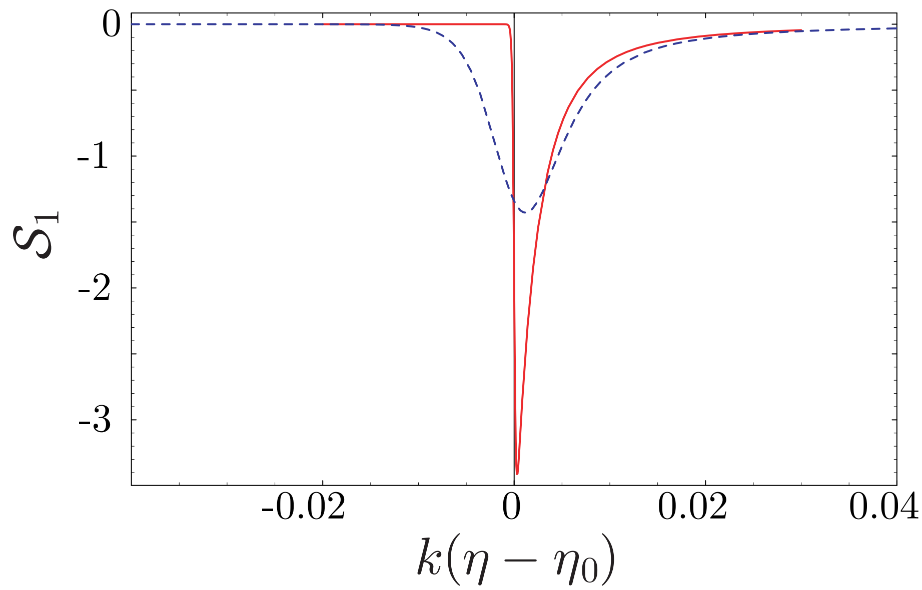

Figure 4.1: Behavior of $\mathcal{S}_1$ around the transition time with the vertical axis in an arbitrary unit. Red solid line shows the case of a sharp transition with $s = 0.02\eta_0$, while blue dashed line represents the case of a smooth transition with $s = 0.5\eta_0$. The wavelength of the mode is chosen to be $k\eta_0 = 0.01$.

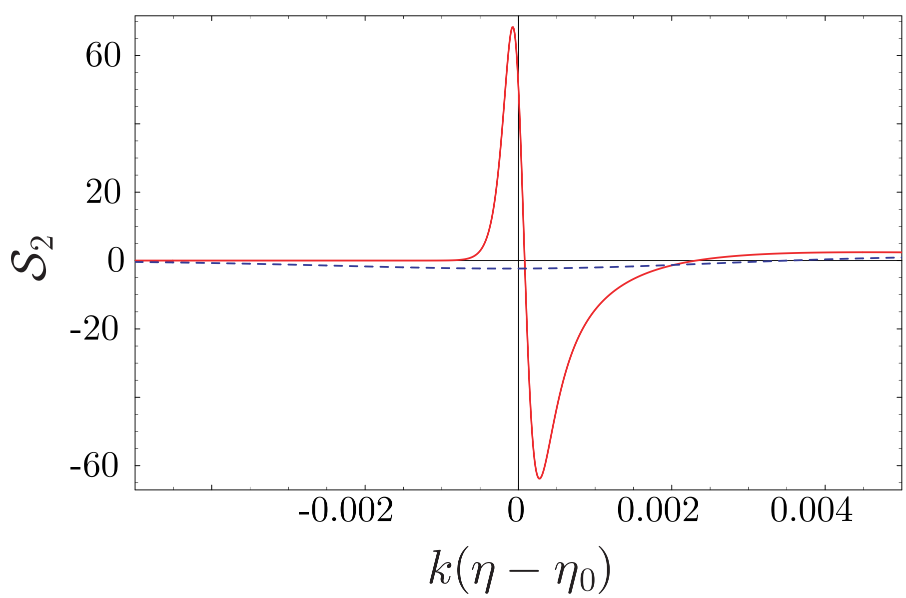

Figure 4.2: Behavior of $\mathcal{S}_2$ around the transition time with the vertical axis in an arbitrary unit. Red solid line shows the case of a sharp transition with $s = 0.02\eta_0$, while blue dashed line represents the case of a smooth transition with $s = 0.5\eta_0$. The choice of the parameter is the same as in Fig. 4.1, $k\eta_0 = 0.01$.

Assuming such a step function-like transition, we can evaluate the corrections from $\mathcal{S}_0$ at $\eta > \eta_0$ as

$$f_0(\eta) \simeq -\frac{k^2\ell^2}{9\overset{(0)}{a}{}^2}\int_{\eta_0}^{\eta}\frac{\eta' d\eta'}{\eta_0^3}\left(1-\frac{\eta_0^3}{\eta^3}\right)\left(1+2\frac{\eta_0^3}{\eta^3}\right), \tag{4.40}$$

$$F_0(\eta) \simeq -\frac{k^2\ell^2}{90a_0^2}\left[5\frac{\eta}{\eta_0}-9+5\left(\frac{\eta_0}{\eta}\right)^2-\left(\frac{\eta_0}{\eta}\right)^5\right]. \tag{4.41}$$

Here the integration constant is chosen so that the correction vanishes before the transition. From this expression, we find that F_0 diverges for a large η, and hence the correction to the final amplitude of fluctuations from $\mathcal{S}_0$ looks infinitely large. However, this is an artifact of the long wavelength approximation. At a late time during the radiation era the zeroth order solution becomes oscillatory. This oscillation suppresses the late time contribution. However, in the long wavelength approximation, this oscillation is not taken into account. In the succeeding section, we will develop another method which takes into account this oscillation. There we will find a finite correction.

As for $\mathcal{S}_1$, we have

$$f_1(\eta) \simeq -\frac{2}{3}\frac{k^2\ell^2}{\overset{(0)}{a}{}^2}\int_{\eta_0}^{\eta}\frac{d\eta'}{\eta'^2}\left(1+5\frac{\eta_0^3}{\eta'^3}\right), \tag{4.42}$$

$$F_1(\eta) \simeq -\frac{2}{3}\frac{k^2\ell^2}{a_0^2}\left[\frac{3}{2}-\frac{9}{4}\left(\frac{\eta_0}{\eta}\right)+\frac{1}{2}\left(\frac{\eta_0}{\eta}\right)^2+\frac{1}{4}\left(\frac{\eta_0}{\eta}\right)^5\right]. \tag{4.43}$$

Long time after the transition (but of course before horizon re-entry), the correction becomes time independent,

$$\lim_{\eta\to\infty} F_1(\eta) = -\frac{k^2\ell^2}{a_0^2}. \tag{4.44}$$

The next step is to evaluate the correction coming from $\mathcal{S}_2$. This can be done as follows. First, for the instantaneous transition $\hat{p}^4 \overset{(0)}{\phi}$ becomes

$$\hat{p}^4 \overset{(0)}{\phi} \simeq -4A_k k^2\left[\frac{1}{\eta_0}\delta(\eta-\eta_0)-4\theta(\eta-\eta_0)\frac{\eta_0^3}{\eta^5}\right] = -4A_k k^2\partial_\eta\left[\theta(\eta-\eta_0)\frac{\eta_0^3}{\eta^4}\right]. \tag{4.45}$$

Then, integrating by parts, the Fourier transform of $\hat{p}^4 \overset{(0)}{\phi}$ is obtained as

$$\begin{aligned} p^4 \overset{\widetilde{(0)}}{\phi} &= \frac{2A_k k^2}{\pi} i\omega \int d\eta' \theta(\eta-\eta_0)\frac{\eta_0^3}{\eta'^4}e^{i\omega\eta'} \\ &= \frac{2A_k k^2}{\pi} i\omega e^{i\omega\eta_0} q(\omega\eta_0), \end{aligned} \tag{4.46}$$

with

$$q(\omega\eta_0) := \int_1^\infty dy \frac{e^{i\omega\eta_0(y-1)}}{y^4}. \tag{4.47}$$

We separate $\mathcal{S}_2$ into two parts,

$$\mathcal{S}_2 = -\frac{1}{4}\int d\omega p^4 \overset{\widetilde{(0)}}{\phi} e^{-i\omega\eta}\ln\left(\frac{p^2}{k^2}\right) - \frac{1}{2}\left[\gamma+\ln\left(\frac{k\ell}{2a_0}\right)+\ln\left(\frac{\eta_0}{\eta}\right)\right]\hat{p}^4 \overset{(0)}{\phi},$$

and, by substituting the expressions obtained above, we have

$$\begin{aligned}\mathcal{S}_2 &= \frac{A_k k^2}{2\pi}\int d\omega\ q(\omega\eta_0)\partial_\eta\left[e^{-i\omega T}\ln\left(\frac{p^2}{k^2}\right)\right]\\ &\qquad +2A_k k^2\left[\gamma+\ln\left(\frac{k\ell}{2a_0}\right)+\ln\left(\frac{\eta_0}{\eta}\right)\right]\partial_\eta\left[\theta(T)\frac{\eta_0^3}{\eta^4}\right],\end{aligned} \tag{4.48}$$

where $T := \eta-\eta_0$. Then, it can be integrated to give

$$\begin{aligned}f_2(\eta) &= \frac{k^2\ell^2}{\overset{(0)}{a}{}^2}\left\{\frac{1}{2\pi}\int d\omega\ q(\omega\eta_0)\left[e^{-i\omega T}\ln\left(\frac{p^2}{k^2}\right)\right]\right.\\ &\qquad \left.+2\left[\gamma+\ln\left(\frac{k\ell}{2a_0}\right)+\ln\left(\frac{\eta_0}{\eta}\right)\right]\theta(T)\frac{\eta_0^3}{\eta^4}+\int_{-\infty}^{\eta}d\eta'\theta(\eta'-\eta_0)\frac{2}{\eta'}\frac{\eta_0^3}{\eta'^4}\right\}.\end{aligned} \tag{4.49}$$

The branch cuts of log function and the path of ω integration in the first term should be chosen so that the retarded boundary condition is ensured, which are presented in Fig. 4.3. The integration is dominated by $\omega^2 \approx k^2$, where we may approximate $q(\omega\eta_0)\approx q(k\eta_0)\approx 1/3$ because $k\eta_0$ is assumed to be small. Then, the first term can be written by using the formula presented in Sec. 4.6, and can be integrated to give a part of $F_2(\eta)$ as

$$\begin{aligned}&-\frac{2k^2\ell^2}{3a_0^2}\int_{-\infty}^{\eta}d\eta'\frac{\eta_0^2}{\eta'^2}\left\{\theta(\eta'-\eta_0)\frac{\cos[k(\eta'-\eta_0)]-1}{\eta'-\eta_0}\right.\\ &\qquad\qquad \left.+\partial_{\eta'}[\theta(\eta'-\eta_0)\ln(k(\eta'-\eta_0))]+\gamma\delta(\eta'-\eta_0)\right\}\\ &\simeq -\frac{2k^2\ell^2}{3a_0^2}\left\{\frac{\eta_0^2}{\eta^2}\ln(k(\eta-\eta_0))+\ln(k\eta_0)\left(1-\frac{\eta_0^2}{\eta^2}\right)\right.\\ &\qquad\qquad \left.-1+\frac{\eta_0}{\eta}+\ln\left(\frac{\eta-\eta_0}{\eta}\right)-\frac{\ln(\eta/\eta_0-1)}{(\eta/\eta_0)^2}+\gamma\right\}\\ &\underset{\eta\to\infty}{\longrightarrow} -\frac{2k^2\ell^2}{3a_0^2}\left[\gamma-1+\ln(k\eta_0)\right],\end{aligned}$$

where we have neglected the term proportional to $\cos[k(\eta'-\eta_0)]-1$ in the integrand because this term brings the higher order contributions in k. The other part of $F_2(\eta)$ is obtained by integrating the remaining parts in Eq. (4.49) as

$$\begin{aligned}&\frac{2k^2\ell^2}{a_0^2}\left\{\left[\gamma+\ln\left(\frac{k\ell}{2a_0}\right)\right]\frac{1}{5}\left(1-\frac{\eta_0^5}{\eta^5}\right)+\frac{1}{25}\left[\frac{\eta_0^5}{\eta^5}-1+5\frac{\ln(\eta/\eta_0)}{(\eta/\eta_0)^5}\right]\right.\\ &\qquad\qquad \left.+\frac{1}{4}\left(1-\frac{\eta_0}{\eta}\right)-\frac{1}{20}\left(1-\frac{\eta_0^5}{\eta^5}\right)\right\}\\ &\underset{\eta\to\infty}{\longrightarrow} \frac{2k^2\ell^2}{a_0^2}\left\{\frac{1}{5}\left[\gamma+\ln\left(\frac{k\ell}{2a_0}\right)\right]+\frac{4}{25}\right\}.\end{aligned}$$

Combining the above two, we finally obtain the correction from $\mathcal{S}_2$,

$$\lim_{\eta\to\infty}F_2(\eta)=\frac{2k^2\ell^2}{a_0^2}\left\{\frac{37}{75}-\frac{2}{15}\left[\gamma+\ln\left(\frac{k}{a_0H_0}\right)\right]+\frac{1}{5}\ln\left(\frac{\ell H_0}{2}\right)\right\}. \tag{4.50}$$

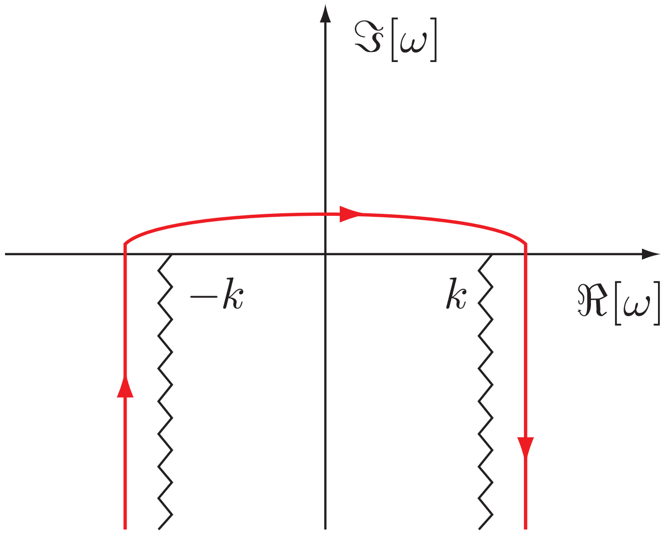

Figure 4.3: Branch cuts and the contour of the integration in Eq. (4.49) for $T > 0$. For $T < 0$ we close the contour on the upper half plane.

4.4 Corrections in the radiation stage

So far we have worked in the long wavelength approximation, and found k-dependent small corrections to the spectrum of tensor perturbations due to $\mathcal{S}_1$ and $\mathcal{S}_2$, which correspond to $E_{\mu\nu}$ term. It might be possible, however, that further corrections arise after the mode re-enters the horizon. We will show that such corrections are highly suppressed. A basic observation supporting this conclusion is that the contributions from $E_{\mu\nu}$ term in the right hand side of Eq. (4.18) become significantly large only at the transition time. It decreases in powers of $a(\eta)$ after the transition, and will be negligible when the long wavelength approximation brakes down.

In the long wavelength approximation, we could not obtain a meaningful estimate for the leading order correction caused by unconventional cosmic expansion, $\mathcal{S}_0$. Here we will give a more rigorous treatment, and will resolve this drawback of the approach taken in the preceding section.

In the radiation stage,

$$u(\eta) := a(\eta)\phi(\eta), \tag{4.51}$$

is a convenient variable. In terms of this new variable, we can rewrite the equation of motion (4.18) as

$$u'' + k^2 u = \frac{\ell^2}{a}\left(\bar{\mathcal{S}}_0 + \mathcal{S}_1 + \mathcal{S}_2\right), \tag{4.52}$$

with

$$\bar{\mathcal{S}}_0 = \frac{a''}{\ell^2} u. \tag{4.53}$$

If we neglect the deviation of the expansion law from the standard one, we have $a \propto \eta$ in the radiation stage and so $a'' = 0$. This means that the first term on the right hand side gives a contribution of $\mathcal{O}(\ell^2)$ from the modification of the expansion law as $\mathcal{S}_0$ in the preceding section. Thus, all the terms collected on the right hand side are of $O(\ell^2)$, and at zeroth order u behaves just like a harmonic oscillator. The zeroth order solution is expressed by a linear combination of the standard modes:

$$\overset{(0)}{u}(\eta) = A_k \left[\alpha \frac{e^{-ik\eta}}{\sqrt{2k}} + \beta \frac{e^{ik\eta}}{\sqrt{2k}}\right], \tag{4.54}$$

where the coefficients α and β are determined by matching the solutions at the onset of the radiation stage.

We define "energy" of the harmonic oscillator as

$$\mathcal{E} := \frac{1}{2}\left|u'\right|^2 + \frac{k^2}{2}|u|^2. \tag{4.55}$$

Then, this energy is conserved at the zeroth order, i.e.,

$$\overset{(0)}{\mathcal{E}} = \frac{k}{2}(|\alpha|^2 + |\beta|^2)|A_k|^2 = \text{ const.}$$

Taking into account the corrections of $\mathcal{O}(\ell^2)$, the total variation of the energy can be estimated as

$$\Delta\mathcal{E} = \sum_{j=0}^{2} \Delta\mathcal{E}_j = \sum_{j=0}^{2} \int_{\eta_0}^{\infty} \mathcal{E}_j' d\eta, \tag{4.56}$$

with

$$\begin{aligned} \mathcal{E}_0' &= \frac{\ell^2}{2a}(u')^* \bar{\mathcal{S}}_0 + \text{ c.c.}, \\ \mathcal{E}_j' &= \frac{\ell^2}{2a}(u')^* \mathcal{S}_j + \text{ c.c.}, \qquad \text{for } j = 1, 2. \end{aligned} \tag{4.57}$$

The amplitude of the oscillator $|A_k|e^{\mathrm{Re}[F]}$ is proportional to $\mathcal{E}^{1/2}\overset{(0)}{a}/a$. The factor $\overset{(0)}{a}/a$ here is to be attributed to unconventional cosmic expansion. Hence we have

$$\begin{aligned} \lim_{\eta\to\infty}\left\{\mathrm{Re}[F_0(\eta)] + \frac{\delta a(\eta)}{\overset{(0)}{a}(\eta)}\right\} &= \frac{\Delta\mathcal{E}_0}{2\overset{(0)}{\mathcal{E}}}, \\ \lim_{\eta\to\infty}\mathrm{Re}[F_j(\eta)] &= \frac{\Delta\mathcal{E}_j}{2\overset{(0)}{\mathcal{E}}}, \qquad \text{for } j = 1, 2. \end{aligned} \tag{4.58}$$

4.4.1 Corrections due to the unconventional expansion law

We start with computing the correction due to the unconventional cosmic expansion, F_0, which we could not determine in the long wavelength approximation. To do so, we first need to determine the coefficients in Eq. (4.54) by requiring that $u^{(0)}$ and its derivative are continuous across the transition from the de Sitter stage to the radiation stage. We assume for a moment that the perturbation in the initial phase is given by the de Sitter growing mode,

$$v(\eta) \simeq \frac{A_k a_0 \eta_0}{2\eta_0 - \eta}\left[1 + \frac{k^2(2\eta_0 - \eta)^2}{2}\right]. \tag{4.59}$$

Solving

$$\begin{aligned} v(\eta_0) &= \overset{(0)}{u}(\eta_0), \\ \partial_\eta v(\eta_0) &= \partial_\eta \overset{(0)}{u}(\eta_0), \end{aligned}$$

we obtain the coefficients as

$$\alpha \simeq \frac{i a_0}{\eta_0 \sqrt{2k}} \left[1 - \frac{2i}{3}(k\eta_0)^3 \right], \quad \beta = \alpha^*. \tag{4.60}$$

The energy of the harmonic oscillator at zeroth order is approximately given by

$$\overset{(0)}{\mathcal{E}} \simeq \frac{|A_k|^2 a_0^2}{2\eta_0^2}. \tag{4.61}$$

Next we integrate

$$\begin{aligned} \mathcal{E}_0' &= \frac{i}{4}\frac{a''}{a}(|\alpha|^2 - |\beta|^2 - \alpha\beta^* e^{-2ik\eta} + \alpha^*\beta e^{2ik\eta})|A_k|^2 + \text{ c.c.} \\ &= \frac{a''}{a}\mathrm{Im}[\alpha\beta^* e^{-2ik\eta}]|A_k|^2. \end{aligned} \tag{4.62}$$

The scale factor correct up to $\mathcal{O}(\ell^2)$ is given in Eq. (4.7). Using this expression, we have

$$\frac{a''}{a} \simeq -\frac{\ell^2 H_0^2}{2}\frac{\eta_0^4}{\eta^6}. \tag{4.63}$$

Then, integrating Eq. (4.62), we have

$$\begin{aligned} \frac{\Delta\mathcal{E}_0}{2\overset{(0)}{\mathcal{E}}} &\simeq -\frac{\ell^2 H_0^2 \eta_0^2}{2a_0^2}\,\mathrm{Im}\left[\alpha\beta^* \int_{\eta_0}^{\infty} d\eta \frac{\eta_0^4}{\eta^6} e^{-2ik\eta}\right] \\ &\simeq \frac{\ell^2 H_0^2}{4k\eta_0}\,\mathrm{Im}\left[\left[1 - \frac{2i}{3}(k\eta_0)^3\right]^2 \left[\frac{1}{5} - \frac{i}{2}k\eta_0 - \frac{2}{3}(k\eta_0)^2 + \frac{2i}{3}(k\eta_0)^3 + \mathcal{O}\left((k\eta_0)^4\right)\right]\right] \\ &\simeq -\frac{1}{8}\ell^2 H_0^2 + \frac{1}{10}\frac{k^2\ell^2}{a_0^2}. \end{aligned} \tag{4.64}$$

The first term in the last line is not suppressed for the $k \to 0$ limit. This is because this term arises due to the asymptotic behavior of $\delta a(\eta)$. In fact one can see from Eq. (4.58) that the expression for the physical amplitude of the perturbations e^{F_0} does not have this term, and finally we have

$$\lim_{\eta\to\infty} \mathrm{Re}[F_0(\eta)] = \frac{1}{10}\frac{k^2\ell^2}{a_0^2}. \tag{4.65}$$

4.4.2 Suppression at sub-horizon scales

Here we shall show that corrections from the $E_{\mu\nu}$ term are suppressed at sub-horizon scales and hence the result obtained in the preceding section by using the long wavelength approximation suffices for our purpose.

Substituting $\overset{(0)}{\phi} = \overset{(0)}{u} / \overset{(0)}{a}$ into $\mathcal{S}_1[\overset{(0)}{\phi}]$, energy gain or loss due to this term is obtained as

$$\mathcal{E}_1' = \frac{\ell^2}{2\overset{(0)}{a}{}^2}\left[\frac{5}{\eta^3}|\overset{(0)}{u}{}'|^2 + \left(-\frac{5}{\eta^4}+\frac{k^2}{\eta^2}\right)(\overset{(0)}{u}{}')^*\overset{(0)}{u}\right] + \text{c.c.}. \tag{4.66}$$

Note that $|\overset{(0)}{u}{}'|^2$ and $(\overset{(0)}{u}{}')^*\overset{(0)}{u}$ consist of just constant parts and oscillating parts with constant amplitudes. Therefore, it is manifest that when we integrate $\mathcal{E}_1'$ the dominant contribution comes from around the lower boundary of the integral, $\eta \approx \eta_0$, and so corrections arising in the sub-horizon regime are suppressed compared to those imprinted beforehand in the super-horizon regime.

A similar expression for $\mathcal{E}_2'$ can be obtained as

$$\begin{aligned}\mathcal{E}_2' &= -\frac{\ell^2}{8\overset{(0)}{a}}(\overset{(0)}{u}{}')^*\int d\omega p^4\, \overset{(0)}{\tilde{\phi}}\, e^{-i\omega\eta}\ln\left(\frac{p^2}{k^2}\right)\\ &\quad +\frac{2\ell^2}{\overset{(0)}{a}{}^2}\left[\ln\left(\frac{k\ell}{2\overset{(0)}{a}}\right)+\gamma\right]\left[\frac{3}{\eta^3}|\overset{(0)}{u}{}'|^2+\left(-\frac{3}{\eta^4}+\frac{k^2}{\eta^2}\right)(\overset{(0)}{u}{}')^*\overset{(0)}{u}\right]+\text{c.c.}.\end{aligned} \tag{4.67}$$

The second term is local and suppressed for large η for the same reason above. To examine the late time behavior of the first term, we substitute the expression (4.54) without limitation $\eta > \eta_0$. This approximation is justified because the kernel $\int d\omega e^{-i\omega(\eta-\eta')}p^4\ln(p^2/k^2)$ decays at least as fast as $1/(\eta-\eta')^3$ for a large time separation, as will be shown in Sec. 4.6. Under this approximation for $\phi(\eta)$, it is easy to calculate its Fourier transform[3]

$$\overset{(0)}{\tilde{\phi}}(\omega) = \frac{i\eta_0}{2a_0}\frac{1}{\sqrt{2k}}\times\begin{cases} -(\alpha+\beta) & \text{for} \quad \omega\le -k,\\ (\alpha-\beta) & \text{for} \quad |\omega|\le k,\\ (\alpha+\beta) & \text{for} \quad \omega \ge k.\end{cases} \tag{4.68}$$

The contour of the ω-integration in the first term of Eq. (4.67) is such shown in Fig. 4.3, and hence it can be separated into three parts as

$$\text{The first term in (4.67)} = -(\overset{(0)}{u}{}')^*\frac{k^2}{16}\frac{k^2\ell^2}{a_0\overset{(0)}{a}}k\eta_0\left(\mathcal{I}^{-k-i\infty}_{\to -k}+\mathcal{I}^{-k}_{\to k}+\mathcal{I}^{k}_{\to k-i\infty}\right), \tag{4.69}$$

where

$$\begin{aligned}\mathcal{I}^{k}_{\to k-i\infty} &= (\mathcal{I}^{-k-i\infty}_{\to -k})^*\\ &= (\alpha+\beta)\frac{e^{-ik\eta}}{\sqrt{2k}}\int_0^\infty dy(y^2+2iy)^2e^{-k\eta\cdot y}\left[\ln(2y-iy^2)-\frac{\pi i}{2}\right],\end{aligned} \tag{4.70}$$

$$\mathcal{I}^{-k}_{\to k} = (\alpha-\beta)\frac{i}{\sqrt{2k}}\int_{-1}^{1}dy(1-y^2)^2e^{-ik\eta\cdot y}\ln(1-y^2). \tag{4.71}$$

We can see that all $\mathcal{I}$s are suppressed for large values of $k\eta$, since their integrand becomes a product of a sooth function and a rapidly oscillating function. The dominant contribution in the integration of $\mathcal{E}'$ therefore comes from around the lower boundary of the integral.

Based on the above analysis, we conclude that all the dominant corrections arise in the super-horizon regime, and thus it is sufficient to evaluate the corrections in the long wavelength approximation.

[3]Depending on the choice of the integration path near $\eta = 0$, the Fourier transform of $\phi(\eta)$ changes by a constant independent of ω. Here we took the principal values.

4.5 Summary

We have investigated leading order corrections to the tensor perturbations in the RS2 braneworld cosmology by using the perturbative expansion scheme of Ref. [139]. We have studied a model composed of slow-roll inflation on the brane, followed by a radiation dominant era. The unperturbed five dimensional bulk is AdS space with curvature radius ℓ. In our expansion scheme the asymptotic boundary conditions in the bulk are imposed by choosing outgoing solutions of bulk perturbations, whose general expression is known in the Poincaré coordinate system. Hence, the issue of bulk boundary conditions is handled without introducing an artificial regulator brane. This is one of the notable advantages of the present scheme.

We set the initial condition when the wave length ak^{-1} is already longer than the Hubble radius during slow-roll inflation. We will not lose much by neglecting the modes which are already inside the Hubble scale at the time of transition to the radiation era, since they are not cosmologically so interesting. We compared the resulting amplitudes of fluctuations after the horizon re-entry with and without the effect of an extra dimension. To do so, we normalized the amplitude so that the late time amplitude of fluctuations becomes identical in two cases if the slow-roll inflation lasts forever. As the reference without the effect of an extra dimension, we took a growing mode solution. The corrections due to gravity propagation through the fifth dimension dominantly come from the contribution around the transition time. Therefore they can be estimated by using the long wavelength approximation with a sufficient accuracy. Since the correction due to the modified cosmic expansion comes from relatively late epoch, it is necessary to take into account the oscillatory behavior of the solution after the re-entry to the Hubble horizon.

Combining the results obtained in Eqs. (4.44), (4.50) and (4.65), we find that the amplitude of a fluctuation with comoving wave number k is modified by a factor $e^{\mathrm{Re}[F]}$ due to the effect of an extra dimension with

$$\lim_{\eta\to\infty}\mathrm{Re}[F(\eta)] = \frac{k^2\ell^2}{a_0^2}\left\{\frac{13}{150}-\frac{4}{15}\left[\gamma+\ln\left(\frac{k}{a_0H_0}\right)\right]+\frac{2}{5}\ln\left(\frac{\ell H_0}{2}\right)\right\}, \tag{4.72}$$

where a_0 and H_0 are the scale factor and the Hubble parameter at the transition time, and γ is the Euler's constant. Leading corrections are proportional to $k^2\ell^2/a_0^2$ or $k^2\ell^2/a_0^2\log \ell k$, as is expected from the dimensional analysis. However, *our calculation here determined the precise numerical factors analytically.*

Throughout this chapter, we have dropped the contribution from the decaying mode for simplicity. Since the decaying mode during the initial slow-roll inflation phase is usually suppressed when the wavelength is longer than the Hubble scale, it does not give a significant effect. However, in the context of braneworld, we have not yet understood how to fix the initial conditions for fluctuations. Therefore, in principle, there is a possibility that the contamination of the decaying mode can be extremely large, although it seems quite unlikely.

It is quite easy to estimate the leading order correction due to the decaying mode. The effect is not due to the non-trivial evolution of a solution but totally due to unconventional initial condition. Adding a decaying mode will modify v given in Eq. (4.59) to

$$v(\eta) \simeq \frac{A_k a_0 \eta_0}{2\eta_0-\eta}\left[1+\frac{k^2(2\eta_0-\eta)^2}{2}+C_d k^3(2\eta_0-\eta)^3\right],$$

where C_d is a complex number which parameterizes the amplitude of the decaying component. Repeating a similar calculation leading to Eq. (4.65), we find that the relative change in the

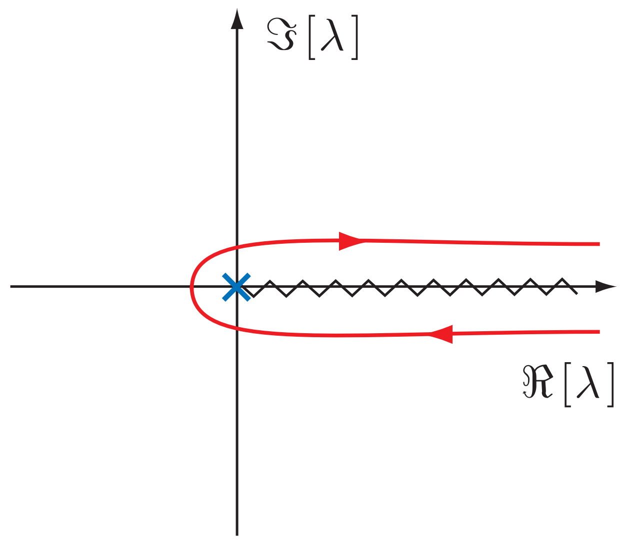

Figure 4.4: Branch cut and the contour of the integration in Eq. (4.76).

amplitude due to the decaying mode is given by

$$\mathrm{Re}[C_d]\frac{\ell^2 H_0^2}{2}(k\eta_0)^3. \tag{4.73}$$

Thus the correction due to the decaying component is more suppressed in the sense of the power of k, but it might be more important than the other corrections for $k\eta_0 > \mathrm{Re}[C_d]^{-1}$ if $\mathrm{Re}[C_d]$ is extremely large.

Lastly we would like to mention the limitation of the present work. The perturbative expansion scheme that we have adopted in this chapter is valid for scales larger than the AdS curvature length ($k\ell/a \ll 1$) at low energies ($\ell H \ll 1$); besides our study is restricted to initially super-horizon perturbations. We gave an initial condition for a solution by hand, but of course such an approach is not satisfactory. To determine the initial condition, we need to investigate the evolution of perturbations at the initial phase where the wavelength of the mode is too short and reduction to an effective four dimensional problem is not possible. Furthermore, to seek for interesting effects that might arise at high energies ($\ell H \gtrsim 1$), we have to develop a new formalism, which is a next challenge and will be addressed in the rest of this thesis.

4.6 Appendix: Details of calculations

In this appendix, we will show the following formula:

$$\int d\omega e^{-i\omega T}\ln\left(\frac{p^2}{k^2}\right) = -4\pi\left\{\theta(T)\frac{\cos(kT)-1}{T} + \partial_T\left[\theta(T)\ln(kT)\right] + \gamma\delta(T)\right\}, \tag{4.74}$$

where $p^2 = k^2 - \omega^2$ and γ is Euler's constant. The branch cuts are all running on the lower half complex plane of ω (Fig. 4.3), which ensures the retarded boundary conditions.

We divide the above integration into two parts:

$$\int d\omega e^{-i\omega T}\ln\left(\frac{p^2}{k^2}\right)=\int d\omega e^{-i\omega T}\ln\left(\frac{-p^2}{\omega^2}\right)+\int d\omega e^{-i\omega T}\ln\left(\frac{-\omega^2}{k^2}\right). \tag{4.75}$$

The first term of the right hand side can be rewritten as

$$\int d\omega e^{-i\omega T}\ln\left(\frac{-p^2}{\omega^2}\right)=\int d\omega e^{-i\omega T}\left[\ln(\omega-k)+\ln(\omega+k)-2\ln\omega\right].$$

The integration can be performed as

$$\begin{aligned}\int d\omega e^{-i\omega T}\ln(\omega-k) &= \theta(T)\Big[\int_{\infty}^{0}(-i)dre^{-iT(k-ir)}\ln(re^{3\pi i/2})\\ &\qquad +\int_0^{\infty}(-i)dre^{-iT(k-ir)}\ln(re^{-\pi i/2})\Big]\\ &= -2\pi\theta(T)\frac{e^{-ikT}}{T},\end{aligned}$$

and the other two integrals are obtained by setting $k\to -k$ and $k\to 0$. Thus we have

$$\int d\omega e^{-i\omega T}\ln\left(\frac{-p^2}{\omega^2}\right)=-4\pi\theta(T)\frac{\cos(kT)-1}{T}.$$

This expression is regular at $T=0$. To evaluate the second term in the right hand side of Eq. (4.75), we need a trick because the integrand suffers from a slow fall-off at large values of $|\omega|$. It should be interpreted as

$$2i\partial_T\left[\int d\omega\frac{e^{-i\omega T}}{\omega}\ln\left(\frac{-i\omega}{k}\right)\right]=2i\partial_T\left\{\theta(T)\left[\int d\lambda\frac{e^{-\lambda}}{\lambda}\left(\ln(-\lambda)-\ln(kT)\right)\right]\right\}, \tag{4.76}$$

where we put $\lambda=i\omega T$. Now the branch cut and the contour of the integration on λ plane are such shown in Fig. 4.4. Thanks to this trick, the expression for the second term becomes a well-defined one, though the result does not change as long as convergence is guaranteed. Then, performing the integration as

$$\begin{aligned}\int d\lambda\frac{e^{-\lambda}}{\lambda}\left(\ln(-\lambda)-\ln(kT)\right) &= \lim_{\nu\to 0}\partial_\nu\int d\lambda e^{-\lambda}(-\lambda)^{\nu-1}+2\pi i\ln(kT)\\ &= \lim_{\nu\to 0}\partial_\nu\left[(e^{-\pi i\nu}-e^{\pi i\nu})\Gamma(\nu)\right]+2\pi i\ln(kT)\\ &= 2\pi i\left[\ln(kT)+\gamma\right],\end{aligned}$$

we finally obtain the formula (4.74).

For $T>0$ the formula (4.74) reduces to $\int d\omega e^{-i\omega T}\ln\left(p^2/k^2\right)=-4\pi\cos(kT)/T$ and hence the kernel that appears in $\mathcal{S}_2$ becomes

$$\int d\omega e^{-i\omega T}p^4\ln\left(\frac{p^2}{k^2}\right)=32\pi\left[\left(\frac{k^2}{T^3}-\frac{3}{T^5}\right)\cos(kT)-\frac{3k}{T^4}\sin(kT)\right],\qquad \text{for}\quad T>0. \tag{4.77}$$

From this expression we find that the non-local source $\mathcal{S}_2$ does not keep the past history for a long time.

Chapter 5

The junction model

In this chapter we study gravitational waves from an inflating brane. If the Hubble parameter on the brane is constant, the power spectrum becomes scale-invariant [84], as was explained in Chapter 3. However, the Hubble parameter usually changes even during inflation. The change of the Hubble parameter, i.e., the nontrivial motion of the brane in the five-dimensional bulk, "disturbs" the graviton wave function. As a result, zero mode gravitons, which correspond to the four-dimensional gravitational waves, are created from vacuum fluctuations in the Kaluza-Klein modes as well as in the zero mode[1]. It is also possible that gravitons initially in the zero mode escape into the extra dimension as the "dark radiation" [111, 42, 88, 90, 91]. Therefore we expect that the nontrivial motion of the brane may leave characteristic features of braneworld inflation. If so, it is interesting to search for a signature of the extra dimension left on the primordial spectrum. However, there is a technical difficulty. When the Hubble parameter is time dependent, the bulk equations are no longer separable. Then we have to solve a complicated partial differential equation. To cope with this difficulty, we consider a simple model in which two de Sitter branes are joined at a certain time; namely, we assume that the Hubble parameter changes discontinuously. In this model we can calculate the power spectrum almost analytically. This is a milder version of the transition described in the work by Gorbunov *et al.* [36], in which they considered a simplified inflation model in which de Sitter stage of inflation is instantaneously connected to Minkowski space and hence it is possible to solve the perturbation equations including the bulk to some extent. Thus we will closely follow the same technic as Ref. [36].

This chapter is organized as follows. In the next section we describe the setup of our five-dimensional model, and explain the formalism introduced in Ref. [36] to solve the mode functions for gravitational wave perturbations. Using this formalism, we explicitly evaluate the Bogoliubov coefficients in Sec. 5.2. In Sec. 5.3 we translate the results for the Bogoliubov coefficients into the power spectrum of gravitational waves, and its properties are discussed. We show that the power spectrum for our five-dimensional model can be reproduced with good accuracy from that for the corresponding four-dimensional model by applying a simple mapping. Section 5.4 is devoted to conclusion.

[1] The situation here is quite similar to particle production by a moving mirror [12, 36]

5.1 Preliminaries

5.1.1 Background metric, perturbations, and mode functions

Let us start with the simple case in which the background is given by a pure de Sitter brane in AdS_5 bulk spacetime [84]. The situation here is exactly the same as what was introduced in Chapter 3. We solve the five-dimensional Einstein equations for gravitational wave perturbations. For this purpose, it is convenient to use a coordinate system in which the position of the brane becomes a constant coordinate surface. In such a coordinate system the background metric is written as

$$ds^2 = \frac{\ell^2}{\sinh^2 \xi} \left[\frac{1}{\eta^2} (-d\eta^2 + \delta_{ij} dx^i dx^j) + d\xi^2 \right], \tag{5.1}$$

where ℓ is the bulk curvature radius, and the de Sitter brane is placed at $\xi = \text{const.} = \xi_b$. Note that here η is supposed to be negative. On the brane, the scale factor is given by $a(\eta) = 1/(-\eta H)$ and the Hubble parameter is

$$H = \ell^{-1} \sinh \xi_b. \tag{5.2}$$

Note that under the coordinate transformations

$$\begin{aligned} t &= \eta \cosh \xi - \eta_0 \cosh \xi_b, \\ z &= -\eta \sinh \xi, \end{aligned} \tag{5.3}$$

with a constant η_0, the metric (5.1) becomes the AdS_5 metric in the Poincaré coordinates.

The metric with gravitational wave perturbations is written as

$$ds^2 = \frac{\ell^2}{\sinh^2 \xi} \left\{ \frac{1}{\eta^2} \left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j \right] + d\xi^2 \right\}. \tag{5.4}$$

We decompose the transverse-traceless tensor h_{ij} into the spatial Fourier modes as

$$h_{ij}(\eta, \mathbf{x}, \xi) = \frac{\sqrt{2}}{(2\pi M_5)^{3/2}} \int d^3p\; \phi(\eta, \xi; k) e^{i\mathbf{k}\cdot\mathbf{x}} e_{ij}, \tag{5.5}$$

where e_{ij} is the polarization tensor, and the summation over different polarization was suppressed. M_5 represents the five-dimensional Planck mass, and it is related to the four-dimensional Planck mass $M_{\rm Pl}$ by $\ell M_5^3 = M_{\rm Pl}^2$. The factor $\sqrt{2}/(M_5)^{3/2}$ is chosen so that the effective action for ϕ corresponds to the action for the canonically normalized scalar field. Then, the Einstein equations for the gravitational wave perturbations reduce to the Klein-Gordon equation for a massless scalar field, $\Box\phi = 0$, in AdS_5. We assume Z_2-symmetry across the brane. Assuming that anisotropic stress is zero on the brane, Israel's junction condition gives the boundary condition for the perturbations as

$$\partial_\xi \phi|_{\xi=\xi_b} = 0. \tag{5.6}$$

Since the equation is separable, the mode functions are given in the form of $\phi_\nu(\eta, \xi) = \psi_\nu(\eta)\chi_\nu(\xi)$, and they are found in Chapter 3.

As will be seen, we need to evaluate the value of the wave function at the location of the brane $\chi_\nu(\xi_b)$, and in some special cases $\chi_\nu(\xi_b)$ reduces to a rather simple form. For $\sinh\xi_b \ll 1$ *and* $\nu\sinh\xi_b \ll 1$, we have

$$\chi_\nu(\xi_b) \approx \sqrt{\frac{\nu\tanh\pi\nu}{2}}\sqrt{\frac{\nu^2+1/4}{\nu^2+9/4}}(\sinh\xi_b)^2, \tag{5.7}$$

while, for $\sinh\xi_b \gg 1$ *or* $\nu\sinh\xi_b \gg 1$, we have

$$\chi_\nu(\xi_b) \approx \frac{1}{\sqrt{\pi}}(\sinh\xi_b)^{3/2}\frac{\nu}{\sqrt{\nu^2+9/4}}. \tag{5.8}$$

For the derivation of these two expressions, see Ref. [36].

5.1.2 Model with a jump in the Hubble parameter

We consider a model in which the Hubble parameter changes during inflation. As we have explained, in the case of constant Hubble parameter the brane can be placed at a constant coordinate plane. When the Hubble parameter varies, we need to consider a moving brane in the same coordinates. For simplicity, we consider the situation in which the Hubble parameter changes discontinuously at $\eta = \eta_0$ from H_1 to

$$H_2 = H_1 - \delta H. \tag{5.9}$$

Here $\delta H/H_1$ is assumed to be small. For later convenience, we define a small quantity ϵ_H by

$$\begin{aligned}\epsilon_H &= \frac{\ell H_1\sqrt{1+(\ell H_2)^2}-\ell H_2\sqrt{1+(\ell H_1)^2}}{\ell H_2}\\ &= \frac{1}{\sqrt{1+(\ell H_1)^2}}\frac{\delta H}{H_1}+\frac{2+3(\ell H_1)^2}{2[1+(\ell H_1)^2]^{3/2}}\left(\frac{\delta H}{H_1}\right)^2+\mathcal{O}\left(\frac{\delta H}{H_1}\right)^3.\end{aligned} \tag{5.10}$$

To describe the motion of the de Sitter brane after transition, it is natural to introduce a new coordinate system $(\tilde\eta, \tilde\xi)$ defined by

$$\begin{aligned} t &= \tilde\eta\cosh\tilde\xi - \tilde\eta_0\cosh\tilde\xi_b,\\ z &= -\tilde\eta\sinh\tilde\xi. \end{aligned} \tag{5.11}$$

Then, the brane expanding with Hubble parameter H_2 is placed at $\tilde\xi = \tilde\xi_b$ by choosing two constants $\tilde\xi_b$ and $\tilde\eta_0$ so as to satisfy $H_2 = \ell^{-1}\sinh\tilde\xi_b$ and $\eta_0\sinh\xi_b = \tilde\eta_0\sinh\tilde\xi_b$. The trajectory of the brane is shown in Fig. 5.1. Apparently, mode functions in this coordinate system take the same form as those in Chapter 3, but the arguments (ξ, η) and the Hubble parameter H_1 are replaced by $(\tilde\xi, \tilde\eta)$ and H_2. We refer to these second set of modes as $\tilde\phi_0$ and $\tilde\phi_\nu$. The relation between (η, ξ) and $(\tilde\eta, \tilde\xi)$ is

$$\begin{aligned} \tilde\eta &= -\sqrt{\eta^2+2\epsilon_H\eta_0\eta\cosh\xi+\epsilon_H^2\eta_0^2},\\ \tanh\tilde\xi &= (\eta\cosh\xi+\epsilon_H\eta_0)^{-1}\eta\sinh\xi. \end{aligned} \tag{5.12}$$

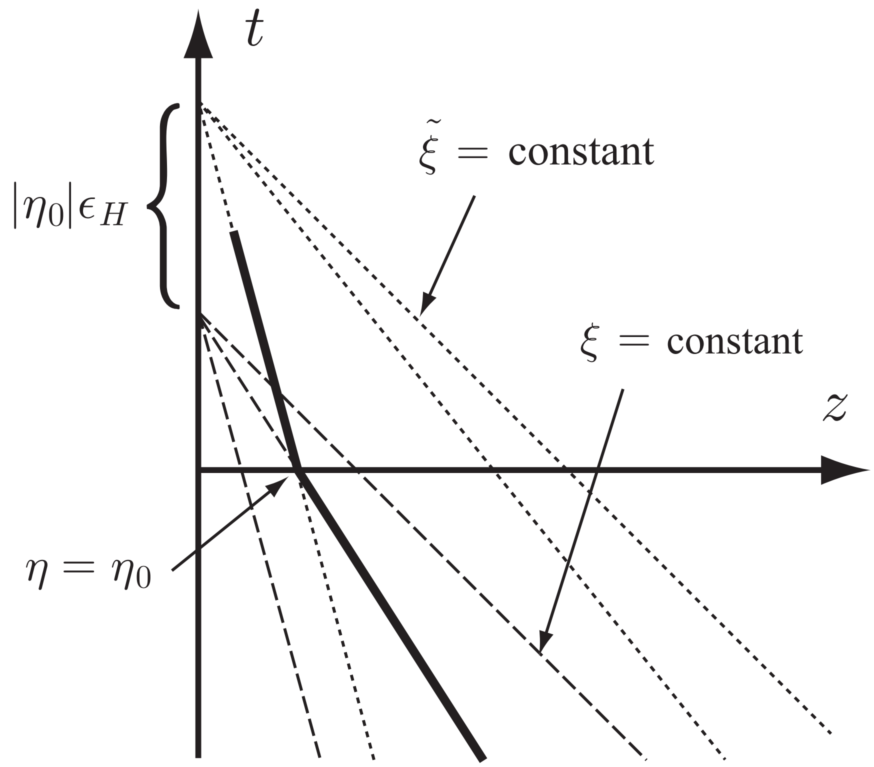

Figure 5.1: Trajectory of the brane (thick solid line) in static coordinates. Dashed (respectively dotted) lines represent surfaces of $\xi =$ constant (respectively $\tilde{\xi} =$ constant).

As explained above, the variation of the Hubble parameter is assumed to be small. For a technical reason, we further impose a weak restriction that the wavelength of the perturbations concerned is larger than $\delta H/H^2$. These conditions are summarized as follows:

$$\frac{\delta H}{H} \ll 1\,, \tag{5.13}$$

$$k|\eta_0|\frac{\delta H}{H} \ll 1\,. \tag{5.14}$$

5.1.3 Method to calculate Bogoliubov coefficients

We consider the graviton wave function φ that becomes the zero mode $\tilde{\phi}_0$ at the infinite future $\tilde{\eta} = 0$. We write the wave function φ as

$$\varphi = \tilde{\phi}_0 + \delta\varphi, \tag{5.15}$$

where the second term $\delta\varphi$ arises because $\tilde{\phi}_0$ does not satisfy the boundary condition for $t < 0$. Writing down $\tilde{\phi}_0$ and $\delta\varphi$ at the infinite past, $\eta = -\infty$, as a linear combination of ϕ_0, ϕ_ν, and their complex conjugates, we can read the Bogoliubov coefficients.

At $\eta \to -\infty$ the first term $\tilde{\phi}_0$ is expanded as

$$\tilde{\phi}_0 \underset{\eta\to-\infty}{\longrightarrow} \sum_{M=0,\nu} \left(U_{0M}\phi_M + V_{0M}\phi^*_M\right), \tag{5.16}$$

where the summation is taken over the zero mode and the KK modes. The coefficients $U_{0\nu}$ and $V_{0\nu}$ are written by using the Wronskian [Eq. (3.31)] as

$$\begin{aligned} U_{0M} &= \lim_{\eta\to-\infty}\left(\tilde{\phi}_0\cdot\phi_M\right), \\ V_{0M} &= -\lim_{\eta\to-\infty}\left(\tilde{\phi}_0\cdot\phi_M^*\right). \end{aligned} \tag{5.17}$$

Evaluation of the Wronskian at $\eta=-\infty$ leads to $V_{00}=V_{0\nu}=0$ [36], while

$$\begin{aligned} U_{00} &\approx \frac{H_2C(\ell H_2)}{H_1C(\ell H_1)}e^{-i\epsilon_H k\eta_0\cosh\xi_b}\Big\{1-i\epsilon_H k\eta_0\left[C^2(\ell H_1)-\cosh\xi_b\right] \\ &\qquad -\frac{1}{2}(\epsilon_H k\eta_0)^2\sinh^2\xi_b\Big\}, \end{aligned} \tag{5.18}$$

$$U_{0\nu} \approx \epsilon_H k\eta_0\frac{-2ie^{i\pi/4}}{\nu^2+1/4}\frac{\ell H_2}{(\ell H_1)^2}C(\ell H_2)\chi_\nu(\xi_b). \tag{5.19}$$

These expressions are approximately correct as long as $k|\eta_0|\delta H/H_1\ll 1$ is satisfied. The derivation of these equations is explained in Sec. 5.6.

The second term $\delta\varphi$ in Eq. (5.15) is obtained as follows. Since both φ and $\tilde{\phi}_0$ satisfy the Klein-Gordon equation in the bulk, $\delta\varphi$ also obeys the same equation. The boundary condition for $\delta\varphi$ is derived from Eq. (5.6) as $\partial_\xi\delta\varphi=-\partial_\xi\tilde{\phi}_0$. Therefore, the solution $\delta\varphi$ is given by

$$\delta\varphi=2\int_{-\infty}^{\eta_0}d\eta' G_{\rm adv}(\eta,\xi;\eta',\xi_b)\left[\partial_{\xi'}\tilde{\phi}_0(\eta',\xi')\right]_{\xi'=\xi_b}, \tag{5.20}$$

with the aid of the advanced Green's function that satisfies

$$\left[\eta^2\frac{\partial^2}{\partial\eta^2}-2\eta\frac{\partial}{\partial\eta}+k^2\eta^2-\sinh^3\xi\frac{\partial}{\partial\xi}\frac{1}{\sinh^3\xi}\frac{\partial}{\partial\xi}\right]G_{\rm adv}(\eta,\xi;\eta',\xi')=\delta(\eta-\eta')\delta(\xi-\xi'). \tag{5.21}$$

The explicit form of the Green's function is [36]

$$G_{\rm adv}(\eta,\xi;\eta',\xi')=\sum_M\frac{i\ell^3}{\sinh^3\xi'\eta'^4}\theta(\eta'-\eta)\left[\phi_M^*(\eta,\xi)\phi_M(\eta',\xi')-\phi_M(\eta,\xi)\phi_M^*(\eta',\xi')\right]. \tag{5.22}$$

Taking the limit $\eta\to-\infty$, we can expand $\delta\varphi$ in terms of in-vacuum mode functions,

$$\delta\varphi\underset{\eta\to-\infty}{\longrightarrow}\sum_{M=0,\nu}\left[u_{0M}\phi_M+v_{0M}\phi_M^*\right], \tag{5.23}$$

where the coefficients are given by

$$u_{0M} = -2i\ell^3\int_{-\infty}^{\eta_0}\frac{d\eta}{\sinh^3\xi_b\eta^4}\phi_M^*(\eta,\xi_b)\left[\partial_\xi\tilde{\phi}_0(\eta,\xi)\right]_{\xi=\xi_b}, \tag{5.24}$$

$$v_{0M} = 2i\ell^3\int_{-\infty}^{\eta_0}\frac{d\eta}{\sinh^3\xi_b\eta^4}\phi_M(\eta,\xi_b)\left[\partial_\xi\tilde{\phi}_0(\eta,\xi)\right]_{\xi=\xi_b}. \tag{5.25}$$

To evaluate these coefficients, we need the source term $\partial_\xi\tilde{\phi}_0\Big|_{\xi=\xi_b}$ written in terms of the coordinates (η,ξ), which is

$$\partial_\xi\tilde{\phi}_0\Big|_{\xi=\xi_b}=-\ell^{-3/2}\sinh\tilde{\xi}_b\frac{C(\ell H_2)}{\sqrt{2k}}ik\epsilon_H\eta_0\eta\sinh\xi_b e^{ik\sqrt{\eta^2+2\epsilon_H\eta_0\eta\cosh\xi_b+\epsilon_H^2\eta_0^2}}. \tag{5.26}$$

From Eqs. (5.16) and (5.23), we finally obtain the Bogoliubov coefficients relating the initial zero mode or KK modes to the final zero mode,

$$\varphi = \sum_M (\alpha_{0M}\phi_M + \beta_{0M}\phi_M^*) \,, \tag{5.27}$$

where

$$\begin{aligned} \alpha_{0M} &= U_{0M} + u_{0M}, \\ \beta_{0M} &= v_{0M}. \end{aligned} \tag{5.28}$$

From these coefficients we can evaluate the number and the power spectrum of the generated gravitons.

5.2 Evaluation of Bogoliubov coefficients

Now let us evaluate the expressions for the Bogoliubov coefficients obtained in the preceding section. We concentrate on the two limiting cases: the low energy regime ($\ell H_1 \ll 1$) and high energy regime ($\ell H_1 \gg 1$). We first evaluate the coefficients α_{00} and β_{00}, which relate the initial zero mode to the final zero mode. We keep the terms up to second order in ϵ_H (or equivalently in $\delta H/H_1$).

Substituting Eq. (5.26) into Eq. (5.25), we have

$$\beta_{00} = C(\ell H_1)C(\ell H_2)\frac{H_2}{H_1}\epsilon_H\eta_0\int_{-\infty}^{\eta_0} d\eta\left(\frac{1}{\eta^2} - \frac{i}{k\eta^3}\right)e^{-ik\left(\eta-\sqrt{\eta^2+2\epsilon_H\eta_0\eta\cosh\xi_b+\epsilon_H^2\eta_0^2}\right)}. \tag{5.29}$$

Because there is a factor ϵ_H in front of the integral, we can neglect the correction of $\mathcal{O}(\epsilon_H^2)$ in the integrand. Then we can carry out the integration to obtain

$$\beta_{00} \approx C(\ell H_1)C(\ell H_2)\frac{H_2}{H_1}\epsilon_H\frac{i}{2k\eta_0}e^{-2ik\eta_0-ik\eta_0\epsilon_H\cosh\xi_b}. \tag{5.30}$$

Similarly, we get

$$\alpha_{00} \approx U_{00} + C(\ell H_1)C(\ell H_2)\frac{H_2}{H_1}\epsilon_H\left(1+\frac{i}{2k\eta_0}\right)e^{-ik\eta_0\epsilon_H\cosh\xi_b}, \tag{5.31}$$

where U_{00} is given by Eq. (5.18).

At low energies ($\ell H_1 \ll 1$), the Bogoliubov coefficients α_{00} and β_{00} become

$$\alpha_{00} \approx \left[1+\frac{i}{2k\eta_0}\frac{\delta H}{H_1}\right]e^{-ik\eta_0\delta H/H_2}, \tag{5.32}$$

$$\beta_{00} \approx \frac{i}{2k\eta_0}\frac{\delta H}{H_1}e^{-2ik\eta_0-ik\eta_0\delta H/H_2}. \tag{5.33}$$

It is worth noting that these expressions are correct up to the second order in $\delta H/H_1$. This result agrees with the result of the four-dimensional calculation $\alpha^{(4\mathrm{D})}$ and $\beta^{(4\mathrm{D})}$.

At high energies ($\ell H_1 \gg 1$), the coefficients are

$$\alpha_{00} \approx \left[1+\frac{3}{2}\left(\frac{i}{2k\eta_0}\frac{\delta H}{H_1}\right)+\frac{3}{8}\left(\frac{\delta H}{H_1}\right)^2-\frac{ik\eta_0}{2}\frac{\delta H}{H_1} -\frac{(k\eta_0)^2}{2}\left(\frac{\delta H}{H_1}\right)^2\right]e^{-ik\eta_0\delta H/H_1-ik\eta_0(3/2)(\delta H/H_1)^2}, \tag{5.34}$$

$$\beta_{00} \approx \frac{3}{2}\left(\frac{i}{2k\eta_0}\frac{\delta H}{H_1}\right)e^{-2ik\eta_0-ik\eta_0(\delta H/H_1)-ik\eta_0(3/2)(\delta H/H_1)^2}. \tag{5.35}$$

Here we stress that the last two terms in the square brackets of Eq. (5.34), both coming from U_{00}, are enhanced at $k|\eta_0| \gg 1$.

Next, we calculate the Bogoliubov coefficients $\alpha_{0\nu}$ and $\beta_{0\nu}$, which relate the initial KK modes to the final zero mode, up to the leading first order in $\delta H/H_1$. Although we will calculate the power spectrum up to second order in $\delta H/H_1$ in the succeeding section, the expressions up to first order are sufficient for $\alpha_{0\nu}$ and $\beta_{0\nu}$, in contrast to the case for α_{00} and β_{00}. Again, substituting $\partial_\xi\tilde{\phi}_0(\eta,\xi_b)$ into Eq. (5.25), we have

$$\beta_{0\nu}=\sqrt{2k}\;\ell^{3/2}C(\ell H_2)\chi^*_\nu(\xi_b)\frac{\sinh\tilde{\xi}_b}{\sinh^2\xi_b}\epsilon_H\eta_0\int_{-\infty}^{\eta_0}\frac{d\eta}{\eta^3}\psi_\nu(\eta)e^{ik\sqrt{\eta^2+2\epsilon_H\eta_0\eta\cosh\xi_b+\epsilon_H^2\eta_0^2}}. \tag{5.36}$$

Setting ϵ_H in the integrand to zero, the coefficient reduces to

$$\beta_{0\nu}\approx\sqrt{\frac{\pi}{2}}C(\ell H_2)\chi^*_\nu(\xi_b)\frac{\ell H_2}{(\ell H_1)^2}\epsilon_H k\eta_0 e^{-\pi\nu/2}\int_{\infty}^{k|\eta_0|}dx x^{-3/2}H^{(1)}_{i\nu}(x)e^{ix}, \tag{5.37}$$

where we have introduced the integration variable $x=-k\eta$. Similarly we have

$$\alpha^*_{0\nu}\approx U^*_{0\nu}-\sqrt{\frac{\pi}{2}}C(\ell H_2)\chi^*_\nu(\xi_b)\frac{\ell H_2}{(\ell H_1)^2}\epsilon_H k\eta_0 e^{-\pi\nu/2}\int_{\infty}^{k|\eta_0|}dx x^{-3/2}H^{(1)}_{i\nu}(x)e^{-ix}, \tag{5.38}$$

where $U_{0\nu}$ is given by Eq. (5.19) and is $\mathcal{O}(\epsilon_H)^2$.

Now let us discuss the dependence of $\alpha_{0\nu}$ and $\beta_{0\nu}$ on ℓH_1 and $\delta H/H_1$ in the limiting cases, $\ell H_1 \ll 1$ and $\ell H_1 \gg 1$. At low energies, we find, using Eq. (5.7),

$$|\beta_{0\nu}|^2,\ |\alpha_{0\nu}|^2 \propto (\ell H_1)^2\left(\frac{\delta H}{H_1}\right)^2 \quad (\ell H_1 \ll 1)\,, \tag{5.39}$$

where we have omitted the dependence on ν and $k|\eta_0|$. These coefficients are suppressed by the factors of ℓH_1 and $\delta H/H_1$. Recall that α_{00} and β_{00} agree with the standard four-dimensional result at low energies. Thus, because of the suppression of $\alpha_{0\nu}$ and $\beta_{0\nu}$ at low energies, the four-dimensional result is recovered only by the contribution from the initial zero mode.

At high energies, we obtain from Eq. (5.8)

$$|\beta_{0\nu}|^2,\ |\alpha_{0\nu}|^2 \propto \left(\frac{\delta H}{H_1}\right)^2 \quad (\ell H_1 \gg 1)\,, \tag{5.40}$$

[2] The integral including the Hankel function is written in terms of generalized hypergeometric functions.

where we have again omitted the dependence on ν and $k|\eta_0|$. In contrast to the result in the low energy regime (5.39), this high energy behavior is not associated with any suppression factor.

Integrating $|\beta_{0\nu}|^2$ over the KK continuum, we obtain the total number of zero mode gravitons created from the initial KK vacuum. It can be shown that the coefficients behave as $\beta_{0\nu} \approx -\alpha^*_{0\nu} \sim (k|\eta_0|)^{1/2}$ at $k|\eta_0| \ll 1$, and we have

$$\beta_{0\nu} + \alpha^*_{0\nu} \sim \mathcal{O}(k|\eta_0|)^{3/2}. \tag{5.41}$$

Thus the number of created gravitons is proportional to k outside the horizon and is evaluated as

$$\int_0^\infty |\beta_{0\nu}|^2 d\nu \approx \begin{cases} 0.5 \times k|\eta_0|(\ell H_1)^2 \left(\dfrac{\delta H}{H_1}\right)^2 & (\ell H_1 \ll 1), \\ 0.3 \times k|\eta_0| \left(\dfrac{\delta H}{H_1}\right)^2 & (\ell H_1 \gg 1). \end{cases} \tag{5.42}$$

On the other hand, making use of the asymptotic form of the Hankel function $H^{(1)}_\nu(x) \sim e^{i[x-(2\nu+1)\pi/4]}/\sqrt{x}$ for $x \to \infty$, we can evaluate the integral in Eq. (5.37) in the $k|\eta_0| \to \infty$ limit as

$$\beta_{0\nu} \propto k|\eta_0| \int_\infty^{k|\eta_0|} dx \frac{e^{2ix}}{x^2} \sim \frac{1}{k|\eta_0|} \quad (k|\eta_0| \to \infty), \tag{5.43}$$

where we have carried out the integration by parts and kept the most dominant term. This shows that the creation of gravitons is suppressed well inside the horizon. Since the assumption of the instantaneous transition tends to overestimate particle production at large k [1], the number of particles created from the initial zero mode and KK modes is expected to be more suppressed inside the horizon than Eq. (5.43) if we consider a realistic situation in which the Hubble parameter changes smoothly. Note that $u_{0\nu} \propto k|\eta_0| \int_\infty^{k|\eta_0|} x^{-2} dx$ is constant for $k|\eta_0| \to \infty$. Therefore, the $\alpha_{0\nu}$ coefficient is dominated by $U_{0\nu}$ at large k, which behaves like $|\alpha_{0\nu}|^2 \sim |U_{0\nu}|^2 \propto (k\eta_0)^2$.

An example of numerical calculation is shown in Fig. 5.2. The figure shows that the spectrum has a peak around the Hubble scale and then decreases inside the horizon, and we confirmed that the behavior of the number density outside the horizon is well described by Eq. (5.42).

5.3 Power spectrum of generated gravitational waves

So far, we have discussed the Bogoliubov coefficients to see the number of created gravitons. However, our main interest is in the power spectrum of gravitational waves because the meaning of "particle" is obscure at the super Hubble scale.

5.3.1 Mapping formula: Pure de Sitter brane

Gravitational waves generated from pure de Sitter inflation on a brane have the scale invariant spectrum

$$\mathcal{P}_{5\mathrm{D}} = \frac{2C^2(\ell H)}{M^2_{\mathrm{Pl}}} \left(\frac{H}{2\pi}\right)^2, \tag{5.44}$$

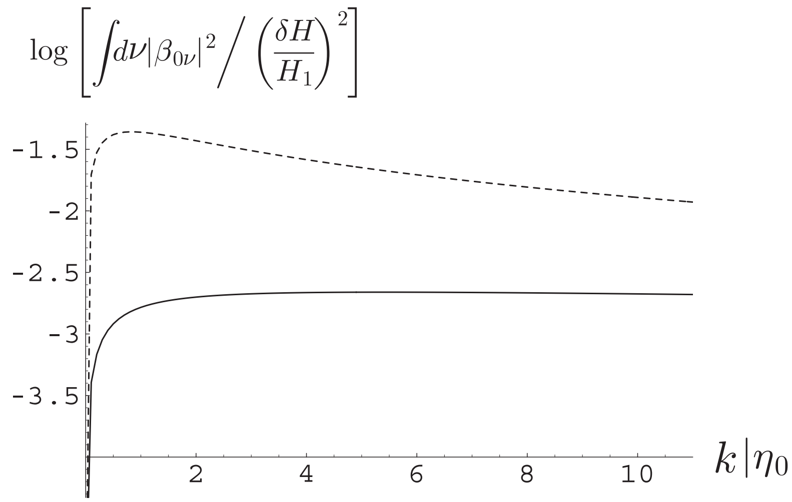

Figure 5.2: Spectra of number density of zero mode gravitons created from vacuum fluctuations in the initial KK modes. Integration over ν is performed numerically. Solid line represents the low energy case with $\ell H_1 = 0.1$, while dashed line shows the high energy case. The latter should be understood as the limiting case $\ell H_1 \to \infty$ since at high energies the number does not depend on ℓH_1; see Eq. (5.40).

which is defined by the expectation value of the squared amplitude of the vacuum fluctuation, $8\pi k^3|\phi_0|^2/(2\pi M_5)^3$, evaluated at a late time. Since $C^2 \sim 1$ at $\ell H \ll 1$, this power spectrum agrees at low energies with the standard four-dimensional spectrum [94, 95]

$$\mathcal{P}_{4\mathrm{D}} = \frac{2}{M_{\mathrm{Pl}}^2}\left(\frac{H}{2\pi}\right)^2. \tag{5.45}$$

At high energies, however, the power spectrum (5.44) is enhanced due to the factor $C(\ell H)$, and is much greater than the four-dimensional counterpart. This amplification effect was found in Ref. [84]. These results say that the difference between Eq. (5.44) and Eq. (5.45) is absorbed by the transformation

$$H \mapsto HC(\ell H). \tag{5.46}$$

5.3.2 Model with variation of the Hubble parameter

Now let us turn to the case in which the Hubble parameter is not constant. Time variation of the Hubble parameter during inflation brings a small modification to the spectrum, and the resulting spectrum depends on the wavelength. Here we consider the amplitude of vacuum fluctuation of the zero mode in the final state. It will be a relevant observable for the observers on the brane at a late epoch because the KK mode fluctuations at super Hubble scale rapidly decay in the expanding universe due to its four-dimensional effective mass. Since the behavior of the zero mode at the infinite future $\tilde{\eta} \to 0$ is known from the explicit form of the mode function [see Eq. (3.21)], we find that the vacuum fluctuation for zero mode at a late epoch is given by

$$\lim_{\tilde{\eta}\to 0}\left(\left|\alpha_{00}\tilde{\phi}_0^* - \beta_{00}^*\tilde{\phi}_0\right|^2 + \int d\nu \left|\alpha_{0\nu}\tilde{\phi}_0^* - \beta_{0\nu}^*\tilde{\phi}_0\right|^2\right)$$
$$= \frac{C^2(\ell H_2)}{\ell}\frac{H_2^2}{2k^3}\left(|\alpha_{00}+\beta_{00}^*|^2 + \int d\nu\, |\alpha_{0\nu}+\beta_{0\nu}^*|^2\right). \tag{5.47}$$

Multiplying this by $(2/M_5^3)(k^3/2\pi^2)$ and recalling the relation $\ell M_5^3 = M_{\rm Pl}^2$, we obtain the power spectrum

$$\mathcal{P}_{\rm 5D}(k) = \mathcal{P}_{\rm 5D}^{\rm zero}(k) + \mathcal{P}_{\rm 5D}^{\rm KK}(k), \tag{5.48}$$

with

$$\mathcal{P}_{\rm 5D}^{\rm zero}(k) = \frac{2C^2(\ell H_2)}{M_{\rm Pl}^2}\left(\frac{H_2}{2\pi}\right)^2 |\alpha_{00}+\beta_{00}^*|^2,$$
$$\mathcal{P}_{\rm 5D}^{\rm KK}(k) = \frac{2C^2(\ell H_2)}{M_{\rm Pl}^2}\left(\frac{H_2}{2\pi}\right)^2 \int_0^\infty d\nu |\alpha_{0\nu}+\beta_{0\nu}^*|^2. \tag{5.49}$$

The power spectrum in the four-dimensional theory, computed in the same way, is given by

$$\mathcal{P}_{\rm 4D}(k) = \frac{2}{M_{\rm Pl}^2}\left(\frac{H_2}{2\pi}\right)^2 \left|\alpha^{\rm (4D)} + \beta^{*\rm (4D)}\right|^2. \tag{5.50}$$

The appearance of the coefficient α in the power spectrum may look unusual. This is due to our setup in which the final state of the universe is still inflating. In such a case, the fluctuations that have left the Hubble horizon never reenter it. Outside the horizon, the number of particle created, does not correspond to the power spectrum.

There are two apparent differences between $\mathcal{P}_{\rm 5D}$ and $\mathcal{P}_{\rm 4D}$; the normalization factor $C(\ell H)$ and the contribution from the KK modes $\mathcal{P}_{\rm 5D}^{\rm KK}$. In the low energy regime, however, the two spectra agree with each other:

$$\mathcal{P}_{\rm 5D} \approx \mathcal{P}_{\rm 4D} \quad (\ell H_1 \ll 1). \tag{5.51}$$

This is because, as is seen from the discussion about the Bogoliubov coefficients in the preceding section, the zero mode contribution $\mathcal{P}_{\rm 5D}^{\rm zero}$ is exactly the same as $\mathcal{P}_{\rm 4D}$ (up to the normalization factor), and the Kaluza-Klein contribution $\mathcal{P}_{\rm 5D}^{\rm KK}$ is suppressed by the factor $(\ell H_1)^2$. On the other hand, when ℓH_1 is large, the amplitude of gravitational waves deviates from the four-dimensional one owing to the amplification of the factor $C(\ell H)$.

We have observed for pure de Sitter inflation that the correspondence between the five-dimensional power spectrum and the four-dimensional one is realized by the map (5.46). It is interesting to investigate whether the correspondence can be generalized to the present case. It seems natural to give the transformation in this case by

$$h \mapsto hC(\ell h), \tag{5.52}$$

where

$$h(k) = H_2\left|\alpha^{\rm (4D)} + \beta^{*\rm (4D)}\right|; \tag{5.53}$$

namely, the rescaled power spectrum $\mathcal{P}_{\rm res}(k)$ is defined as

$$\mathcal{P}_{\rm res}(k) = \frac{2C^2(\ell h)}{M_{\rm Pl}^2}\left(\frac{h}{2\pi}\right)^2. \tag{5.54}$$

We will see that this transformation works well and mostly absorbs the difference between the five- and four-dimensional amplitudes.

We examine the differences between the five-dimensional spectrum and the rescaled four-dimensional spectrum by expanding them with respect to $\delta H/H_1$ as

$$\mathcal{P}_{\rm 5D}(k) = \mathcal{P}_{\rm 5D}^{(0)} + \mathcal{P}_{\rm 5D}^{(1)} + \mathcal{P}_{\rm 5D}^{(2)} + \mathcal{O}\left(\frac{\delta H}{H}\right)^3 + \mathcal{P}_{\rm 5D}^{\rm KK}, \tag{5.55}$$

$$\mathcal{P}_{\rm res}(k) = \mathcal{P}_{\rm res}^{(0)} + \mathcal{P}_{\rm res}^{(1)} + \mathcal{P}_{\rm res}^{(2)} + \mathcal{O}\left(\frac{\delta H}{H}\right)^3. \tag{5.56}$$

Here the quantity associated with the superscript (n) represents the collection of the terms of $\mathcal{O}\left((\delta H/H)^n\right)$. On the right hand side of Eq. (5.55), all the terms except for the last one come from the initial zero mode. The direct expansion shows that the leading terms in Eqs. (5.55) and (5.56) exactly agree with each other up to the first order in $\delta H/H_1$,

$$\begin{aligned}\mathcal{P}_{\rm 5D}^{(0)} + \mathcal{P}_{\rm 5D}^{(1)} &= \mathcal{P}_{\rm res}^{(0)} + \mathcal{P}_{\rm res}^{(1)} \\ &= \frac{2C^2(\ell H_1)}{M_{\rm Pl}^2}\left(\frac{H_1}{2\pi}\right)^2\left\{1 + \left[\frac{\sin(2k\eta_0)}{k\eta_0} - 2\right]\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\frac{\delta H}{H_1}\right\}. \end{aligned}\tag{5.57}$$

These terms contain only the contribution from the initial zero mode.

The contribution from the initial KK modes $\mathcal{P}_{\rm 5D}^{\rm KK}$ is of order $(\delta H/H_1)^2$ because both $\alpha_{0\nu}$ and $\beta_{0\nu}$ are $\mathcal{O}(\delta H/H_1)$. Thus, to examine whether the agreement of the spectrum continues to hold even after including the KK modes, we investigate the second order part of the spectrum. The second order terms $\mathcal{P}_{\rm 5D}^{(2)}$ and $\mathcal{P}_{\rm res}^{(2)}$ are given by

$$\begin{aligned}\mathcal{P}_{\rm 5D}^{(2)}(k) &= \frac{2C^2(\ell H_1)}{M_{\rm Pl}^2}\left(\frac{H_1}{2\pi}\right)^2\left\{(k\eta_0)^2\left[\frac{C^{-2}(\ell H_1) + C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}} - 2\right]\right. \\ &\quad + \cos(k\eta_0)\left[\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}} + 1\right] + \frac{\sin^2(k\eta_0)}{(k\eta_0)^2}\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}} \\ &\quad + \frac{\sin(2k\eta_0)}{2k\eta_0}\left[\frac{2+3(\ell H_1)^2}{1+(\ell H_1)^2} - \frac{6C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\right] \\ &\quad \left. + \frac{4C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}} - \frac{3+4(\ell H_1)^2}{1+(\ell H_1)^2}\right\}\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\left(\frac{\delta H}{H_1}\right)^2, \end{aligned}\tag{5.58}$$

$$\begin{aligned}\mathcal{P}_{\rm res}^{(2)}(k) &= \frac{2C^2(\ell H_1)}{M_{\rm Pl}^2}\left(\frac{H_1}{2\pi}\right)^2\left\{2\cos(k\eta_0) + \frac{\sin^2(k\eta_0)}{(k\eta_0)^2}\right. \\ &\quad + \frac{\sin(2k\eta_0)}{k\eta_0}\left[\frac{2+3(\ell H_1)^2}{1+(\ell H_1)^2} - \frac{4C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\right] \\ &\quad - \frac{\sin^2(2k\eta_0)}{(2k\eta_0)^2}\left[\frac{4+5(\ell H_1)^2}{1+(\ell H_1)^2} - \frac{4C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\right] \end{aligned}$$

$$+\frac{4C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}-\frac{3+4(\ell H_1)^2}{1+(\ell H_1)^2}\Bigg\}\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\left(\frac{\delta H}{H_1}\right)^2. \tag{5.59}$$

From these equations, we notice that $\mathcal{P}^{(2)}_{\rm res}$ and $\mathcal{P}^{(2)}_{\rm 5D}$ do not agree with each other. We see that the difference is enhanced in particular at $k|\eta_0|\gg 1$. However, as we have mentioned earlier, the KK mode contribution also gives a correction of the same order, and in fact we show that approximate agreement is recovered even in this order by adding the KK mode contribution.

First we observe the power spectrum at $k|\eta_0|\ll 1$. Expanding $\mathcal{P}^{(2)}_{\rm 5D}$ and $\mathcal{P}^{(2)}_{\rm res}$ with respect to $k|\eta_0|$, we have

$$\frac{\mathcal{P}^{(2)}_{\rm 5D}(k)}{\mathcal{P}^{(0)}}\,,\ \frac{\mathcal{P}^{(2)}_{\rm res}(k)}{\mathcal{P}^{(0)}}\sim(k\eta_0)^2\left(\frac{\delta H}{H_1}\right)^2. \tag{5.60}$$

On the other hand, Eq. (5.41) leads to

$$\frac{\mathcal{P}^{\rm KK}_{\rm 5D}(k)}{\mathcal{P}^{(0)}}\sim(k|\eta_0|)^3\left(\frac{\delta H}{H_1}\right)^2. \tag{5.61}$$

Therefore the difference is small outside the horizon as

$$\left|\frac{\mathcal{P}^{(2)}_{\rm 5D}+\mathcal{P}^{\rm KK}_{\rm 5D}-\mathcal{P}^{(2)}_{\rm res}}{\mathcal{P}^{(0)}}\right|\sim(k\eta_0)^2\left(\frac{\delta H}{H_1}\right)^2\quad(k|\eta_0|\ll 1)\,, \tag{5.62}$$

although the cancellation between $\mathcal{P}^{(2)}_{\rm 5D}(k)$ and $\mathcal{P}^{(2)}_{\rm res}(k)$ does not happen. By a similar argument, at $k|\eta_0|\lesssim 1$, we have

$$\left|\frac{\mathcal{P}^{(2)}_{\rm 5D}+\mathcal{P}^{\rm KK}_{\rm 5D}-\mathcal{P}^{(2)}_{\rm res}}{\mathcal{P}^{(0)}}\right|\sim\left(\frac{\delta H}{H_1}\right)^2\quad(k|\eta_0|\lesssim 1)\,. \tag{5.63}$$

The situation is more interesting when we consider the spectrum inside the Hubble horizon. There is a term proportional to $(k\eta_0)^2$ in $\mathcal{P}^{(2)}_{\rm 5D}$, which is dominant at $k|\eta_0|\gg 1$, while there is no corresponding term in $\mathcal{P}^{(2)}_{\rm res}$. Hence, the difference between $\mathcal{P}^{(2)}_{\rm 5D}$ and $\mathcal{P}^{(2)}_{\rm res}$ is

$$\frac{\mathcal{P}^{(2)}_{\rm 5D}-\mathcal{P}^{(2)}_{\rm res}}{\mathcal{P}^{(0)}}\approx-(k\eta_0)^2\left[2-\frac{C^{-2}(\ell H_1)+C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\right]\frac{C^2(\ell H_1)}{\sqrt{1+(\ell H_1)^2}}\left(\frac{\delta H}{H_1}\right)^2. \tag{5.64}$$

Our approximation is valid for $k|\eta_0|\delta H/H_1\lesssim 1$ [Eq. (5.14)]. Hence, within the region of validity, this difference can be as large as $\mathcal{P}^{(0)}$. We also note that the terms proportional to $(k|\eta_0|)^2$ come from α_{00}, while the contribution from β_{00} is suppressed at $k|\eta_0|\gg 1$. On the other hand, the contribution from initial KK modes, $\mathcal{P}^{\rm KK}_{\rm 5D}(k)$, is dominated by $\alpha_{0\nu}$ at $k|\eta_0|\gg 1$: $\mathcal{P}^{\rm KK}_{\rm 5D}\propto\int d\nu|\alpha_{0\nu}+\beta^*_{0\nu}|^2\sim\int d\nu|\alpha_{0\nu}|^2$. This means that, although the creation of zero mode gravitons from the initial KK modes is negligible inside the horizon, a part of the amplitude of the final zero mode comes from the initial KK modes losing their KK momenta. Since the coefficient $\alpha_{0\nu}$ is proportional to $k|\eta_0|$ at large $k|\eta_0|$, $\mathcal{P}^{\rm KK}_{\rm 5D}$ behaves as

$$\frac{\mathcal{P}^{\rm KK}_{\rm 5D}(k)}{\mathcal{P}^{(0)}}\sim+(k\eta_0)^2\left(\frac{\delta H}{H_1}\right)^2. \tag{5.65}$$

This KK mode contribution cancels $\mathcal{P}_{\rm 5D}^{(2)}$. The cancellation can be proved by looking at the property of the Bogoliubov coefficients

$$|\alpha_{00}|^2 - |\beta_{00}|^2 + \int d\nu (|\alpha_{0\nu}|^2 - |\beta_{0\nu}|^2) = 1. \tag{5.66}$$

This relation together with Eq. (5.47) implies that the power spectrum cannot significantly deviate from $C^2(\ell H_2) H_2^2 / 2\ell k^3$ in the region where β_{00} and $\beta_{0\nu}$ are negligibly small. We can also demonstrate the cancellation by explicit calculation in the low and high energy limits. For $k|\eta_0| \gg 1$, we have $\alpha_{0\nu} \approx U_{0\nu}$. Then from Eqs. (5.7), (5.8), and (5.19), we obtain

$$\int d\nu |\alpha_{0\nu}|^2 \approx \begin{cases} \displaystyle\int_0^\infty \frac{2\nu \tanh(\pi\nu) d\nu}{(\nu^2+1/4)(\nu^2+9/4)} \times (k\eta_0)^2 (\ell H_1)^2 \left(\frac{\delta H}{H_1}\right)^2 & (\ell H_1 \ll 1), \\ \displaystyle\frac{6}{\pi} \int_0^\infty \frac{\nu^2 d\nu}{(\nu^2+1/4)^2(\nu^2+9/4)} \times (k\eta_0)^2 \left(\frac{\delta H}{H_1}\right)^2 & (\ell H_1 \gg 1), \end{cases} \tag{5.67}$$

which gives $(k\eta_0)^2 (\ell H_1)^2 (\delta H / H_1)^2$ in the low energy regime and $(3/4)(k\eta_0)^2 (\delta H/H_1)^2$ in the high energy regime. Comparing these with Eq. (5.64), we see that $\mathcal{P}_{\rm 5D}^{\rm KK}(k)$ cancels $\mathcal{P}_{\rm 5D}^{(2)}(k)$[3].

To summarize, we have observed that the agreement between the rescaled spectrum $\mathcal{P}_{\rm res}(k)$ and the five-dimensional spectrum $\mathcal{P}_{\rm 5D}(k)$ is exact up to first order in $\delta H / H_1$. The agreement is not exact at second order, but we found that the correction is not enhanced at any wavelength irrespective of the value of ℓH_1. Just for illustrative purpose we show the results of numerical calculations in Fig. 5.3 and Fig. 5.4.

5.4 Summary

In this chapter we have investigated the generation of primordial gravitational waves and its power spectrum in the inflationary braneworld model, focusing on the effects of the variation of the Hubble parameter during inflation. For this purpose, we considered a model in which the Hubble parameter changes discontinuously.

In the case of de Sitter inflation with constant Hubble parameter H, the spectrum is known to be given by Eq. (5.44) [84]. It agrees with the standard four-dimensional one [Eq. (5.45)] at low energies $\ell H \ll 1$, but at high energies $\ell H \gg 1$ it significantly deviates from Eq. (5.45) due to the amplification effect of the zero mode normalization factor $C(\ell H)$. One can say, however, the five-dimensional spectrum is obtained from the four-dimensional one by the map $H \mapsto HC(\ell H)$.

In a model with variable Hubble parameter, gravitational wave perturbations are expected to be generated not only from the "in-vacuum" of the zero mode but also from that of the Kaluza-Klein modes. Hence, it is not clear whether there is a simple relation between the five-dimensional spectrum and the four-dimensional counterpart. Analyzing the model with a discontinuous jump in the Hubble parameter, we have shown that this is indeed approximately the case. More precisely, if the squared amplitude of four-dimensional fluctuations is given by $(h/2\pi M_{\rm Pl})^2$, we transform $(h/2\pi M_{\rm Pl})^2$ into $C^2(\ell h)(h/2\pi M_{\rm Pl})^2$, then the resulting rescaled spectrum exactly agrees with the five-dimensional spectrum $\mathcal{P}_{\rm 5D}(k)$ up to first order in $\delta H/H_1$. At second order $\mathcal{O}(\delta H/H)^2$ the agreement is not exact, but the difference is not

[3]The small ℓH expansion $C^2(\ell H) \approx 1 - (\ell H)^2 [1/2 + \ln(\ell H/2)]$ is used here to investigate the low energy case.

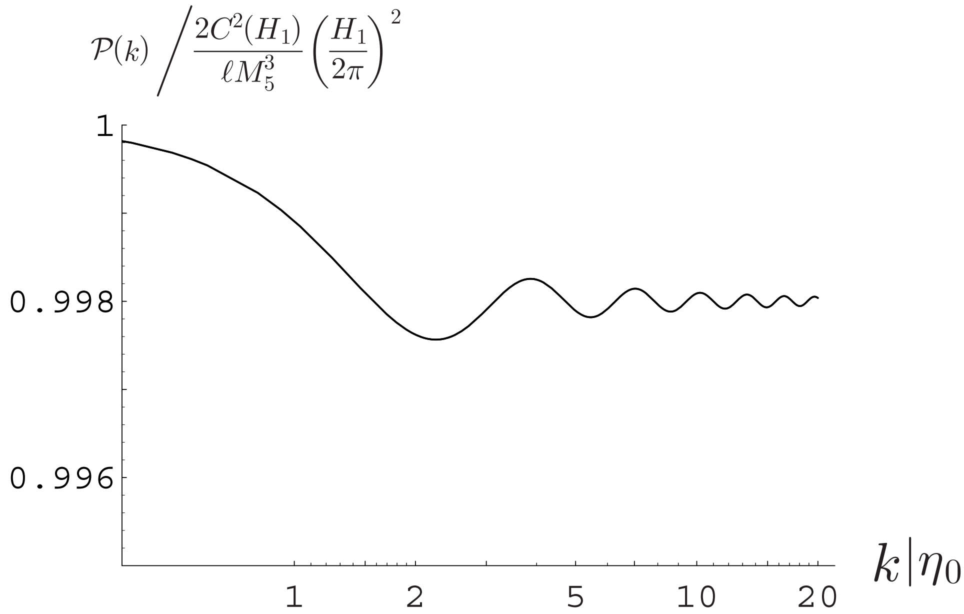

Figure 5.3: Five-dimensional power spectrum of gravitational waves $\mathcal{P}_{\rm 5D}(k)$ and the four-dimensional one $\mathcal{P}_{\rm 4D}(k)$ at low energies ($\ell H_1 = 10^{-2}$) with $\delta H/H_1 = 10^{-3}$. These two agree with each other. In this case initial KK modes give negligible contribution.

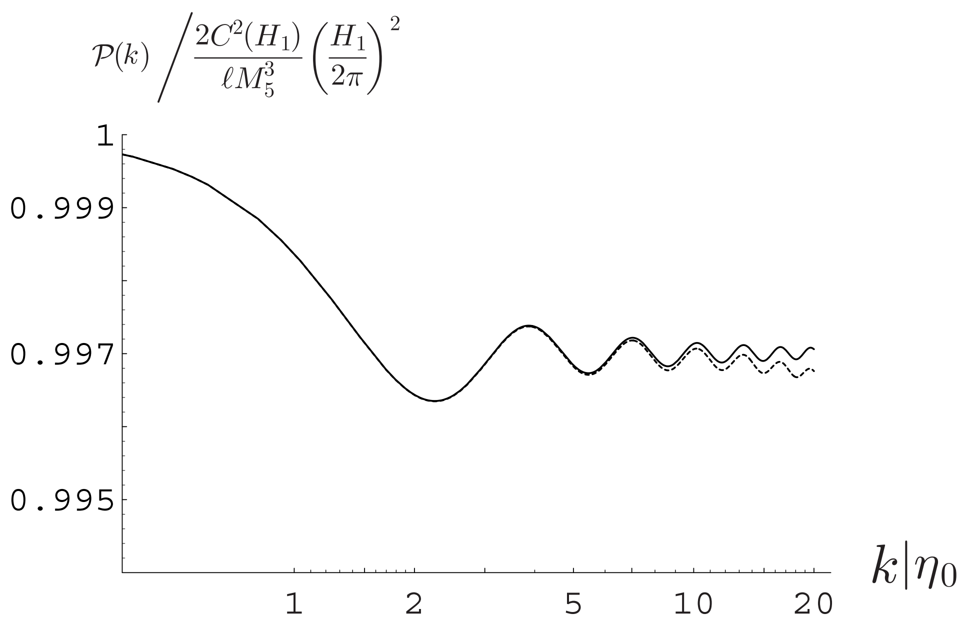

Figure 5.4: Power spectra of gravitational waves at high energies ($\ell H_1 = 10^3$) with $\delta H/H_1 = 10^{-3}$. Five-dimensional spectrum $\mathcal{P}_{\rm 5D}^{\rm zero}(k) + \mathcal{P}_{\rm 5D}^{\rm KK}(k)$ and the rescaled four-dimensional one $\mathcal{P}_{\rm res}(k)$ agree well with each other (solid lines), while only the zero mode contribution $\mathcal{P}_{\rm 5D}^{\rm zero}(k)$ gives the reduced fluctuation amplitude inside the horizon (dotted line).

enhanced at any wavelength irrespective of the value of ℓH_1. Hence, in total, the agreement is not significantly disturbed by the mismatch at second order. As a non-trivial point, we also found that the initial KK mode vacuum fluctuations can give non-negligible contribution to the final zero mode states at second order.

One may expect that the power spectrum of gravitational waves in the braneworld model would reflect the characteristic length scale corresponding to the curvature (or "compactification") scale of the extra dimension ℓ. However, our analysis showed that the resultant power spectrum does not depend on the ratio of the wavelength of gravitational waves to the bulk curvature scale, $k|\eta_0|\ell H$.

Here we should mention the result of Ref. [36] that is summarized in Sec. 5.5. Their setup is the most violent version of the transition $H_1 \to H_2 = 0$. If the wavelength of the gravitational waves is much longer than both the Hubble scale and the bulk curvature scale, the power spectrum is given by Eq. (5.71) and obviously it is obtained from the four-dimensional counterpart by the map $H \mapsto HC(\ell H)$. However, if the wavelength is longer than the Hubble scale but much smaller than the bulk curvature radius, the amplitude is highly damped as is seen from Eq. (5.74) and the map $H \mapsto HC(\ell H)$ does not work at all. This damping of the amplitude can be understood in the following way. In the high energy regime $\ell H \gg 1$, the motion of the brane with respect to the static bulk is ultrarelativistic. At the moment of transition to the Minkowski phase, the brane abruptly stops. Zero mode gravitons with wavelength smaller than the bulk curvature scale can be interpreted as "particles" traveling in the five dimensions. These gravitons make a "hard hit" with the brane at the moment of this transition, and get large momenta in the fifth direction relative to the static brane. As a result, these gravitons escape into the bulk as KK gravitons, and thus the amplitude (5.74) is damped. On the other hand, in our model the change of the Hubble parameter is assumed to be small, and hence such violent emission of KK gravitons does not happen. If this interpretation is correct, the mapping rule $h \mapsto hC(\ell h)$ will generally give a good estimate for the prediction of inflationary braneworld models as far as time variation of the Hubble parameter is smooth. If we can confirm the validity of this prescription in more general cases, the analysis of gravitational wave perturbations will be simplified a lot, and in fact it will be confirmed by a numerical study in the next chapter.

5.5 Appendix: Particle creation when connected to Minkowski brane

Here for comparison with our results we briefly summarize the results obtained by Gorbunov *et al.* [36] focusing on the power spectrum of gravitational waves. Their method is basically the same that we have already explained in the main text. They considered the situation that de Sitter inflation on the brane with constant Hubble parameter H suddenly terminates at a conformal time $\eta = \eta_0$, and is followed by a Minkowski phase. The power spectrum of gravitational waves is expressed in terms of the Bogoliubov coefficients as

$$\mathcal{P}_{5\mathrm{D}}(k) = \frac{2}{M_{\mathrm{Pl}}^2}\left(\frac{H}{2\pi}\right)^2 (k\eta_0)^2 \left(1 + 2|\beta_{00}|^2 + 2\int d\nu |\beta_{0\nu}|^2\right), \tag{5.68}$$

where $|\beta_{00}|^2$ and $|\beta_{0\nu}|^2$ are the number of zero mode gravitons created from initial zero mode and KK modes, respectively. At super Hubble scale ($k|\eta_0| \ll 1$) we can neglect the first term in the parentheses, which corresponds to the vacuum fluctuations in Minkowski space.

According to Ref. [36], when $k|\eta_0|\ell H \ll 1$ [i.e., the wavelength of gravitational wave $(k/a)^{-1}$ is much larger than the bulk curvature scale ℓ] and $k|\eta_0| \ll 1$, the coefficients are given by

$$|\beta_{00}|^2 \approx \frac{C^2(\ell H)}{4(k\eta_0)^2}, \tag{5.69}$$

$$\int d\nu |\beta_{0\nu}|^2 \sim \begin{cases} k|\eta_0|(\ell H)^2 & (\ell H \ll 1), \\ k|\eta_0|\ell H & (\ell H \gg 1). \end{cases} \tag{5.70}$$

One can see that the contribution from initial KK modes is suppressed irrespective of the expansion rate ℓH. Therefore the power spectrum is evaluated as

$$\mathcal{P}_{5D} \approx \frac{C^2(\ell H)}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2. \tag{5.71}$$

Equation (5.71) is half of the power spectrum on the de Sitter brane [Eq. (5.44)], and this result can be understood as follows. The amplitude of fluctuations at the super Hubble scale stays constant. After the sudden transition from the de Sitter phase to the Minkowski phase, the Hubble scale becomes infinite. Therefore, those fluctuation modes are now inside the Hubble horizon, and they begin to oscillate. As a result, the mean-square vacuum fluctuation becomes half of the initial value.

On the other hand, when $k|\eta_0|\ell H \gg 1$ and $k|\eta_0| \ll 1$ (these conditions require $\ell H \gg 1$), the Bogoliubov coefficients are given by

$$|\beta_{00}|^2 \approx \frac{C^2(\ell H)}{(k\eta_0)^2}\frac{1}{(k\eta_0\ell H)^2}, \tag{5.72}$$

$$\int d\nu |\beta_{0\nu}|^2 \sim \frac{1}{(k\eta_0\ell H)^2}. \tag{5.73}$$

As before, the contribution from initial KK modes is negligible, and that from $|\beta_{00}|^2$ dominates the power spectrum,

$$\mathcal{P}_{5D} \approx \frac{C^2(\ell H)}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2 \frac{4}{(k\eta_0\ell H)^2}. \tag{5.74}$$

One can see that the spectrum is suppressed by the factor $4/(k\eta_0\ell H)^2$.

5.6 Appendix: Details of calculations

We derive Eqs. (5.18) and (5.19) by calculating the Wronskian (5.17). Expanding Eq. (5.12) in terms of ϵ_H, we have

$$\tilde{\eta} = \eta + \epsilon_H\eta_0\cosh\xi - \epsilon_H^2\eta_0^2(\sinh\xi)^2/(2\eta) + \cdots, \tag{5.75}$$

$$\tilde{\xi} = \xi - \epsilon_H\eta_0\sinh\xi/\eta + \epsilon_H^2\eta_0^2\cosh\xi\sinh\xi/\eta^2 + \cdots, \tag{5.76}$$

which reduce to the following forms at the infinite past ($\eta \to -\infty$),

$$\tilde{\eta} - \eta = \epsilon_H\eta_0\cosh\xi, \tag{5.77}$$

$$\tilde{\xi} - \xi = 0. \tag{5.78}$$

Then, U_{00} is evaluated at $\eta = -\infty$ as

$$
\begin{aligned}
U_{00} &= -2i\ell^3 \int_{\xi_b}^{\infty} \frac{d\xi}{\eta^2 \sinh^2\xi} \left(\tilde{\phi}_0 \partial_\eta \phi_0^* - \phi_0^* \partial_\eta \tilde{\phi}_0\right)\Big|_{\eta\to-\infty} \\
&= \frac{H_2}{H_1} C(\ell H_1) C(\ell H_2) \cdot 2(\ell H_1)^2 \int_{\xi_b}^{\infty} d\xi \frac{e^{-i\epsilon_H k\eta_0 \cosh\xi}}{\sinh^3\xi} \\
&\approx \frac{H_2 C(\ell H_2)}{H_1 C(\ell H_1)} e^{-i\epsilon_H k\eta_0 \cosh\xi_b} \cdot 2(\ell H_1)^2 C^2(\ell H_1) \times \\
&\quad \times e^{i\epsilon_H k\eta_0 \cosh\xi_b} \int_{\xi_b}^{\infty} d\xi \frac{1}{\sinh^3\xi} \left[1 - i\epsilon_H k\eta_0 \cosh\xi - \frac{1}{2}(\epsilon_H k\eta_0)^2 \cosh^2\xi\right], \quad (5.79)
\end{aligned}
$$

where we expanded the integrand with respect to ϵ_H in the last line. Integrating each term, we finally obtain

$$
\begin{aligned}
U_{00} &\approx \frac{H_2 C(\ell H_2)}{H_1 C(\ell H_1)} e^{-i\epsilon_H k\eta_0 \cosh\xi_b} \\
&\quad \times \left\{1 - i\epsilon_H k\eta_0 \left[C^2(\ell H_1) - \cosh\xi_b\right] - \frac{1}{2}(\epsilon_H k\eta_0)^2 \sinh^2\xi_b\right\}. \quad (5.80)
\end{aligned}
$$

Note that the condition $\epsilon_H k|\eta_0| \cosh\xi_b (\approx k|\eta_0|\delta H/H) \ll 1$ is required in order to justify the expansion of the exponent. Because the integral is saturated at $\xi \approx \xi_b$, we do not have to worry about the validity of the expansion for large $\cosh\xi$.

The explicit form of $U_{0\nu}$ is needed up to the first order in ϵ_H. A similar calculation leads to

$$
\begin{aligned}
U_{0\nu} &= -2i\ell^3 \int_{\xi_b}^{\infty} \frac{d\xi}{\eta^2 \sinh^3\xi} \left(\tilde{\phi}_0 \partial_\eta \phi_\nu^* - \phi_\nu^* \partial_\eta \tilde{\phi}_0\right)\Big|_{\eta\to-\infty} \\
&= -2i\ell^3 \cdot \ell^{-1/2} C(\ell H_2) \frac{H_2}{\sqrt{2k}} \cdot \frac{\ell^{-3/2}}{\sqrt{2k}} \int_{\xi_b}^{\infty} \frac{d\xi}{\eta^2 \sinh^3\xi} \chi_\nu(\xi) \\
&\quad \times \left[\left(\tilde{\eta} - \frac{i}{k}\right) e^{-ik\tilde{\eta}} \cdot (-1 - ik\eta) e^{ik\eta + i\pi/4} - (-\tilde{\eta})\, e^{ik\eta + i\pi/4} \cdot (-ik\eta) e^{-ik\tilde{\eta}}\right]\Big|_{\eta\to-\infty} \\
&= -2e^{i\pi/4} \ell H_2 C(\ell H_2) \int_{\xi_b}^{\infty} d\xi \frac{e^{-i\epsilon_H k\eta_0 \cosh\xi_b}}{\sinh^3\xi} \chi_\nu(\xi). \quad (5.81)
\end{aligned}
$$

The integral in the last line, which we call I, can be calculated as follows. Again, expanding the integrand in terms of ϵ_H, we have

$$
I \approx \int_{\xi_b}^{\infty} d\xi \frac{\chi_\nu(\xi)}{\sinh^3\xi} \left(1 - i\epsilon_H k\eta_0 \cosh\xi\right). \quad (5.82)
$$

Let us consider the first term in the parentheses. The spatial wave function χ_ν satisfies $(\sinh\xi)^{-3}\chi_\nu = -(\nu^2 + 9/4)^{-1}\partial_\xi[(\sinh\xi)^{-3}\partial_\xi\chi_\nu]$ with the boundary condition $\partial_\xi\chi_\nu(\xi_b) = 0$. Therefore, together with the behavior at infinity, $(\sinh\xi)^{-3}\partial_\xi\chi_\nu \sim (\sinh\xi)^{-3}\partial_\xi(\sinh\xi)^{3/2} \to 0$, we find that the integral of the first term vanishes. Then, using the integration by parts twice, we have

$$\begin{aligned}\left(\nu^2+\frac{9}{4}\right)I &\approx i\epsilon_H k\eta_0\int_{\xi_b}^{\infty}d\xi\cosh\xi\frac{\partial}{\partial\xi}\left[\frac{1}{\sinh^3\xi}\frac{\partial}{\partial\xi}\chi_\nu(\xi)\right]\\ &= -i\epsilon_H k\eta_0\int_{\xi_b}^{\infty}d\xi\frac{1}{\sinh^2\xi}\frac{\partial}{\partial\xi}\chi_\nu(\xi)\\ &= i\epsilon_H k\eta_0\frac{\chi_\nu(\xi_b)}{\sinh^2\xi_b}+i\epsilon_H k\eta_0\int_{\xi_b}^{\infty}d\xi\frac{-2\cosh\xi}{\sinh^3\xi}\chi_\nu(\xi)\\ &\approx i\epsilon_H k\eta_0\frac{\chi_\nu(\xi_b)}{(\ell H_1)^2}+2I,\end{aligned}\tag{5.83}$$

from which we can evaluate $U_{0\nu}$. Note that the approximation is valid when $k|\eta_0|\delta H/H_1 \ll 1$.

Chapter 6

Quantum-mechanical generation of gravitational waves

The generation and evolution of perturbations are among the most important issues in cosmology because of their direct link to cosmological observations such as the stochastic gravitational wave background and the temperature anisotropy of the cosmic microwave background (CMB), by which we can probe the early universe. While the cosmological perturbation theory in the conventional four-dimensional universe is rather established [67, 110, 4, 94], calculating cosmological perturbations in the braneworld still remains to be a difficult problem. Although some attempts have been made concerning scalar perturbations [74, 76, 126, 78, 79, 146, 81], further progress is awaited to give a clear prediction about the CMB anisotropy in the Randall-Sundrum braneworld. Almost the same is true for gravitational wave (tensor) perturbations [41, 84, 36, 61, 45, 46, 49, 50, 27, 5, 6, 139, 63]. However, since their generation and evolution depend basically only on the background geometry, they are slightly easier to handle.

During inflation super-horizon gravitational wave perturbations are generated from vacuum fluctuations of gravitons. The pure de Sitter braneworld is the special case that allows definite analytical computation of quantum fluctuations. Thanks to the symmetry of the de Sitter group, the perturbation equation becomes separable and can be solved exactly [84]. The time variation of the Hubble parameter generally causes mixing of a massless zero mode and massive Kaluza-Klein modes, and this effect was investigated based on the "junction" models, which assume an instantaneous transition from a de Sitter to a Minkowski brane [36] or to another de Sitter brane [61], as was explained in the previous chapter. There we discussed the quantum-mechanical generation of gravitational waves within such limited and simplified models.

As for the classical evolution of gravitational waves during the radiation dominated epoch, several studies have been done [45, 46, 49, 50], assuming that a given single initial mode for each comoving wave number dominates. The focus of these works is mainly on the evolution of modes which re-enter the horizon in the high-energy regime. While the late time evolution is worked out analytically in Refs. [63, 78] by resorting to low-energy approximation methods.

In this chapter we consider the generation of primordial gravitational waves during inflation in more general models of the Randall-Sundrum type. For definiteness, we adopt a simple setup in which both initial and final phases are described by de Sitter braneworlds. Two de Sitter phases with different values of the Hubble expansion rate are smoothly interpolated. This work is an extension of the work on the "junction" models in Chapter 5. In the previous analysis the transition of the Hubble rate was abrupt and the gap was assumed to be infinitesimal. Here we extend the previous results to more general models with smooth transition by using numerical calculations with a refined formulation.

This chapter is organized as follows. In the next section we briefly summarize past studies [84, 36, 61] on the generation of gravitational waves via quantum fluctuations during inflation in the Randall-Sundrum braneworld, emphasizing the mapping formula introduced in the previous chapter [61]. In Sec. 6.2 we describe our numerical scheme to investigate the generation of gravitational waves, and then in Sec. 6.3 we present results of our calculations. Section 6.4 is devoted to discussion.

6.1 Gravitational waves in inflationary braneworld

6.1.1 Pure de Sitter brane

The background spacetime that we consider is composed of a five-dimensional AdS bulk, whose metric is given in the Poincaré coordinates by

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + \delta_{ij}dx^i dx^j + dz^2\right), \tag{6.1}$$

and a Friedmann brane at $z = z(t)$.

First let us consider the generation of gravitational waves from pure de Sitter inflation on the brane [84]. The coordinate system appropriate for the present situation is

$$ds^2 = \frac{\ell^2}{\sinh^2\xi}\left[\frac{1}{\eta^2}\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right) + d\xi^2\right], \tag{6.2}$$

which is obtained from Eq. (6.1) by a coordinate transformation

$$t = \eta\cosh\xi + t_0, \tag{6.3}$$
$$z = -\eta\sinh\xi, \tag{6.4}$$

where t_0 is an arbitrary constant and η is the conformal time, which is negative. The de Sitter brane is located at

$$\xi = \xi_b = \text{constant}, \tag{6.5}$$

and the Hubble parameter on the brane is given by

$$H = \ell^{-1}\sinh\xi_b. \tag{6.6}$$

The gravitational wave perturbations are described by the metric

$$ds^2 = \frac{\ell^2}{\sinh^2\xi}\left\{\frac{1}{\eta^2}\left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] + d\xi^2\right\}, \tag{6.7}$$

and we decompose the perturbations into the spatial Fourier modes as

$$h_{ij} = \frac{\sqrt{2}}{(2\pi M_5)^{3/2}}\int d^3k\ \phi_{\mathbf{k}}(\eta,\xi)e^{i\mathbf{k}\cdot\mathbf{x}}e_{ij}, \tag{6.8}$$

where e_{ij} is the transverse-traceless polarization tensor and M_5 is the fundamental mass scale which is related to the four-dimensional Planck mass $M_{\rm Pl}$ by $\ell(M_5)^3 = M_{\rm Pl}^2$. Hereafter we will suppress the subscript $\mathbf{k}$. The linearized Einstein equations give the Klein-Gordon-type equation for ϕ:

$$\left[\frac{\partial^2}{\partial\eta^2} - \frac{2}{\eta}\frac{\partial}{\partial\eta} + k^2 - \frac{\sinh^3\xi}{\eta^2}\frac{\partial}{\partial\xi}\frac{1}{\sinh^3\xi}\frac{\partial}{\partial\xi}\right]\phi = 0. \tag{6.9}$$

Assuming the Z_2-symmetry across the brane, the boundary condition on the brane is given by $\partial_\xi\phi|_{\xi=\xi_b} = 0$. The perturbation equation (6.9) admits one discrete zero mode $\phi_0(\eta)$ as well as massive Kaluza-Klein (KK) modes $\phi_\nu = \psi_\nu(\eta)\cdot\chi_\nu(\xi)$, which were already given in Chapter 3.

In inflationary cosmology, fluctuations in the graviton field (and other fields such as the inflaton) are considered to be generated quantum-mechanically. As explained in Chapter 3, we can quantize the graviton field following the standard canonical quantization scheme. Then the expectation value of the squared amplitude of the vacuum fluctuation in the zero mode is given by

$$\begin{aligned} |\phi_0|^2 &= \ell^{-1}C^2(\ell H)\frac{H^2}{2k^3}(1+k^2\eta^2) \\ &\to \ell^{-1}C^2(\ell H)\frac{H^2}{2k^3}, \end{aligned} \tag{6.10}$$

where the expression in the second line is obtained by evaluating the perturbation in the super-horizon regime/at a late time, and hence this is the amplitude of the *growing mode*. In terms of the power spectrum defined by

$$\mathcal{P} := \frac{4\pi k^3}{(2\pi)^3}\cdot\frac{2}{(M_5)^3}|\phi_0|^2, \tag{6.11}$$

we have

$$\mathcal{P} = \frac{2C^2(\ell H)}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2, \tag{6.12}$$

where we have used $\ell(M_5)^3 = M_{\rm Pl}^2$. This is the standard flat spectrum up to the overall factor $C^2(\ell H)$. In conventional four-dimensional cosmology the power spectrum of the primordial gravitational waves from de Sitter inflation is given by [94, 95]

$$\mathcal{P}_{\rm 4D} = \frac{2}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2. \tag{6.13}$$

Thus just by rescaling the amplitude in four-dimensional cosmology as

$$H \mapsto HC(\ell H), \tag{6.14}$$

the braneworld result (6.12) is exactly obtained.

6.1.2 The "junction" model

Pure de Sitter inflation on the brane described in the previous subsection is a special case where the amplitude of the growing zero mode can be obtained completely analytically. As a next step to understand gravitational waves from more general inflation with $H \neq \text{const}$, a discontinuous change in the Hubble parameter was considered in the previous chapter [61]. In such a "junction" model the Hubble parameter is given by

$$H(\eta) = \begin{cases} H_i, & \eta < \eta_0, \\ H_f = H_i - \delta H, & \eta > \eta_0, \end{cases} \tag{6.15}$$

and

$$\frac{\delta H}{H_i} \ll 1, \tag{6.16}$$

is assumed.

The evolution of gravitational wave perturbations can be analyzed by solving the equation of motion backward using the advanced Green's function [36]. A set of mode functions can be constructed for the initial de Sitter stage, and another for the final de Sitter stage, either of which forms a complete orthonormal basis for the graviton wave function. The final (growing) zero mode may be written as a linear combination of the initial zero and KK modes. The Bogoliubov coefficients give the creation rate of final zero mode gravitons from the initial vacuum fluctuations in the Kaluza-Klein modes as well as in the zero mode. Thus the power spectrum may be written as a sum of two separate contributions,

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_{\text{KK}}, \tag{6.17}$$

where $\mathcal{P}_0$ and $\mathcal{P}_{\text{KK}}$ are the parts coming from the initial zero and Kaluza-Klein modes, respectively.

An important result brought by analyzing this junction model is that $\mathcal{P}$ is well approximated by the rescaled spectrum $\mathcal{P}_{\text{res}}$ obtained by using a simple map speculated by the exact result of pure de Sitter inflation [Eq. (6.14)]. More precisely, let $\mathcal{P}_{\text{4D}}(k)$ be the gravitational wave spectrum evaluated in the standard four-dimensional inflationary universe with the same time dependence of the Hubble parameter (6.15), and then the rescaled spectrum is defined by

$$\mathcal{P}_{\text{4D}} = \frac{2}{M_{\text{Pl}}^2}\left(\frac{h_k}{2\pi}\right)^2 \mapsto \mathcal{P}_{\text{res}} = \frac{2C^2(\ell h_k)}{M_{\text{Pl}}^2}\left(\frac{h_k}{2\pi}\right)^2. \tag{6.18}$$

Comparing the rescaled spectrum with $\mathcal{P}$ obtained from a five-dimensional calculation, one finds that the difference between these two is suppressed to be second order like [61]

$$\left|\frac{\mathcal{P} - \mathcal{P}_{\text{res}}}{\mathcal{P}}\right| \lesssim \left(\frac{\delta H}{H_i}\right)^2 \ll 1. \tag{6.19}$$

It might be worth noting here that the KK contribution $\mathcal{P}_{\text{KK}}$ is necessary for realizing this interesting agreement between the braneworld result and the (basically) four-dimensional result especially at high energies $\ell H \gg 1$. The above agreement (6.19) raised a speculation that the mapping formula $h_k \mapsto C(\ell h_k)$ may work with good accuracy in more general inflation models with a smoothly changing expansion rate. The central purpose of the present chapter is *to check whether this speculation is correct or not.*

6.2 Formulation

6.2.1 Basic equations

Now let us explain the formulation that we use to study the generation of gravitational waves without assuming a pure de Sitter brane. Our formulation is based on double null coordinates, which are presumably the most convenient for numerical calculations.

The metric (6.1) can be rewritten by using double null coordinates

$$u = t - z, \tag{6.20}$$

$$v = t + z, \tag{6.21}$$

in the form of

$$ds^2 = \frac{4\ell^2}{(v-u)^2}\left(-dudv + \delta_{ij}dx^i dx^j\right). \tag{6.22}$$

The trajectory of the brane can be specified arbitrarily by

$$v = q(u). \tag{6.23}$$

By a further coordinate transformation

$$U = u, \tag{6.24}$$

$$q(V) = v, \tag{6.25}$$

we obtain

$$ds^2 = \frac{4\ell^2}{[q(V)-U]^2}\left[-q'(V)dUdV + \delta_{ij}dx^i dx^j\right], \tag{6.26}$$

where a prime denotes differentiation with respect to the argument. Now in the new coordinates the position of the brane is simply given by

$$U = V. \tag{6.27}$$

We will use this coordinate system for actual numerical calculations.

The induced metric on the brane is

$$ds_b^2 = \frac{4\ell^2}{[q(V)-V]^2}\left[-q'(V)dV^2 + \delta_{ij}dx^i dx^j\right], \tag{6.28}$$

from which we can read off the conformal time η and the scale factor a, respectively, as

$$d\eta = \sqrt{q'(V)}\, dV, \tag{6.29}$$

$$a = \frac{2\ell}{q(V)-V}, \tag{6.30}$$

and hence the Hubble parameter on the brane is written as

$$\ell H = \frac{1}{2\sqrt{q'(V)}}[1-q'(V)], \tag{6.31}$$

or equivalently

$$q'(V) = \left(\sqrt{1+\ell^2 H^2} - \ell H\right)^2 . \tag{6.32}$$

Given the Hubble parameter as a function of η, one can integrate Eqs. (6.29) and (6.32) to obtain q as a function of V.

When the brane undergoes pure de Sitter inflation (and thus $q'(V)$ = constant), the following relation between (U, V) and (η, ξ) will be useful:

$$\xi = \xi_b + \frac{1}{2}\ln\left(\frac{t_0 - U}{t_0 - V}\right), \tag{6.33}$$

$$\eta = -e^{-\xi_b}[(t_0 - U)(t_0 - V)]^{1/2}, \tag{6.34}$$

and

$$q(V) = e^{-2\xi_b}(V - t_0) + t_0. \tag{6.35}$$

The Klein-Gordon-type equation for a gravitational wave perturbation ϕ in the (U, V) coordinates reduces to

$$\left[4\partial_U\partial_V + \frac{6}{q(V) - U}(\partial_V - q'(V)\,\partial_U) + q'(V)\,k^2\right]\phi = 0, \tag{6.36}$$

supplemented by the boundary condition

$$[\partial_U - \partial_V]\phi\big|_{U=V} = 0. \tag{6.37}$$

The expression for the Wronskian evaluated on a constant V hypersurface is given by

$$(X \cdot Y) = 2i\int_{-\infty}^{V} dU \left[\frac{2\ell}{q(V) - U}\right]^3 (X\partial_U Y^* - Y^*\partial_U X), \tag{6.38}$$

which is independent of the choice of the hypersurface.

When inflation on the brane deviates from pure de Sitter one, the decomposition into the zero mode and KK modes becomes rather ambiguous. For this reason we require the initial and final phases of inflation to be pure de Sitter, though arbitrary cosmic expansion is allowed in the intermediate stage. In both de Sitter phases, $q(V)$ can be fit by Eq. (6.35), and ξ_b and t_0 are determined, respectively. Hence we have two sets of de Sitter coordinates (η, ξ) and $(\tilde{\eta}, \tilde{\xi})$. We distinguish the coordinates in the final phase by associating them with tilde. In the final de Sitter phase the mode will be well outside the horizon, and hence we expand the graviton field in terms of the growing and decaying zero mode solutions $\tilde{\phi}_g$ and $\tilde{\phi}_d$ as

$$\phi = \hat{A}_g\tilde{\phi}_g + \hat{A}_d\tilde{\phi}_d + \int d\nu\left(\hat{A}_\nu\tilde{\phi}_\nu + \hat{A}_\nu^\dagger\tilde{\phi}_\nu^*\right), \tag{6.39}$$

where

$$\tilde{\phi}_g := \mathrm{Im}\left[\tilde{\phi}_0\right], \tag{6.40}$$

$$\tilde{\phi}_d := \mathrm{Re}\left[\tilde{\phi}_0\right], \tag{6.41}$$

and the mode functions with tilde are defined in the same way as ϕ_0 and ϕ_ν with the substitution of $(\tilde{\xi}, \tilde{\eta})$ for (ξ, η). It can be easily seen that the growing and the decaying modes are normalized as

$$\left(\tilde{\phi}_g \cdot \tilde{\phi}_d\right)_f = \frac{1}{2}, \quad \left(\tilde{\phi}_g \cdot \tilde{\phi}_g\right)_f = \left(\tilde{\phi}_d \cdot \tilde{\phi}_d\right)_f = 0, \tag{6.42}$$

where subscript f means that the expression is to be evaluated in the final de Sitter phase. Notice that de Sitter mode functions thus defined as functions of (U, V) through (ξ, η) [or $(\tilde{\xi}, \tilde{\eta})$] do not satisfy the equation of motion outside the initial (or final) de Sitter phase. Back in the initial de Sitter phase the graviton field can be expanded as

$$\phi = \hat{a}_0 \phi_0 + \hat{a}_0^\dagger \phi_0^* + \int d\nu \left(\hat{a}_\nu \phi_\nu + \hat{a}_\nu^\dagger \phi_\nu^*\right). \tag{6.43}$$

We assume that initially the gravitons are in the de Sitter invariant vacuum state annihilated by $\hat{a}_0$ and $\hat{a}_\nu$,

$$\hat{a}_0 |0\rangle = \hat{a}_\nu |0\rangle = 0. \tag{6.44}$$

We would like to evaluate the expectation value of the squared amplitude of the vacuum fluctuation in the growing mode at a late time,

$$\left|\tilde{\phi}_g\right|^2 \langle 0|\hat{A}_g^2|0\rangle,$$

or equivalently, the power spectrum,

$$\begin{aligned} \mathcal{P}(k) &= \frac{4\pi k^3}{(2\pi)^3} \frac{2}{(M_5)^3} \cdot \left|\tilde{\phi}_g\right|^2 \langle 0|\hat{A}_g^2|0\rangle \\ &\to \frac{2C^2(\ell H_f)}{M_{\rm Pl}^2} \left(\frac{H_f}{2\pi}\right)^2 \langle 0|\hat{A}_g^2|0\rangle, \end{aligned} \tag{6.45}$$

where H_f is the Hubble parameter in the final de Sitter phase. In order to obtain the final amplitude, it is not necessary to solve the evolution of all the (infinite number of) degrees of freedom with their initial conditions set by Eq. (6.43). In fact, we have only to solve the backward evolution of the final decaying mode, as explained below.

Suppose that a solution $\Phi(U, V)$ is chosen so as to satisfy $\Phi = \tilde{\phi}_d$ in the final de Sitter phase. From the Wronskian condition (6.42) we see that

$$\begin{aligned} \frac{1}{2}\hat{A}_g &= \left(\phi \cdot \tilde{\phi}_d\right)_f = (\phi \cdot \Phi) \\ &= (\phi_0 \cdot \Phi)_i \, \hat{a}_0 + \int d\nu (\phi_\nu \cdot \Phi)_i \, \hat{a}_\nu + \text{h.c.}, \end{aligned}$$

where subscript i means that the expression is to be evaluated in the initial de Sitter phase. In the above we used the fact that the Wronskian is constant in time. Thus we obtain

$$\langle 0|\hat{A}_g^2|0\rangle = 4\left[|(\phi_0 \cdot \Phi)_i|^2 + \int d\nu |(\phi_\nu \cdot \Phi)_i|^2\right], \tag{6.46}$$

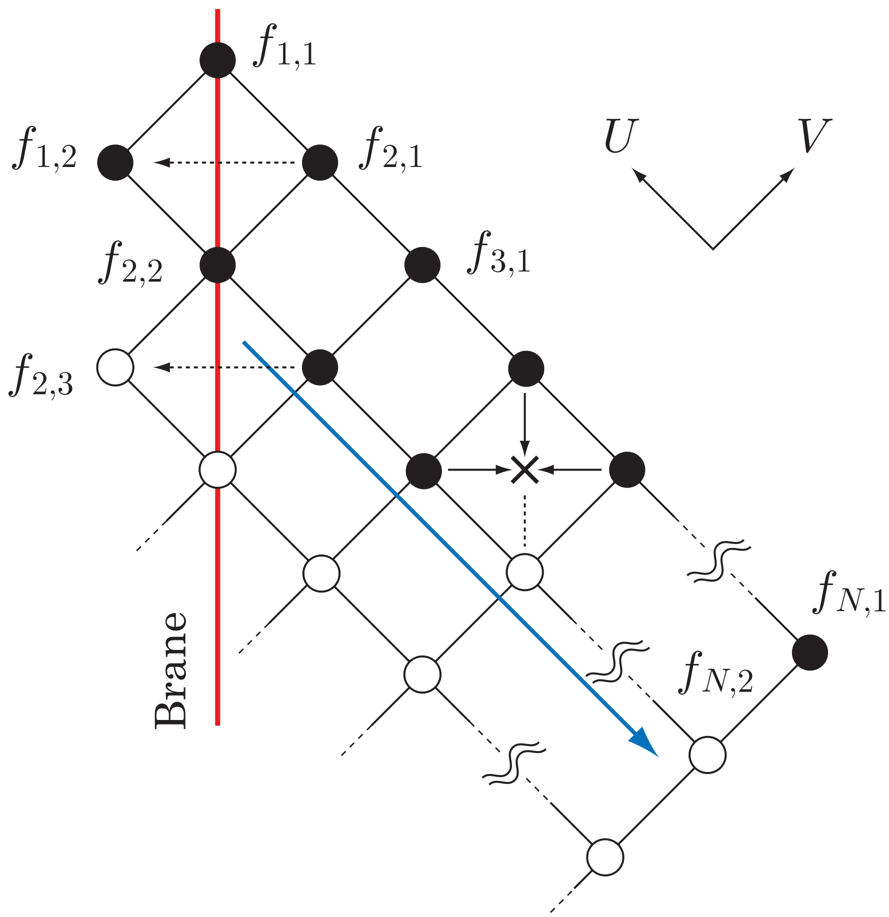

Figure 6.1: Numerical (backward) evolution scheme.

from which we can calculate the power spectrum of the primordial gravitational waves. It is obvious that the spectrum is written in the form of Eq. (6.17) with

$$\mathcal{P}_0 := \frac{8C^2(\ell H_f)}{M_{\rm Pl}^2}\left(\frac{H_f}{2\pi}\right)^2 |(\phi_0 \cdot \Phi)_i|^2, \tag{6.47}$$

$$\mathcal{P}_{\rm KK} := \frac{8C^2(\ell H_f)}{M_{\rm Pl}^2}\left(\frac{H_f}{2\pi}\right)^2 \int d\nu |(\phi_\nu \cdot \Phi)_i|^2. \tag{6.48}$$

6.2.2 Numerical scheme

Here we describe the algorithm that we employ to solve the backward evolution of the gravitational perturbations numerically. We set the boundary conditions for the mode Φ so as to be identical to a decaying zero mode $\tilde{\phi}_d$ at a time $V = V_f$ in the final de Sitter phase. We decompose Φ as

$$\Phi(U,V) = \tilde{\phi}_d(U,V) + \delta\Phi(U,V), \tag{6.49}$$

and solve the equation for $\delta\Phi(U,V)$ instead of $\Phi(U,V)$. The equation of motion for $\delta\Phi$ is obtained as

$$\left[4\partial_U\partial_V + \frac{6}{q(V)-U}(\partial_V - q'(V)\partial_U) + q'(V)k^2\right]\delta\Phi$$
$$= -\left[4\partial_U\partial_V + \frac{6}{q(V)-U}(\partial_V - q'(V)\,\partial_U) + q'(V)\,k^2\right]\tilde{\phi}_d. \tag{6.50}$$

Since both Φ and $\tilde{\phi}_d$ satisfy the boundary condition of the form of Eq. (6.37), the boundary condition for $\delta\Phi$ is also written as

$$[\partial_U - \partial_V]\delta\Phi\big|_{U=V} = 0. \tag{6.51}$$

We immediately find that we do not have to solve the backward evolution in the final de Sitter phase, since $\delta\Phi$ identically vanishes there.

In some cases Φ is not disturbed so much from its final configuration $\tilde{\phi}_d(U,V_f)$, especially when we discuss a small deviation from the pure de Sitter case. It is advantageous then to use a small quantity $\delta\Phi$ as a variable in numerical calculations. Of course, there is no problem even when $\delta\Phi$ does not stay small.

Our scheme to obtain the values of $\delta\Phi$ at the initial surface $V = V_i$ is as follows. We give the boundary conditions at $V = V_f$ as $f_{1,1}, f_{2,1}, \cdots, f_{N,1} = 0$, where

$$f_{n,m} := \delta\Phi(U = V_f - \varepsilon(n-1), V = V_f - \varepsilon(m-1)).$$

A sketch of the numerical grids is shown in Fig. 6.1. Here N is taken to be sufficiently large, and we use the same grid spacing ε both in the U and V directions. At a virtual site $(U,V) = (V_f, V_f - \varepsilon)$ we set

$$f_{1,2} = f_{2,1}, \tag{6.52}$$

so that the boundary condition (6.51) is satisfied at $(V_f - \varepsilon/2, V_f - \varepsilon/2)$. From $f_{1,1}$, $f_{2,1}$, and $f_{1,2}$ we can determine the value of $f_{2,2}$ by using the equation of motion (6.50) at $(V_f - \varepsilon/2, V_f - \varepsilon/2)$, and then from $f_{2,1}$, $f_{3,1}$, and $f_{2,2}$ we can determine $f_{3,2}$, and so on. We repeat the same procedure in the subsequent time steps until we obtain $f_{M,M}, f_{M+1,M}, \cdots, f_{N,M}$ at $V_i = V_f - \varepsilon(M-1)$.

6.2.3 A toy model

As stated above, to make the problem well posed, we consider models which have initial and final de Sitter phases. To perform numerical calculations, as a concrete example, we adopt a simple toy model in which the Hubble parameter is given by an analytic form

$$H(\eta) = \bar{H} - \Delta\tanh\left(\frac{\eta-\eta_0}{s}\right), \tag{6.53}$$

where

$$\bar{H} := \frac{H_i + H_f}{2}, \quad \Delta := \frac{H_i - H_f}{2}, \tag{6.54}$$

with H_i and H_f the initial and final values of the Hubble parameter, s is a parameter that controls the smoothness of the transition, and η_0 indicates a transition time. Taking the

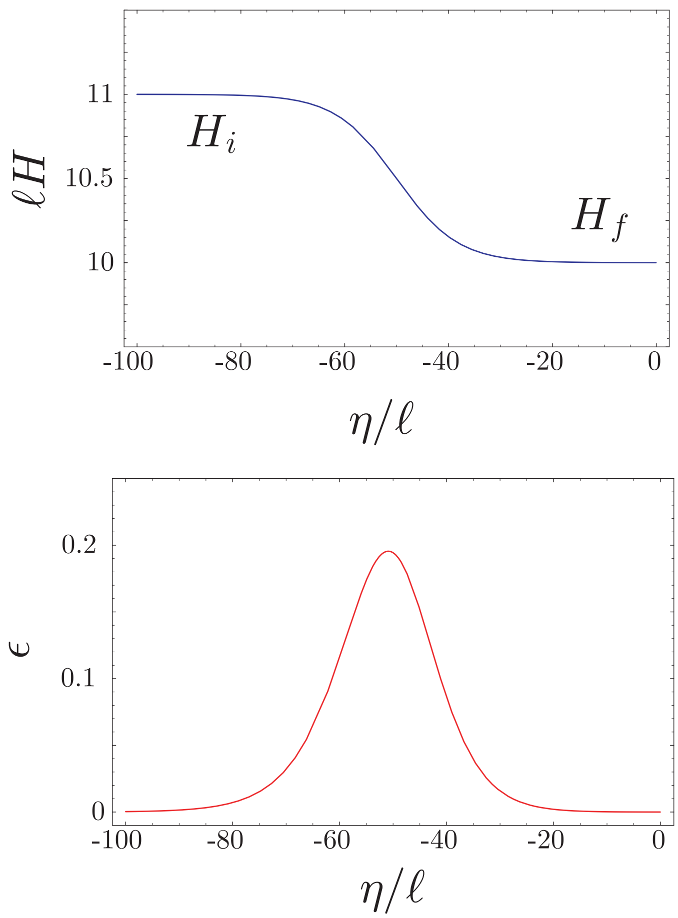

Figure 6.2: The Hubble parameter and the slow-roll parameter ϵ in our model. This is a plot for the model of Fig. 6.9.

initial time η_i and the final time η_f so that $\eta_0 - \eta_i \gg s$ and $\eta_f - \eta_0 \approx -\eta_0 \gg s$, we have $H \approx H_i$ for $\eta \to \eta_i$ and $H \approx H_f$ for $\eta \to \eta_f$. The scale factor for this inflation model is given by

$$a(\eta) = \left\{ s\Delta \ln \left[2\cosh\left(\frac{\eta - \eta_0}{s}\right)\right] - \bar{H}\eta + \Delta\eta_0 \right\}^{-1}, \tag{6.55}$$

where the integration constant is determined by imposing that $a \approx (-H_f\eta)^{-1}$ at $\eta \approx \eta_f \approx 0$. The slow-roll parameter, $\epsilon := -\partial_\eta H/(aH^2)$, is obtained as

$$\epsilon = \frac{\Delta\{s\Delta \ln[2\cosh((\eta - \eta_0)/s)] - \bar{H}\eta + \Delta\eta_0\}}{s[\bar{H}\cosh((\eta - \eta_0)/s) - \Delta \sinh((\eta - \eta_0)/s)]^2}, \tag{6.56}$$

which takes maximum at $\eta = \eta_0$. The maximum value is

$$\epsilon(\eta_0) = \epsilon_{\max} = \frac{\Delta}{s\bar{H}^2 a(\eta_0)}. \tag{6.57}$$

The behavior of the Hubble parameter and the slow-roll parameter is shown in Fig. 6.2.

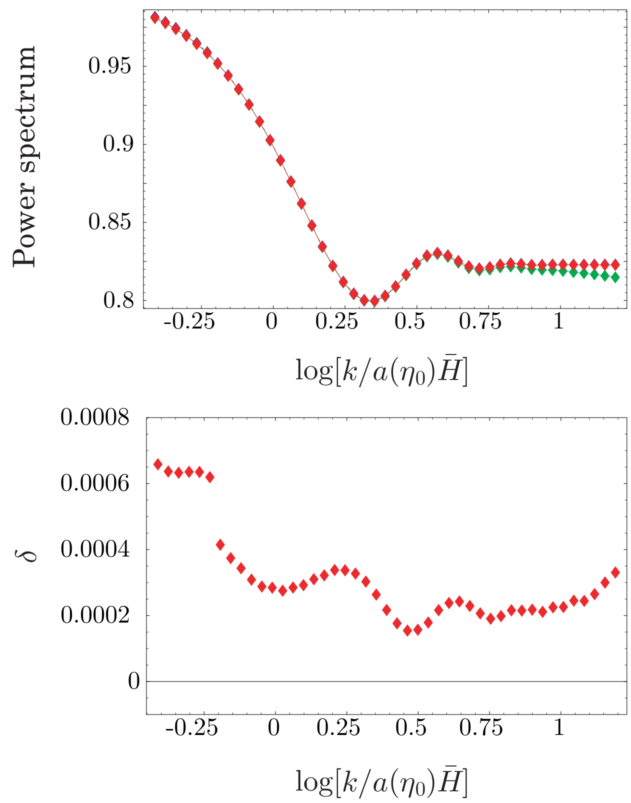

Figure 6.3: Top panel: power spectra of gravitational waves normalized by $(2C^2(\ell H_i)/M_{\rm Pl}^2)(H_i/2\pi)^2$. Red diamonds (upper ones) indicate the result including the contributions from the initial Kaluza-Klein modes, while green diamonds (lower ones) represent the contribution from the initial zero mode. Although the rescaled four-dimensional spectrum is shown by blue diamonds, they are almost hidden by the red ones since the result of the five-dimensional calculation is well approximated by the rescaled four-dimensional spectrum. Bottom panel: difference between the five-dimensional and rescaled four-dimensional spectra.

Here we should mention the limitation of this simple toy model. For fixed values of H_i, H_f, and $\epsilon_{\rm max}$ one can prolong the period of the transition $\sim s$ measured in conformal time by taking large s. However, using the definition of $\epsilon_{\rm max}$ (6.57), the transition time scale in *proper* time is found to be given by $\sim a(\eta_0)s = \bar{H}^2\epsilon_{\rm max}/\Delta$. Therefore we cannot change the transition time scale independently of the other parameters, H_i, H_f, and $\epsilon_{\rm max}$.

6.3 Numerical results

Using various parameters shown in Table. 6.1 we performed numerical calculations, the results of which are presented in Figs. 6.3–6.10. Numbers of grids are $N \times M = 50000 \times 1000$, and the grid separation ε is chosen to be $(0.11 - 2.5) \times \ell$ depending on the energy scale of inflation. Note here that the step width in conformal time η is given by $\Delta\eta = \sqrt{q'(V)}\ \varepsilon$ [Eq. (6.29)], and $q'(V)$ becomes smaller for a larger value of ℓH [Eq. (6.32)]. Our choice of ε makes $\Delta\eta$ about the same size, $\Delta\eta \approx 0.1$, in all the calculations. Integration over ν is performed up

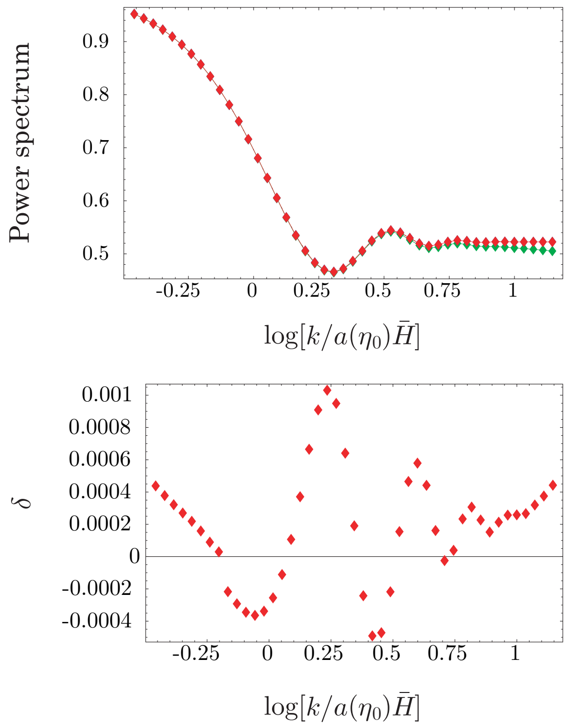

Figure 6.4: Same as Fig. 6.3, but the parameters are different.

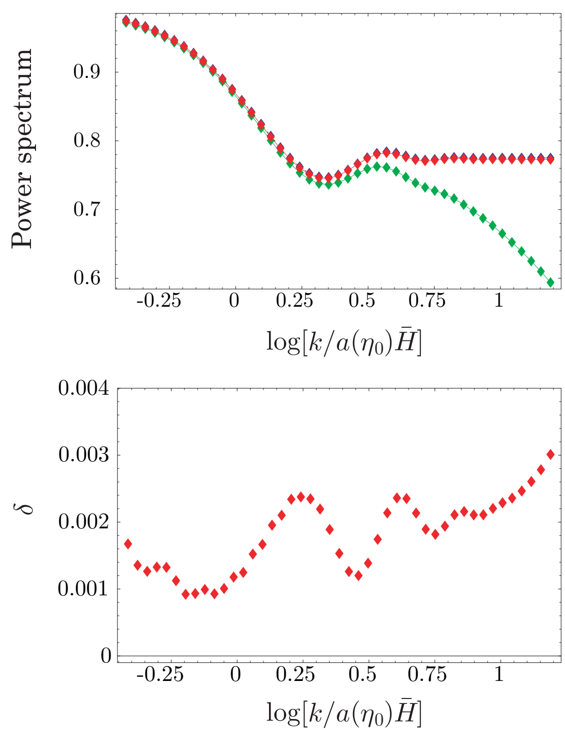

Figure 6.5: Same as Fig. 6.3, but the parameters are different.

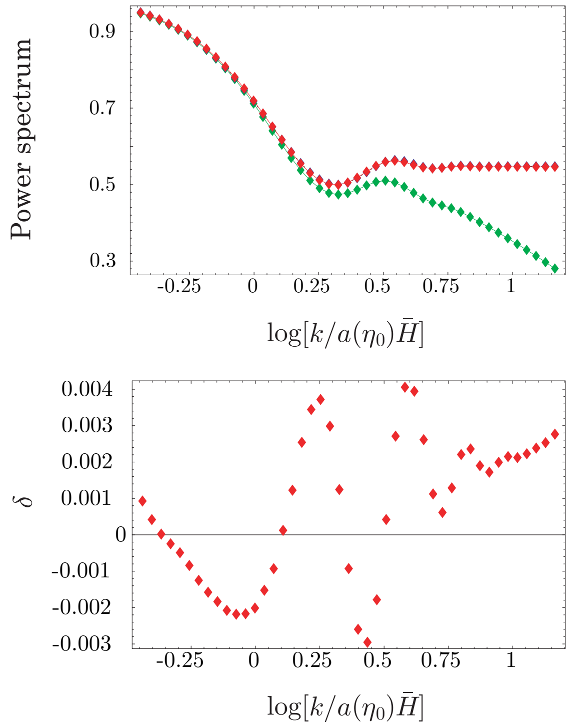

Figure 6.6: Same as Fig. 6.3, but the parameters are different.

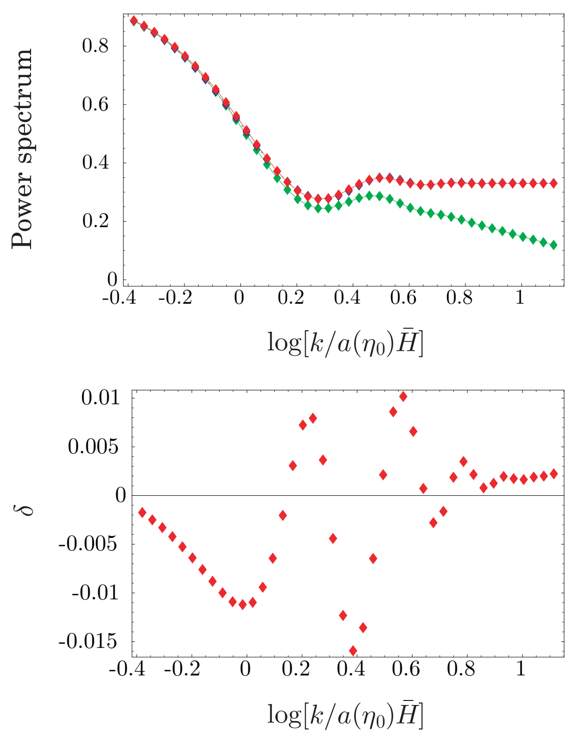

Figure 6.7: Same as Fig. 6.3, but the parameters are different.

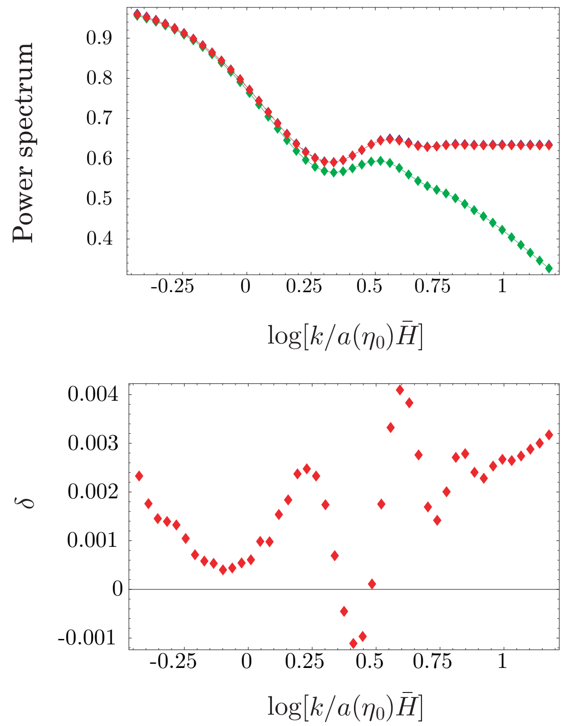

Figure 6.8: Same as Fig. 6.3, but the parameters are different.

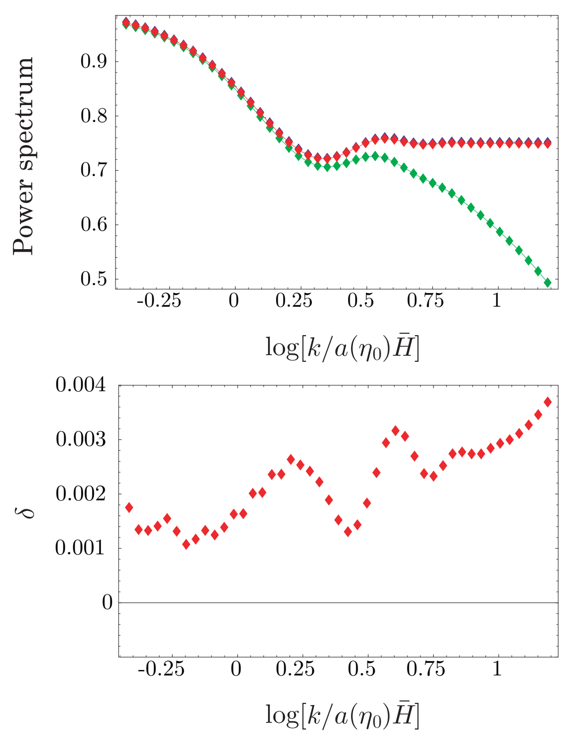

Figure 6.9: Same as Fig. 6.3, but the parameters are different.

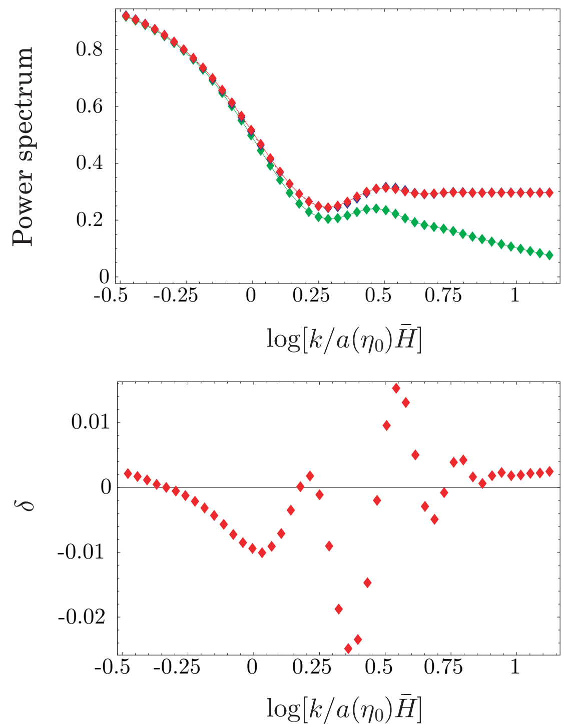

Figure 6.10: Same as Fig. 6.3, but the parameters are different.

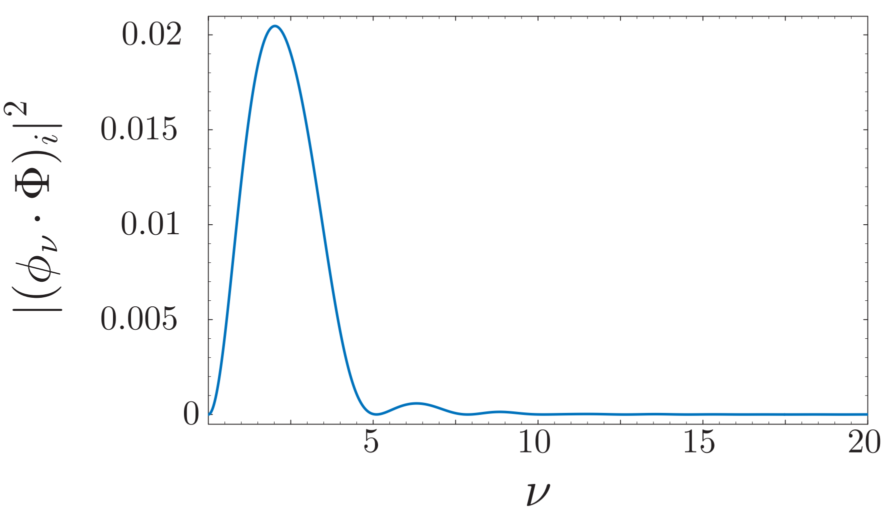

Figure 6.11: Wronskian $|(\phi_\nu \cdot \Phi)_i|^2$ as a function of ν. The parameters are given by those of Fig. 6.9 and $k/a(\eta_0)\bar{H} = 10$.

Table 6.1: Parameters used for the numerical calculations presented in the figures.

	ℓH_i	ℓH_f	$\epsilon_{\max}$	ℓ/s	η_0/ℓ
Figure 6.3	0.11	0.10	0.21	0.09	−50.45
Figure 6.4	0.11	0.08	0.69	0.1	−50.46
Figure 6.5	1.1	1.0	0.21	0.09	−50.21
Figure 6.6	1.25	1.0	0.45	0.09	−50.21
Figure 6.7	1.5	1.0	0.75	0.09	−50.21
Figure 6.8	3.5	3.0	0.34	0.095	−50.08
Figure 6.9	11	10	0.19	0.085	−50.02
Figure 6.10	15	10	0.75	0.09	−50.02

to $\nu = 20 - 40$ with an equal grid spacing of 0.15. The typical behavior of the integrand $|(\phi_\nu \cdot \Phi)_i|^2$ is shown in Fig. 6.11.

In order to compare the five-dimensional power spectrum with a four-dimensional counterpart, we solve the conventional evolution equation for gravitational waves in four-dimensions,

$$\left(\partial_\eta^2 + 2aH\partial_\eta + k^2\right)\Phi = 0, \tag{6.58}$$

where $H(\eta)$ is given by the same function as used for the corresponding five-dimensional computation, and then we calculate the *rescaled* power spectrum obtained from the four-dimensional 'bare' spectrum by using the mapping formula (6.18).

We introduce a parameter that represents a difference between the five-dimensional power spectrum and the rescaled four-dimensional spectrum:

$$\delta(k) := \frac{\mathcal{P}_{\rm res} - \mathcal{P}}{\mathcal{P}}. \tag{6.59}$$

In all the cases of our calculations we clearly see that $|\delta| \ll 1$ and thus we conclude that *the primordial spectrum of the gravitational waves in the braneworld is quite well approximated by the rescaled four-dimensional spectrum.* We also find that the difference becomes larger for larger values of the slow-roll parameter ϵ, but $|\delta|$ is no greater than $\mathcal{O}(10^{-2})$ even for models with $\epsilon_{\max} \simeq 0.75$. Notice that the universe is not inflating any more if ϵ is greater than unity. At high energies $\ell H \gtrsim 1$ it can be seen that dependence of δ on ℓH is weak, while at low energies like $\ell H \sim 0.1$, δ is further suppressed compared to the high energy cases.

Contributions from the initial KK modes can be significant for small wavelength modes. For example, for the modes with $k/a(\eta_0)\bar{H} \simeq 10$ we see that $\mathcal{P}_{\rm KK}/\mathcal{P} \sim \mathcal{O}(10^{-2})$ at low energies, but the KK contribution can become as large as $\mathcal{P}_{\rm KK}/\mathcal{P} \sim 0.6$ in our most 'violent' model with $\ell H \sim 10$ and $\epsilon_{\max} \simeq 0.75$.

6.4 Summary

In this chapter we have investigated the generation of gravitational waves during inflation on a brane and computed the primordial spectrum. Extending the previous work [61], we have considered a model composed of the initial and final de Sitter stages, and the transition region connecting them smoothly. We have numerically solved the backward evolution of the final *decaying* mode to obtain the amplitude of the *growing* zero mode at a late time by making use of the Wronskian method,

We found that the power spectrum $\mathcal{P}$ is well approximated by the rescaled spectrum $\mathcal{P}_{\rm res}$ basically calculated in standard four-dimensional inflationary cosmology. Here the rescaling formula for the amplitude is given by a simple map $h_k \mapsto h_k C(\ell h_k)$, where C is the normalization factor of the zero mode. Although the difference of the two spectra, $\delta := (\mathcal{P}_{\rm res} - \mathcal{P})/\mathcal{P}$, depends on the energy scale of inflation and the slow-roll parameter, our numerical analysis clearly shows that in any case the mapping formula works with quite good accuracy, yielding $|\delta| \lesssim \mathcal{O}(10^{-2})$. This implies that the mapping relation between the two spectra holds quite generally in the Randall-Sundrum braneworld, which we believe is a useful formula.

We have taken into account the vacuum fluctuations in initial Kaluza-Klein modes as well as the zero mode, and found that both of them contribute to the final amplitude of the zero mode. The amount of the initial KK contribution can be large on small scales, and it increases as the energy scale of inflation ℓH becomes higher. This gives rise to a quite interesting picture. When the expansion rate changes during inflation, zero mode gravitons escape into the bulk as KK gravitons, but at the same time bulk gravitons come onto the brane to compensate for the loss, and these two effects almost cancel each other. This seems to happen, irrespective of the energy scale, in a wide class of the inflation models even with a not-so-small slow-roll parameter. It is suggested that this is not the case in the radiation dominated decelerating universe [45, 46, 49, 50], where the decay into KK gravitons reduces the amplitude of the gravitational waves on the brane. The junction model of Ref. [36] joining de Sitter and Minkowski branes also show the suppression of the gravitational wave amplitude at high energies.

What is the reason for the remarkable agreement of the braneworld spectrum and the rescaled four-dimensional spectrum? Extremely long wavelength modes, which leave the horizon during the initial de Sitter stage much before the Hubble parameter changes, have a squared amplitude of $(2C^2(\ell H_i)/M_{\rm Pl}^2)(H_i/2\pi)^2$, and the amplitude of the perturbations stays constant during the subsequent stages [1]. The same is true for four-dimensional inflationary cosmology, and so the mapping formula is applicable to these long wavelength modes. On the other hand, a mode whose wavelength is much shorter than the Hubble horizon scale at the transition time will feel the transition as adiabatic and hence the particle production is exponentially suppressed, $\langle 0|\hat{A}_g^\dagger \hat{A}_g|0\rangle \approx 1$, leading to $\mathcal{P} \approx (2C^2(\ell H_f)/M_{\rm Pl}^2)(H_f/2\pi)^2$. The same argument can be applied to the conventional cosmology. Thus it is not surprising that the mapping formula works for such short wavelength modes. However, at present there seems no simple reason for the amplitude of the modes with $k \sim \mathcal{O}(a(\eta_0)\bar{H})$ to coincide with the rescaled one.

[1] In the $k^2 \to 0$ limit, ϕ=const is a growing solution of Eq. (6.36).

Chapter 7

The spectrum of gravitational waves

Cosmological inflation predicts the gravitational wave background arising due to quantum fluctuations in the graviton field. Gravitational wave fluctuations are stretched beyond the horizon radius by rapid expansion during inflation, and at a later stage they come back inside the horizon possibly with rich information on the early universe and hence on high energy physics. Though yet undetected, gravitational waves will provide us with a powerful tool to probe fundamental physics in near future [106].

Gravitational waves from inflation on the brane was first studied by Langlois *et al.* [84], under an assumption that inflation is exactly de Sitter (Chapter 3). In this special case, the perturbation equation is separable and analytically solvable. A toy model called the "junction model" [36, 61] is an extended version of the pure de Sitter braneworld, which allows a sudden change of the Hubble parameter H by joining two maximally symmetric (i.e., de Sitter or Minkowski) branes at some time (Chapter 5). Later, the junction model is extended to a more general inflation model with a smooth expansion rate [65] (Chapter 6). To make the cosmological model more realistic, one should take into account the radiation-dominated phase that follows after inflation, and, at least in the low energy regime ($\ell H \ll 1$), corrections to the evolution of gravitational waves are shown to be small [139, 63] (Chapter 4). In a much more general and interesting case, i.e., in the high energy ($\ell H \gg 1$) radiation-dominated phase, the perturbation equation no longer has a separable form and hence one cannot even define a "zero mode" and "Kaluza-Klein modes" without ambiguity. To understand the evolution of gravitational waves in that regime, numerical studies have been done by Hiramatsu *et al.* [45, 46] and by Ichiki and Nakamura [49, 50]. Their results give us a lot of implications, for example, on the damping nature of the gravitational wave amplitude due to the Kaluza-Klein mode generation, but the initial condition they adopt is naive, neglecting initial quantum fluctuations in the Kaluza-Klein modes. Hence, its validity is open to question.

The goal of the present chapter is *to clarify the late time power spectrum of gravitational waves in the Randall-Sundrum brane cosmology, evolving through the radiation-dominated stage after their generation during inflation.* We closely follow the same line in the previous chapter [65], in which, using the Wronskian formulation, we have formulated a numerical scheme for the braneworld cosmological perturbations. The initial condition in our analysis is imposed quantum-mechanically, and therefore we will be able to obtain a true picture of the generation and evolution of gravitational perturbations in the braneworld.

This chapter is organized as follows. In Section 7.1, we start with giving the background cosmological model and summarize basic known results concerning the gravitational wave mode functions in the de Sitter and Minkowski braneworlds. In Section 7.2, we describe the Wronskian formulation to obtain the power spectrum of gravitational waves, and then we show our numerical results in Sec. 7.3. In Section 7.4 we discuss an amount of the dark radiation generated due to excitation of Kaluza-Klein modes. Finally we conclude in Sec. 7.5.

7.1 Preliminaries

7.1.1 The background model

Now we describe a model for the background. We shall work in the cosmological setting of the Randall-Sundrum braneworld, and so the bulk is given by a five-dimensional AdS spacetime. The AdS metric in the Poincoré coordinates is

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2 + \delta_{ij}dx^i dx^j + dz^2), \tag{7.1}$$

where ℓ is the bulk curvature scale and constrained by table-top experiments as $\ell \lesssim 0.1$ mm [99, 22]. A cosmological brane moves in this static bulk, the trajectory of which is given by $z = z(t)$. The scale factor of the universe is related to the position of the brane as $a(t) = \ell/z(t)$.

We consider the following cosmological model on the brane. The initial stage of the model is given by de Sitter inflation with a constant Hubble parameter $H = H_i$, which is smoothly connected to the radiation-dominated phase. (In order to join the two phases smoothly, the brane is not exactly de Sitter at the very last stage of inflation.) In the radiation stage the scale factor evolves subject to the modified Friedmann equation [9, 10, 111, 71, 51]

$$H^2 = \frac{\rho_r}{3M_{\rm Pl}^2}\left(1 + \frac{\rho_r}{2\sigma}\right), \tag{7.2}$$

where ρ_r is the radiation energy density and $\sigma = 6M_{\rm Pl}^2/\ell^2$ is the tension of the brane. Since the conventional conservation law holds on the brane, we have $\rho_r \propto a^{-4}$. Thus, in terms of the proper time τ on the brane we obtain

$$a(\tau) = a(\tau_1)\left[\left(\frac{\tau}{\tau_1}\right)^2 + 2c\left(\frac{\tau}{\tau_1}\right) - 2c\right]^{1/4}, \tag{7.3}$$

where τ_1 is a fiducial time, $c := \sqrt{1 + [\rho_r(\tau_1)/2\sigma]} - 1$, and $\rho_r(\tau_1)/\sigma = \ell^2/8\tau_1^2$. After a period of time the energy scale of the universe becomes sufficiently low, and the radiation-dominated phase is then smoothly connected to the Minkowski phase. This artificial connection will not cause any unexpected problems on our final result (i.e., power spectra of gravitational waves) because at the end of the radiation stage the brane universe is already in the low energy regime. Just for simplicity we assume that the Minkowski brane is located at $z = \ell$. Namely, the scale factor is normalized so that $a = a_0 = 1$ when the universe ceases expanding. The motion of the brane is shown in Fig. 7.1.

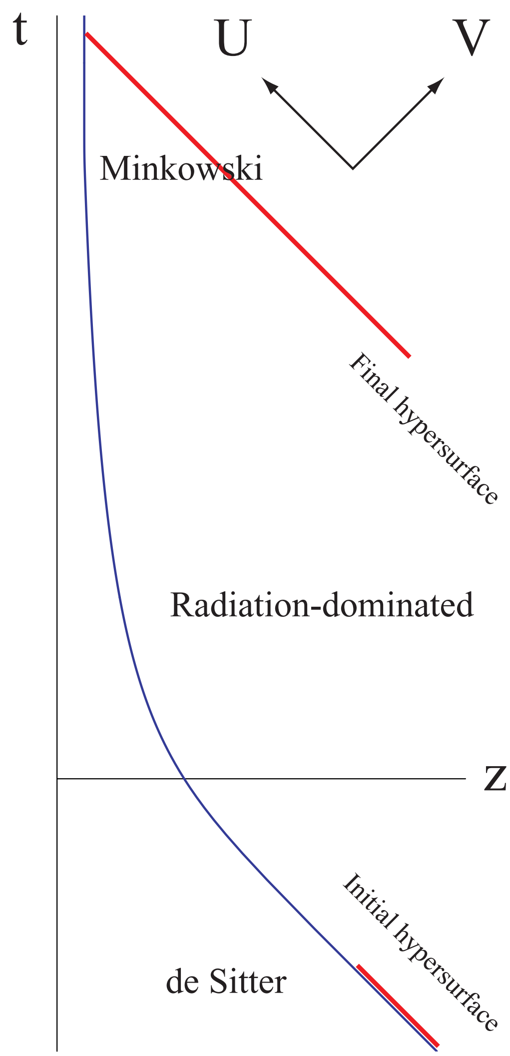

Figure 7.1: Brane trajectory in static coordinates.

7.1.2 Minkowski and de Sitter braneworlds

Let us consider tensor perturbations in AdS spacetime bounded by a brane. We can decompose the graviton field into a zero mode and Kaluza-Klein (KK) modes without ambiguity when the brane is maximally symmetric. This is the reason why the initial and final stages of the background model are given by the de Sitter and Minkowski phases, respectively.

We write the perturbed metric as

$$ds^2 = \frac{\ell^2}{z^2}\left[-dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j + dz^2\right], \tag{7.4}$$

where h_{ij} is the transverse-traceless metric perturbation. We decompose it into the spatial Fourier modes as usual.

For the analysis of perturbations from the Minkowski brane, the above Poincaré coordinate system will be best suited, and the perturbation equation is

$$\left(\frac{\partial^2}{\partial t^2}+k^2-\frac{\partial^2}{\partial z^2}+\frac{3}{z}\frac{\partial}{\partial z}\right)\phi=0, \tag{7.5}$$

subject to the boundary condition

$$\partial_z\phi\big|_{z=\ell}=0. \tag{7.6}$$

The mode solutions of Eq. (7.5) were already found in Chapter 3. Going to quantum theory, the graviton field can be expanded in terms of the zero mode and KK modes as

$$\phi=\hat{A}_0\varphi_0+\hat{A}_0^\dagger\varphi_0^*+\int_0^\infty dm\left(\hat{A}_m\varphi_m+\hat{A}_m^\dagger\varphi_m^*\right), \tag{7.7}$$

where $\hat{A}_n$ and $\hat{A}_n^\dagger$ $(n=0,m)$ are the annihilation and creation operators, respectively, of their corresponding modes. The normalized zero mode function is given by

$$\varphi_0(t)=\frac{1}{\sqrt{2k\ell}}e^{-ikt}, \tag{7.8}$$

while the normalized KK mode function is

$$\varphi_m(t,z)=\frac{1}{\sqrt{2\omega\ell^3}}e^{-i\omega t}u_m(z), \tag{7.9}$$

with

$$u_m(z):=z^2\sqrt{\frac{m}{2}}\frac{Y_1(m\ell)J_2(mz)-J_1(m\ell)Y_2(mz)}{\sqrt{[Y_1(m\ell)]^2+[J_1(m\ell)]^2}}, \tag{7.10}$$

and

$$\omega=\sqrt{k^2+m^2}. \tag{7.11}$$

The normalization here is determined by the Wronskian conditions

$$\begin{aligned}
&(\varphi_0\cdot\varphi_0)=-(\varphi_0^*\cdot\varphi_0^*)=1,\\
&(\varphi_m\cdot\varphi_{m'})=-(\varphi_m^*\cdot\varphi_{m'}^*)=\delta(m-m'),\\
&(\varphi_0\cdot\varphi_m)=(\varphi_0^*\cdot\varphi_m^*)=0,\\
&(\varphi_n\cdot\varphi_{n'}^*)=0,\qquad\text{for}\quad n,n'=0,m,
\end{aligned} \tag{7.12}$$

where the Wronskian is defined by [36]

$$(X\cdot Y):=-2i\int_\ell^\infty dz\left(\frac{\ell}{z}\right)^3\left(X\partial_tY^*-Y^*\partial_tX\right). \tag{7.13}$$

In the de Sitter braneworld we introduce another set of coordinates (η,ξ), which is related to (t,z) as

$$t=\eta\cosh\xi+t_0,\quad z=-\eta\sinh\xi, \tag{7.14}$$

where t_0 is an arbitrary constant. In (η, ξ) frame the de Sitter brane is located at a fixed coordinate position $\xi = \xi_b =$ constant, and the Hubble parameter on the brane is given by $H_i = \ell^{-1} \sinh \xi_b$. The perturbation equation again has a separable form subject to the Neumann boundary condition at the brane. Treating ϕ as an operator, the graviton field can be expanded as

$$\phi = \hat{a}_0 \phi_0 + \hat{a}_0^\dagger \phi_0^* + \int_0^\infty d\nu \left(\hat{a}_\nu \phi_\nu + \hat{a}_\nu^\dagger \phi_\nu^* \right), \tag{7.15}$$

where $\hat{a}_n$ and $\hat{a}_n^\dagger$ $(n = 0, \nu)$ are the annihilation and creation operators of each mode. The explicit form of the normalized mode functions was already given before in Chapter 3.

7.2 Wronskian formulation

Due to the presence of an infinite tower of Kaluza-Klein modes, cosmological perturbations in the braneworld have infinite degrees of freedom. Instead of solving an initial value problem for such a system, it would be better to use the Wronskian formulation in order to take necessary degrees of freedom out of infinite information. In the present case, we would like to know the final amplitude of the zero mode, and therefore in fact what we need to do is solving the (backward) evolution of a single degree of freedom [36, 61, 65]. Following the same line in the previous chapter [65], we shall compute the amplitude of gravitational waves in the final Minkowski phase using the double null coordinates and the Wronskian.

As explained in Section 7.1, our cosmological model is composed of the de Sitter inflationary phase followed by the radiation-dominated epoch, which is connected smoothly to the final Minkowski phase. In the initial de Sitter phase, the graviton field can be expanded as Eq. (7.15), while in the final Minkowski phase it can be expanded as Eq. (7.7). We assume that initially the gravitons are in the de Sitter invariant vacuum state annihilated by $\hat{a}_0$ and $\hat{a}_\nu$,

$$\hat{a}_0 |0\rangle = \hat{a}_\nu |0\rangle = 0. \tag{7.16}$$

The expectation value of the squared amplitude of the zero mode in the final stage is

$$\begin{aligned} \langle 0| \left(\varphi_0 \hat{A}_0 + \varphi_0^* \hat{A}_0^\dagger \right)^2 |0\rangle &= |\varphi_0|^2 \langle 0| \left(1 + 2\hat{A}_0^\dagger \hat{A}_0 \right) |0\rangle + \text{ oscillating part} \\ &\simeq \frac{1}{k\ell} N_f, \end{aligned}$$

where $N_f := \langle 0|\hat{A}_0^\dagger \hat{A}_0|0\rangle$ is the number of created zero mode gravitons. Here we used the commutation relation $\left[\hat{A}_0, \hat{A}_0^\dagger \right] = 1$ and assumed that $N_f \gg 1$. The final power spectrum is then given by

$$\begin{aligned} \mathcal{P}(k) &:= \frac{4\pi k^3}{(2\pi)^3} \frac{2}{(M_5)^3} \cdot \frac{1}{k\ell} N_f \\ &= \frac{k^2}{\pi^2 M_{\rm Pl}^2} N_f. \end{aligned} \tag{7.17}$$

The operator $\hat{A}_0$ can be projected out by making use of the Wronskian relations. Noting that the Wronskian is constant in time, we have

$$\begin{aligned} \hat{A}_0 &= (\phi\cdot\varphi_0)_f = (\phi\cdot\Phi) \\ &= (\phi_0\cdot\Phi)_i\hat{a}_0 + \int d\nu(\phi_\nu\cdot\Phi)_i\hat{a}_\nu + \text{ h.c.}, \end{aligned} \tag{7.18}$$

where Φ is a solution of the Klein-Gordon equation (6.36) whose final configuration is the zero mode function φ_0 in the Minkowski phase, and subscript f and i denote the quantities evaluated on the final and initial hypersurfaces, respectively. Thus we clearly see that final zero mode gravitons are created from the vacuum fluctuations both in the initial zero mode and in the KK modes:

$$N_f = |(\phi_0^*\cdot\Phi)_i|^2 + \int d\nu|(\phi_\nu^*\cdot\Phi)_i|^2. \tag{7.19}$$

Correspondingly, the power spectrum (7.17) can be written as a sum of the two contributions:

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_{\rm KK}, \tag{7.20}$$

where

$$\mathcal{P}_0 := \frac{k^2}{\pi^2 M_{\rm Pl}^2}|(\phi_0^*\cdot\Phi)_i|^2, \tag{7.21}$$

$$\mathcal{P}_{\rm KK} := \frac{k^2}{\pi^2 M_{\rm Pl}^2}\int d\nu|(\phi_\nu^*\cdot\Phi)_i|^2. \tag{7.22}$$

7.3 Spectrum of gravitational waves

De Sitter inflation on the brane predicts the flat primordial spectrum [84]

$$\delta_T^2 := \frac{2C^2(\ell H_i)}{M_{\rm Pl}^2}\left(\frac{H_i}{2\pi}\right)^2. \tag{7.23}$$

During inflation the gravitational wave perturbations are stretched to super-horizon scales, and then they stays constant until horizon reentry, with their amplitude given by Eq. (7.23). (ϕ = constant is a growing solution of Eq. (6.36) in the limit $k^2 \to 0$.) The primordial spectrum of gravitational waves from non-de Sitter inflation is studied extensively in Chapter 6 [65], and so in this chapter we concentrate on the simple case where inflation is given by the exact de Sitter model.

For long wavelength modes with $k \ll k_*$, where

$$k_* := a_* H_* = a_*/\ell \tag{7.24}$$

labels the mode that reenters the horizon when $\ell H = 1$, the amplitude will decay as $h_{\mu\nu} \propto a^{-1}$ after horizon reentry, because gravity on the brane is basically described by four-dimensional general relativity in the low energy regime ($\ell H \ll 1$). In fact, it is explicitly shown that leading order corrections to the cosmological evolution of gravitational waves are suppressed by ℓ^2 and $\ell^2 \ln \ell$ at low energies [139, 63]. In particular, modes with $k < k_0$, where

$$k_0 := a_0 H_0 = H_0, \tag{7.25}$$

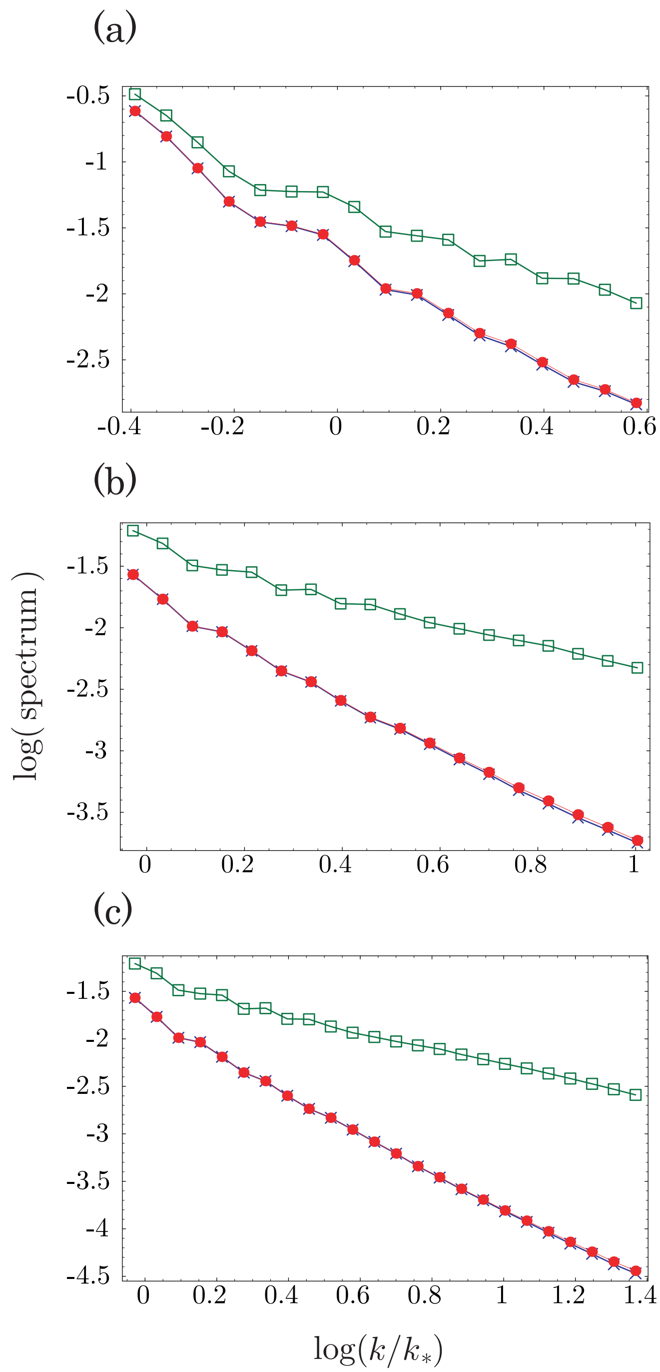

Figure 7.2: Power spectra of gravitational waves from inflation with $\ell H_i = 10$ (a), 42 (b), and 100 (c), normalized by $\delta_T^2/2$. The total power spectrum is shown by red circles, while blue crosses represent the contribution only from the initial zero mode. Green squires indicate results from basically four-dimensional calculations, only including the effect of the modification of the background expansion rate.

and H_0 is the Hubble parameter evaluated at the end of the radiation-dominated phase (i.e., just before the Minkowski phase)[1], reenter the "horizon" in the final Minkowski phase, and then they begin to oscillate (but their amplitude will not decay). As a result, the mean-square vacuum fluctuations of such modes become half of the initial value, leading to

$$\mathcal{P} = \frac{\delta_T^2}{2}, \quad \text{for } k < k_0. \tag{7.26}$$

In Ref. [36], the same spectrum is obtained for scales larger than the AdS and horizon scales by using the "junction model", in which an instantaneous transition from a de Sitter to a Minkowski brane is assumed. For the reason mentioned above, modes with $k_0 < k \ll k_*$ have the standard spectrum,

$$\mathcal{P} = \frac{\delta_T^2}{2}\left(\frac{k}{k_0}\right)^{-2}. \tag{7.27}$$

Something nontrivial may happen to gravitational waves with $k \gtrsim k_*$. If the effects of mode mixing are neglected, only the modification of the background expansion rate alters the spectrum for these short wavelength modes to

$$\tilde{\mathcal{P}} \simeq \frac{\delta_T^2}{2}\left(\frac{k}{k_*}\right)^{-2/3}\left(\frac{k_*}{k_0}\right)^{-2}. \tag{7.28}$$

However, this evaluation will not be correct because mode mixing is expected to be efficient at high energies.

Our procedure to obtain a correct power spectrum is as follows. We solve the perturbation equation (6.36) with its boundary condition set to be $\Phi(U, V_f) = \varphi_0$ on the final hypersurface. The numerical backward evolution scheme we use here is the same as that used in the previous chapter [65], and the detailed description of the scheme is found there. After obtaining the configuration on the initial hypersurface, we evaluate the Wronskian to get the power spectrum.

We performed numerical calculations for three different values of the inflationary Hubble parameter, $\ell H_i = 10$, 42, and 100. The radiation-dominated phase is terminated when the Hubble parameter decreases down to $H \simeq 0.03/\ell =: H_0$, and then it is connected smoothly to the Minkowski phase, so that we can see the amplitude of the well-defined zero mode. The numbers of grids are 90000 in the U-direction and 12000 in the V-direction, and the grid separation is chosen to be $\sim 0.005 \times \ell$. Integration over the KK index ν is performed up to $\nu = 25$ with an equal grid spacing of 0.05.

The power spectra of gravitational waves are shown in Fig. 7.2. Assuming that the power spectrum is of the form

$$\mathcal{P} = A\left(\frac{k}{k_0}\right)^{n}, \tag{7.29}$$

[1] To be more precise, the scale factor in Eq. (7.25) should be replaced by a bit smaller one than $a_0 = 1$, because a_0 is defined as the scale factor in the Minkowski phase (where the Hubble parameter vanishes). However, as the smooth period connecting the radiation and Minkowski phases is taken to be very short, this point is not important.

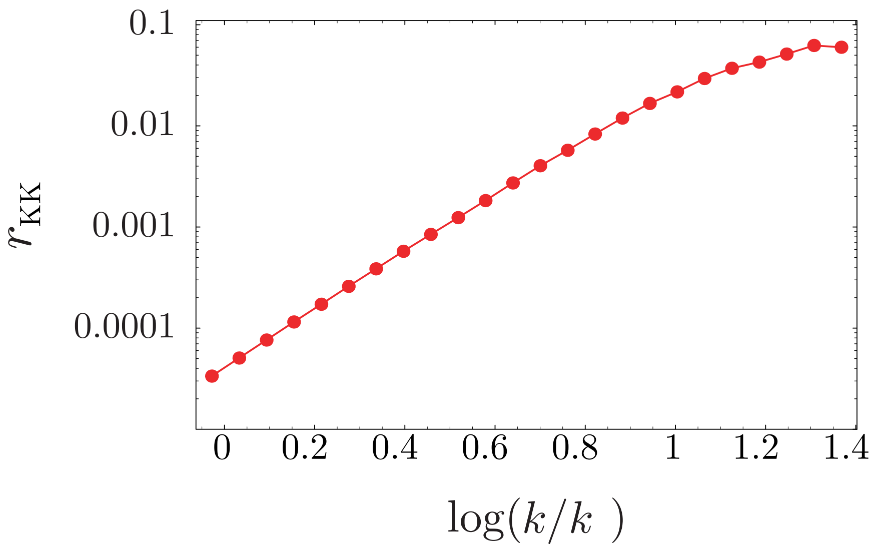

Figure 7.3: Contribution of initial Kaluza-Klein fluctuations, $r_{\rm KK}$, for the model with $\ell H_i = 100$.

we find that irrespective of the inflationary energy scale, the parameters are approximately given by

$$A \simeq \frac{\delta_T^2}{2}, \quad (7.30)$$

$$n \simeq -2. \quad (7.31)$$

Namely, we have the same spectrum as the standard one [Eq. (7.27)] even for short wavelength modes with $k \gtrsim k_*$. As is shown in Fig. 7.3, the contribution of the vacuum fluctuations in the initial KK modes to the final spectrum,

$$r_{\rm KK}(k) := \frac{\mathcal{P}_{\rm KK}}{\mathcal{P}_0 + \mathcal{P}_{\rm KK}}, \quad (7.32)$$

never exceeds 10% so far as the present calculations are concerned, and hence it gives a subdominant effect. On the other hand, the excitation of KK modes suppresses the amplitude of the gravitational waves relative to Eq. (7.28), and our result implies that the effect of the modification of the background Friedmann equation compensate this suppression, leading to approximately the same spectral tilt as that in conventional four-dimensional cosmology. This is consistent with the numerical study by Hiramatsu *et al.* [46], in which they assume the initial configuration in the bulk to be a de Sitter zero mode and obtain $\mathcal{P}_0 \propto k^{-2}$.

Unfortunately, due to the limited number of grids in the U-direction, it is difficult to evaluate accurate values of $r_{\rm KK}$; the convergence is not so good (Fig. 7.4). However, since $r_{\rm KK}$ decreases with an increasing number of grids, it is strongly expected that the effect of the initial KK fluctuations is negligibly small. To obtain a more accurate evaluation of the contribution of the initial KK modes, we need an improved numerical formulation, though it seems quite unlikely that $r_{\rm KK}$ turns to increase at a much larger number of grids[2]. (We confirmed that the convergence of the zero mode part $\mathcal{P}_0$ is sufficiently good.)

[2] After completing the original contribution (hep-th/0511186 [66]), we have improved the numerical code

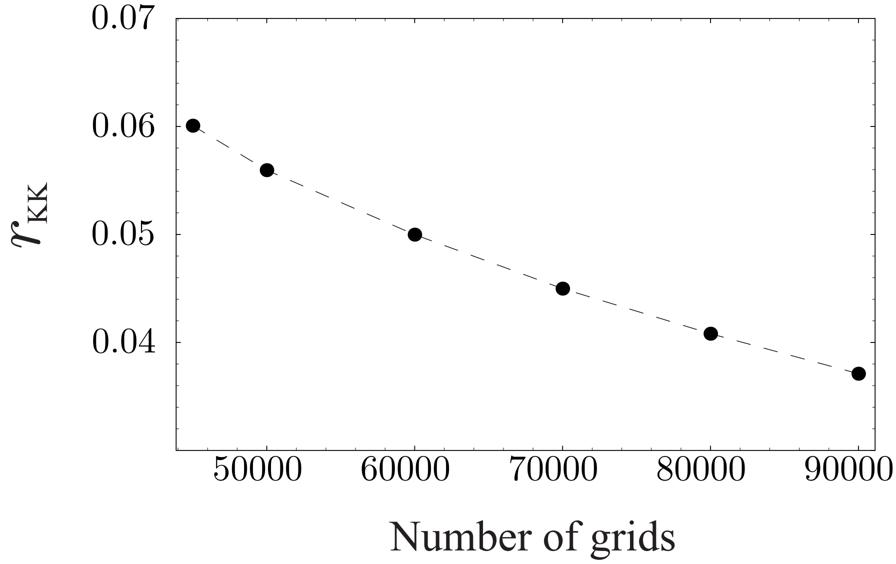

Figure 7.4: $r_{\rm KK}$ for $\ell H_i = 100$ and $\log(k/k_*) = 1.12$, versus the number of grids in the U-direction.

7.4 Generation of dark radiation

So far we have concentrated on final zero mode gravitons created from initial vacuum fluctuations. In this section we shall discuss the generation of KK gravitons. In particular, we are interested in the KK mode gravitons created from initial fluctuations in the zero mode, because from the five-dimensional point of view they are interpreted as gravitons that escape from the brane. At sufficiently late times, all the emitted gravitons fall deep into the bulk, and then the bulk spacetime is described as an AdS-Schwarzschild black hole, the mass of which divided by a^4 is viewed as the "dark radiation" from a brane observer [42, 88, 90, 91].

Before going to the estimation of the energy density of final KK mode gravitons, first let us take a look at the energy density of zero mode gravitons $\rho_{\rm GW}$. Since at low energies gravitational waves evolve in a standard manner, their energy density behaves as $\rho_{\rm GW} \propto a^{-4}$. Therefore, the ratio $\rho_{\rm GW}/\rho_r$ is an invariant quantity in the low energy regime, *irrespective of the cosmic expansion.* Evaluating it at the end of the radiation stage, we have

$$\frac{\rho_{\rm GW}}{\rho_r} = \frac{\rho_{{\rm GW},0}}{\rho_{r,0}} \simeq \ln\left(\frac{k_i}{k_0}\right)\cdot\frac{\delta_T^2}{6} \sim \delta_T^2. \tag{7.33}$$

Note that $\delta_T < 10^{-5}$. In deriving the estimate (7.33), we used the formula

$$\rho_{{\rm GW},0} = M_{\rm Pl}^2 \int_{k_0}^{k_i} d\ln k\; k^2 \mathcal{P}(k), \tag{7.34}$$

and substituted the numerical result obtained in the previous section, $\mathcal{P} \simeq (\delta_T^2/2)(k/k_0)^{-2}$, where k_0 can be eliminated in favor of $\rho_{r,0}$ by using the Friedmann equation at low energies

and obtained more accurate values of $r_{\rm KK}$. The result is not so different from the above. In the case of $\ell H_i = 100$, for example, we have $r_{\rm KK} \simeq 0.022$ for $\log(k/k_*) = 1.12$ and $r_{\rm KK} \simeq 2.4\times10^{-4}$ for $\log(k/k_*) = 0.40$. Thus we safely conclude that the effect of the initial KK fluctuations is negligibly small.

$k_0^2 = H_0^2 \simeq \rho_{r,0}/(3M_{\rm Pl}^2)$. The upper limit of the integral may be given by the inverse horizon scale at the end of inflation,

$$k_i := a_i H_i, \tag{7.35}$$

because the particle production is exponentially suppressed on sub-horizon scales.

Let $\tilde\rho_{\rm GW}$ be the energy density of gravitational waves obtained by neglecting the mode mixing effect. More precisely, $\tilde\rho_{\rm GW}$ is the energy density of gravitational waves $h_{\mu\nu}$ where $h_{\mu\nu}$ is a solution of the conventional perturbation equation $(\partial_\tau^2 + 3H\partial_\tau + k^2/a^2)h_{\mu\nu} = 0$ with the cosmic expansion given by a solution of the *modified* Friedmann equation. This would be much greater than $\rho_{\rm GW}$. Then, $\Delta\rho := \tilde\rho_{\rm GW} - \rho_{\rm GW} \simeq \tilde\rho_{\rm GW}$ is the energy density that leaks from the brane, and by definition $\Delta\rho$ is proportional to a^{-4} as long as it is evaluated in the low energy regime. Thus, $\Delta\rho/\rho_r$ is an invariant quantity. Now $\Delta\rho_{,0}$ can be calculated from the spectrum of the form (7.28), and we have an estimate

$$\frac{\Delta\rho}{\rho_r} = \frac{\Delta\rho_{,0}}{\rho_{r,0}} \simeq \frac{\delta_T^2}{8} \times \ell H_i, \tag{7.36}$$

where we used $a^4\rho_r = a_*^4\sigma = \rho_{r,0}$ and the Friedmann equation at high energies, $H^2 \simeq \rho^2/6M_{\rm Pl}^2\sigma$. The estimate (7.36) implies that a large amount of energy (compared to $\rho_{\rm GW}$) is lost from the brane. Is the escaped energy $\Delta\rho$ directly transferred to the final bulk gravitons? To discuss this point, we compare it with the energy density of the generated dark radiation.

Since KK modes are excited dominantly at high energies but not at low energies, the dark radiation, as is deduced from its name, behaves like a radiation component, $\rho_{\rm DR} \propto a^{-4}$, at late times. Hence, we shall see the ratio $\rho_{\rm DR}/\rho_r$ but it may be evaluated at the end of the radiation stage.

The total number of the created bulk gravitons is given again by the Wronskian as

$$\int dm \langle 0|\hat A_m^\dagger \hat A_m|0\rangle = \int dm \left[|(\phi_0^*\cdot\Phi_m)_i|^2 + \int d\nu |(\phi_\nu^*\cdot\Phi_m)_i|^2\right],$$

where Φ_m is a solution of the Klein-Gordon equation (6.36) whose final configuration is a KK mode function φ_m in the Minkowski phase. Concentrating on the first part, $|(\phi_0^*\cdot\Phi_m)|^2 dm$ is identified as the number of KK gravitons coming from the initial zero mode fluctuations, with their mass between m and $m + dm$. Thus the energy density is expressed as

$$\begin{aligned}\rho_{\rm DR,0} &:= \int \frac{d^3k}{(2\pi)^3}\int dm\, \omega |(\phi_0^*\cdot\Phi_m)_i|^2 \\ &= \int d\ln k \int d\ln m\, \frac{mk^3\omega}{2\pi^2}|(\phi_0^*\cdot\Phi_m)_i|^2. \end{aligned}\tag{7.37}$$

As a side remark, in the junction model of Ref. [36], the second part, i.e., the number of KK gravitons created from the initial fluctuations in KK modes, is suppressed compared to the first one.

We numerically calculated the Wronskian $|(\phi_0^*\cdot\Phi_m)|^2$ in a similar manner to the previous section. The Hubble parameter during inflation is chosen to be $\ell H_i = 42$. In this case, the number of grids are 150000 in the U-direction and 20000 in the V-direction, and the grid separation is about $0.0036 \times \ell$.

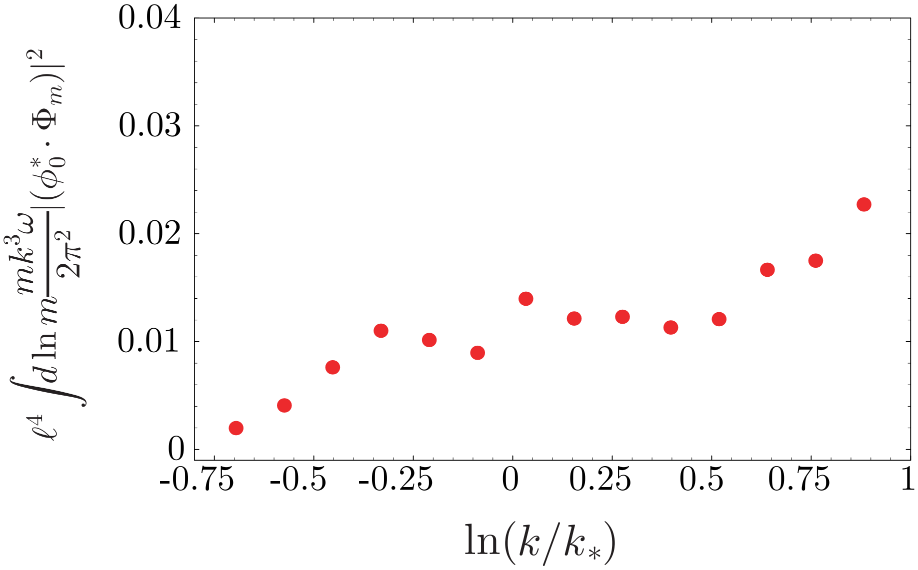

Figure 7.5: Energy of bulk gravitons multiplied by the phase space factor.

The integrand

$$\int d\ln m \; \frac{mk^3\omega}{2\pi^2}|(\phi_0^* \cdot \Phi_m)_i|^2$$

is plotted in Fig. 7.5. For each k, integration over m is performed up to $m \sim 2/\ell$ with a grid spacing of $\Delta \ln m \simeq 0.17$. Performing the integration over k, we obtain

$$\rho_{\mathrm{DR},0} \approx 0.04 \times \ell^{-4}, \tag{7.38}$$

where one should note that contributions from modes with $k < k_*$ are suppressed. The radiation energy density can be written as

$$\rho_{r,0} = a_i^4 \rho_{r,i} \simeq a_i^4 (6M_{\mathrm{Pl}}^2 \sigma)^{1/2} H_i \simeq \frac{9}{2\pi^2}\frac{1}{\delta_T^2}(a_i H_i)^4, \tag{7.39}$$

which can be obtained by using the modified Friedmann equation and noting that $\delta_T^2 \simeq 3\ell H_i^3/4\pi^2 M_{\mathrm{Pl}}^2$ for $\ell H_i \gg 1$. From Eqs. (7.38) and (7.39) we have an estimate

$$\frac{\rho_{\mathrm{DR}}}{\rho_r} < \mathcal{O}(1) \times \delta_T^2. \tag{7.40}$$

This result indicates that the energy density of the generated dark radiation is not larger than that of zero mode gravitons [Eq. (7.33)]. Of course, this is a completely harmless amount of an extra radiation component [48]. The scattering of particles on the brane in the early universe, discussed in [42, 88, 90, 91], can be a more efficient way to produce bulk gravitons.

Although a large amount of energy is lost from the brane via excitation of KK modes in the high energy regime, the final energy density of the dark radiation is much smaller than that, without an enhancement factor like ℓH_i in Eq. (7.36). This discrepancy is explained as follows [42, 90]. In the high energy regime, the motion of the brane is so relativistic (in the frame defined by the static bulk coordinates) that emitted gravitons run almost parallel to the brane trajectory. These gravitons stay in the vicinity of the brane and bounce off it many times during the high energy stage, until eventually they are reflected by the non-relativistic brane to fall off into the bulk. During this process, the gravitons lose a large portion of their momentum transverse to the brane because they repeatedly hit the retreating brane. This qualitatively accounts for the smallness of the final energy density of the dark radiation. To justify the above interpretation quantitatively, a more rigorous analysis will be needed in the direction of Refs. [90, 109], which includes calculating the pressure to the brane due to the effective energy-momentum tensor of the bulk gravitational waves.

Here we should comment on the result of the junction model obtained by Gorbunov *et al.* [36]. In terms of the power spectrum, their result is summarized as

$$\mathcal{P} \approx \begin{cases} \dfrac{\delta_T^2}{2}, & (k \ll k_*), \\ \dfrac{\delta_T^2}{2}\dfrac{4}{(k/k_*)^2}, & (k_* \ll k \ll k_i), \end{cases} \tag{7.41}$$

where $k_* = a_*/\ell$, $k_i = a_i H_i$, and $a_* = a_i (= 1)$ because a de Sitter inflationary stage is directly joined to a Minkowski phase in the junction model (see also Appendix A of Ref. [61]). From this we can estimate the energy density that leaks from the brane as

$$\Delta\rho \approx M_{\rm Pl}^2 H_i^2 \delta_T^2 \approx \rho_{e.i.}\delta_T^2 \times \ell H_i, \tag{7.42}$$

where $\rho_{e.i.}$ is the energy density at the "end of inflation". On the other hand, according to Appendix D of Ref. [36], the energy density of created KK gravitons is given by

$$\rho_{\rm DR} \approx H_i^4 \approx \rho_{e.i.}\delta_T^2. \tag{7.43}$$

Thus, we find that $\Delta\rho \sim \rho_{\rm DR} \times \ell H_i$, which is consistent with our present result. [Note that in the junction model the energy density of final zero mode gravitons is estimated as $\rho_{\rm GW} \sim (M_{\rm Pl}^2/\ell^2)\delta_T^2$ and hence $\rho_{\rm GW} \ll \rho_{e.i.}\delta_T^2 \sim \rho_{\rm DR}$.]

7.5 Summary

We have examined the power spectrum of the gravitational wave background in the cosmological scenario of the Randall-Sundrum braneworld. There are three possible ingredients which may lead the power spectrum to a non-standard one: the unconventional background expansion rate due to the ρ^2 term in the Friedmann equation, the excitation of KK modes during the radiation-dominated stage at high energies, and the effect of initial vacuum fluctuations in KK modes. Previous estimates are based on a rather simple toy model [36] or numerical studies about the classical evolution of perturbations, neglecting the initial KK fluctuations [45, 46]. In the present analysis, initial conditions are set in a quantum-mechanical manner and hence the effect of the initial KK fluctuations is included. Along the same line in Ref. [65], we make use of the Wronskian formulation to obtain the final amplitude of the zero mode gravitational waves numerically. We have found that the effect of initial KK vacuum fluctuations are subdominant: $r_{\rm KK} < 0.1$. Our result confirms that the damping of the amplitude due to the KK mode excitation and the enhancement due to the modification of the background expansion rate mainly work, but almost cancel each other. Consequently, the power spectrum is basically the same as the standard one obtained in conventional four-dimensional cosmology. We believe that the cancellation between the two effects is a phenomenon peculiar to the radiation-dominated phase. To make the particularity of the radiation stage clear, it would be interesting to investigate consequences of a different equation of state parameter $w(= p/\rho)$ after the inflationary stage. This is the next issue we plan to report in a future publication.

We have also estimated the energy density of the generated dark radiation numerically, and shown that only a tiny amount is generated. It is smaller than the energy density of zero mode gravitational waves.

Chapter 8

Cosmological perturbations in the bulk inflaton model

So far we have elaborated on cosmological tensor perturbations in the Randall-Sundrum braneworld scenario [123, 124], where the bulk is empty except for a negative cosmological constant. In this chapter, we shall consider a different class of a braneworld model so called the bulk inflaton model. A conservative construction of inflation models in the context of the RS model assumes that slow-roll inflation is driven by a scalar field confined to the brane, and such a model is considered, e.g., in Ref. [100]. An empty bulk, however, seems less likely from the point of view of unified theories, which often require various fields in addition to gravity. Considering a bulk scalar field, Himemoto *et al.* [43, 44, 130, 108] have shown that, interestingly, a bulk scalar field can mimic the standard slow-roll inflation on the brane under a certain condition (see also Ref. [60]).

Also in the context of heterotic M theory, cosmological solutions has been studied [97, 125, 98, 133]. In the model discussed in Refs. [97, 125], the scalar field has an exponential potential in the bulk and the tensions of the two branes are also exponential functions of the scalar field. In this model the power-law expansion (but not inflation) is realized on the brane. A single-brane model with such exponential-type potentials is also interesting, and it has been investigated for a static brane case [54, 23, 13] and a dynamical (cosmological) case [118, 29, 87, 19]. Recently an inflationary solution was found in a similar setup by Koyama and Takahashi [75, 77], extending the results of Refs. [23, 13, 118, 29]. A striking feature of their model is that cosmological perturbations can be solved analytically.

As we have stressed through this thesis, for the purpose of giving a prediction in braneworld models it is essential to take into account perturbations in the bulk. To do so, generally we cannot avoid solving partial differential equations in the bulk with discouragingly complicated boundary conditions. Only a few cases are known where perturbation equations can be analytically solved [84, 36, 61]. One of them is the special class of bulk inflaton models mentioned above [75, 77]. In this chapter we clarify the reason why the perturbation equations are soluble in this special case. Based on this notion, we present a new systematic method to find a wider class of background cosmological solutions and to analyze perturbations from them. *The essence as to why our method works is deeply connected to the fact that tensor perturbations in the vacuum RS braneworld with a de Sitter brane are exactly solvable.*

This chapter is organized as follows. In the next section, we explain our basic ideas of constructing background solutions and of analyzing cosmological perturbations. In Sec. 8.2

we consider a model with a single scalar field in the bulk, which is the main interest of this work, and derive an effective theory on the brane. Then, in Sec. 8.3 we present some examples of exact solutions for the background cosmology obtained by making use of the ideas explained in Sec. 8.1. Section 8.4 deals with cosmological perturbations. Section 8.5 is devoted to discussion.

8.1 Basic ideas

8.1.1 Solutions in the Randall-Sundrum vacuum braneworld

We begin with a model whose action is given by

$$S = S_g + S_b, \tag{8.1}$$

where

$$S_g = \frac{1}{2\kappa_{D+1}^2} \int d^{D+1}X\sqrt{-G}\left(R[G] - 2\Lambda_{D+1}\right), \tag{8.2}$$

is the action of $(D+1)$-dimensional Einstein gravity with a negative cosmological constant $\Lambda_{D+1} = -D(D-1)/2\ell^2$,

$$S_b = -\int d^D X\sqrt{-g}\ \sigma, \tag{8.3}$$

is the action of a vacuum brane with a tension σ, and g is the determinant of the induced metric on the brane.

We assume Z_2-symmetry across the brane, so that the tension of the brane is determined by the junction condition as

$$H_0^2 = \frac{\kappa_{D+1}^4\sigma^2}{4(D-1)^2} - \frac{1}{\ell^2}, \tag{8.4}$$

where H_0 is related to the D-dimensional cosmological constant induced on the brane Λ_b by

$$\Lambda_b = \frac{1}{2}(D-1)(D-2)H_0^2, \tag{8.5}$$

and it represents the deviation of σ from the fine-tuned Randall-Sundrum value $2(D-1)/\ell\kappa_{D+1}^2$.

One of the key ideas in the present chapter is to make use of the following well-known fact. If a metric $ds_{(D)}^2$ is a solution of the D-dimensional vacuum Einstein equations with a cosmological constant Λ_b, then

$$ds_{(D+1)}^2 = e^{2\omega(z)}\left(dz^2 + ds_{(D)}^2\right), \tag{8.6}$$

is a solution of the $(D+1)$-dimensional model defined by Eq. (8.1), and the warp factor $e^{\omega(z)}$ is given by

$$e^{\omega(z)} = \frac{\ell H_0}{\sinh(H_0 z)}. \tag{8.7}$$

(For a Ricci-flat brane, we have $\Lambda_b = 0$. In this case the warp factor reduces to $e^{\omega(z)} = \ell/z$.) Namely, we can construct a $(D+1)$-dimensional solution in the Randall-Sundrum braneworld from a vacuum solution of the D-dimensional Einstein equations. A well-known example is the five-dimensional black string solution obtained from the four-dimensional Schwarzschild solution [18].

8.1.2 Bulk inflaton models from dimensional reduction

We now explain how to obtain an $(n+2)$-dimensional braneworld model with bulk scalar fields from $(n+2+\sum j_i)[= D+1]$-dimensional spacetime by dimensional reduction. We use n to represent the number of uncompactified spatial dimensions on the brane, which is three in realistic models. Let us consider $(n+2+\sum j_i)$-dimensional spacetime whose metric is given by

$$ds^2_{(D+1)} = G_{AB}dX^A dX^B = \mathcal{G}_{ab}(x)dx^a dx^b + \sum e^{2\phi_i(x)} d\sigma_i^2, \tag{8.8}$$

where $d\sigma_i^2$ is the line element of a j_i-dimensional constant curvature space with the volume $\mathcal{V}_i$. Here the indices a and b run from 0 to $(n+1)$, and ϕ_i is assumed to depend only on the $(n+2)$-dimensional coordinates x^a.

Then dimensional reduction to $(n+2)$ dimensions yields

$$\begin{aligned} S_g^{(n+2)} &= \frac{1}{2\kappa_{n+2}^2}\int d^{n+2}x\sqrt{-\mathcal{G}}\; e^Q\Bigg[R[\mathcal{G}] - \sum j_i \mathcal{G}^{ab}\partial_a\phi_i\partial_b\phi_i + \mathcal{G}^{ab}\partial_a Q\partial_b Q \\ &\qquad -2\Lambda_{D+1} + \sum K_i j_i(j_i-1)e^{-2\phi_i}\Bigg], \end{aligned} \tag{8.9}$$

where

$$Q := \sum j_i\phi_i, \tag{8.10}$$

$\kappa_{n+2}^2 := \kappa_{D+1}^2/\prod \mathcal{V}_i$ and K_i represents the signature of the curvature of the metric $d\sigma_i^2$: -1 (open), 0 (flat), or 1 (closed). Making a conformal transformation to the "Einstein frame,"

$$\tilde{\mathcal{G}}_{ab} = e^{2Q/n}\mathcal{G}_{ab}, \tag{8.11}$$

we have

$$\begin{aligned} S_g^{(n+2)} &= \frac{1}{2\kappa_{n+2}^2}\int d^{n+2}x\sqrt{-\tilde{\mathcal{G}}}\Bigg[R[\tilde{\mathcal{G}}] - \sum j_i\tilde{\mathcal{G}}^{ab}\partial_a\phi_i\partial_b\phi_i - \frac{1}{n}\tilde{\mathcal{G}}^{ab}\partial_a Q\partial_b Q \\ &\qquad -2\Lambda_{D+1}e^{-2Q/n} + \sum K_i j_i(j_i-1)e^{-2\phi_i-2Q/n}\Bigg]. \end{aligned} \tag{8.12}$$

Notice that the kinetic term of each scalar field has an appropriate signature since $j_i > 0$. A parallel calculation gives

$$S_b^{(n+2)} = -\int d^{n+1}x\sqrt{-q}\;\tilde{\sigma}e^{-Q/n}, \tag{8.13}$$

where $\tilde{\sigma} = \sigma\prod\mathcal{V}_i$, so that $\kappa_{n+2}^2\tilde{\sigma} = \kappa_{D+1}^2\sigma$, and q is the determinant of the induced metric on the brane, $q_{\hat{a}\hat{b}} := \tilde{\mathcal{G}}_{\hat{a}\hat{b}}|_{z=z_b} = e^{2Q/n}\mathcal{G}_{\hat{a}\hat{b}}|_{z=z_b}$. The caret upon indices represents restriction to the subspace parallel to the brane. Hence $\hat{a}$ and $\hat{b}$ run from 0 to n. Here z_b represents the location of the brane. In this manner, we can derive $(n+2)$-dimensional braneworld models with bulk scalar fields which have exponential-type potentials both in the bulk and on the brane.

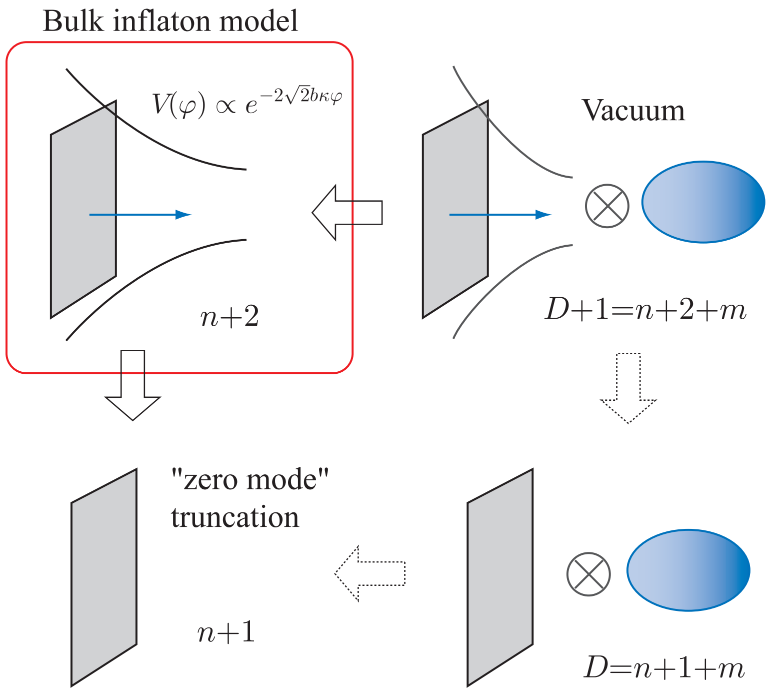

Figure 8.1: Schematic of the higher dimensional vacuum description of an $(n+2)$-dimensional bulk inflaton model and the derivation of the $(n+1)$-dimensional effective theory on the brane. The top left picture represents the bulk inflaton model that we are interested in. To analyze cosmological background solutions (and "zero mode" perturbations), we use the $(n+1)$-dimensional description shown in the bottom left corner. On the other hand, we make use of the $(D+1)$-dimensional description presented in the top right corner to simplify the perturbation analysis.

8.2 Single scalar field in the bulk

From now on, for simplicity, we focus on models with a single bulk scalar field. The m-dimensional space represented by $e^{2\phi}d\sigma^2$ is compactified on either a torus ($K=0$), a sphere ($K=1$), or a compact hyperboloid ($K=-1$). Using a canonically normalized field $\varphi := \kappa_{n+2}^{-1}\sqrt{m(m+n)/n}\;\phi$, the $(n+2)$-dimensional reduced action is written as

$$S^{(n+2)} = \int d^{n+2}x\sqrt{-\tilde{\mathcal{G}}}\left\{\frac{1}{2\kappa_{n+2}^2}R[\tilde{\mathcal{G}}] - \frac{1}{2}\tilde{\mathcal{G}}^{ab}\partial_a\varphi\partial_b\varphi - V(\varphi)\right\} - \int d^{n+1}x\sqrt{-q}\;U(\varphi), \tag{8.14}$$

where the potentials are

$$V(\varphi) = -\frac{(m+n)(m+n+1)}{2\kappa_{n+2}^2\ell^2}e^{-2\sqrt{2}b\kappa_{n+2}\varphi} - \frac{Km(m-1)}{2\kappa_{n+2}^2}e^{-\sqrt{2}\kappa_{n+2}\varphi/nb}, \tag{8.15}$$

$$U(\varphi) = \tilde{\sigma}e^{-\sqrt{2}b\kappa_{n+2}\varphi}, \tag{8.16}$$

with

$$b := \sqrt{\frac{m}{2n(m+n)}}. \tag{8.17}$$

If we assume that the metric $ds^2_{(D+1)}$ is given in the form of (8.6), the action can further reduced to the $(n+1)$-dimensional effective one on the brane. We write the D-dimensional part of the metric in the form of

$$ds^2_{(D)} = g_{\hat{A}\hat{B}} dx^{\hat{A}} dx^{\hat{B}} = g^{(n+1)}_{\hat{a}\hat{b}}(x) dx^{\hat{a}} dx^{\hat{b}} + e^{2\alpha(t)} d\sigma^2. \tag{8.18}$$

Then, comparing the coefficient in front of $d\sigma^2$, it follows that

$$\phi(t,z) := \alpha(t) + \omega(z). \tag{8.19}$$

Also, the $(n+2)$-dimensional metric $\tilde{\mathcal{G}}_{ab}$ is written as

$$\tilde{\mathcal{G}}_{ab} dx^a dx^b = e^{2m\phi/n} \mathcal{G}_{ab} dx^a dx^b = e^{2(m+n)\omega/n} \left(q_{\hat{a}\hat{b}} dx^{\hat{a}} dx^{\hat{b}} + e^{2m\alpha/n} dz^2 \right), \tag{8.20}$$

where we have set $e^{\omega(z_b)} = 1$. Substituting the above expression into (8.14), we perform the integration over z to obtain

$$\begin{aligned} S^{(n+1)} = \frac{1}{2\kappa^2_{n+1}} \int d^{n+1}x \sqrt{-q} \Big\{ & e^{m\alpha/n} R[q] - \frac{m(m+n)}{n} e^{m\alpha/n} q^{\hat{a}\hat{b}} \partial_{\hat{a}}\alpha \partial_{\hat{b}}\alpha \\ & -2\Lambda_b e^{-m\alpha/n} + Km(m-1) e^{-2\alpha - m\alpha/n} \Big\}, \end{aligned} \tag{8.21}$$

where $\kappa^2_{n+1} = \kappa^2_D / \mathcal{V}$ with

$$\kappa^2_D := \kappa^2_{D+1} \left[2 \int_{z_b}^{\infty} e^{(m+n)\omega} dz \right]^{-1},$$

and $\mathcal{V}$ is the volume of the m-dimensional compactified space.

The above reduction to $(n+1)$ dimensions can be done more easily, starting with the $(D+1)[= n+2+m]$-dimensional action. First we perform the integration over z and obtain a D-dimensional effective action,

$$S^{(D)} = \frac{1}{2\kappa^2_D} \int d^D x \sqrt{-g} (R[g] - 2\Lambda_b), \tag{8.22}$$

where Λ_b is, as before, the one defined by Eq. (8.5). The obtained effective action is that for D-dimensional pure gravity with a cosmological constant Λ_b. Compactifying m dimensions further, and taking into account

$$q_{\hat{a}\hat{b}} = e^{2m\alpha/n} g^{(n+1)}_{\hat{a}\hat{b}}, \tag{8.23}$$

the same expression for the $(n+1)$-dimensional effective action (8.21) can be recovered in a parallel way as we did for the reduction from $(D+1)$ dimensions to $(n+2)$ dimensions. Using $\varphi_{BD} := e^{m\alpha/n}$ and $\omega_{BD} := \sqrt{n(m+n)/m}$, we can rewrite the action in a familiar form:

$$\begin{aligned} S^{(n+1)} = \frac{1}{2\kappa^2_{n+1}} \int d^{n+1}x \sqrt{-q} \Big\{ & \varphi_{BD} R - \frac{\omega_{BD}}{\varphi_{BD}} (\partial \varphi_{BD})^2 \\ & -2\Lambda_b \varphi_{BD}^{-1} + Km(m-1) \varphi_{BD}^{-(m+2n)/m} \Big\}, \end{aligned} \tag{8.24}$$

which implies that the effective theory on the brane is described by a scalar-tensor theory. The action (8.21), or equivalently (8.24), describes not only the background unperturbed cosmology but also the zero mode perturbation, both of which are independent of the extra-dimensional coordinate, z, apart from the overall factor $e^{2\omega}$.

The induced metric on the brane q_{ab} is the metric in the Jordan frame. If we use the metric in the Einstein frame,

$$\tilde{q}_{\hat{a}\hat{b}} = e^{\frac{2m}{n(n-1)}\alpha} q_{\hat{a}\hat{b}}, \tag{8.25}$$

the effective action becomes

$$S^{(n+1)} = \int d^{n+1}x\sqrt{-\tilde{q}}\left\{\frac{1}{2\kappa_{n+1}^2}R[\tilde{q}] - \frac{1}{2}\tilde{q}^{\hat{a}\hat{b}}\partial_{\hat{a}}\tilde{\varphi}\partial_{\hat{b}}\tilde{\varphi} - \tilde{V}(\tilde{\varphi})\right\}, \tag{8.26}$$

where $\tilde{\varphi} := \kappa_{n+1}^{-1}\sqrt{m(m+n)/n}\ \alpha$ and the potential is

$$\kappa_{n+1}^2\tilde{V}(\tilde{\varphi}) = \Lambda_b e^{-\frac{2\sqrt{2}n}{n-1}b\kappa_{n+1}\tilde{\varphi}} - \frac{Km(m-1)}{2}e^{-\frac{2\sqrt{2}n(m+n-1)}{m(n-1)}b\kappa_{n+1}\tilde{\varphi}}. \tag{8.27}$$

Obviously, the system defined by the above action is equivalent to Einstein gravity with a scalar field. A discussion of this type of potential can be found in Ref. [117].

8.3 Examples of the background spacetime

In this section, we give some examples of D-dimensional vacuum solutions, which generate $(D+1)$-dimensional braneworld solutions by making use of the prescription described in Sec. 8.1.1. Here we discuss models with a single scalar field and investigate their cosmological evolution in detail. Generalization to the case of multiple scalar fields is given in Sec. 8.7.

8.3.1 Kasner-type solutions

We first consider the following Kasner-type solution as an example of the Ricci-flat case, $\Lambda_b = 0$,

$$g_{\hat{A}\hat{B}}dx^{\hat{A}}dx^{\hat{B}} = e^{2\alpha(\eta)}\left[-d\eta^2 + \gamma_{\mu\nu}dy^\mu dy^\nu\right] + e^{2\beta(\eta)}\delta_{ij}dx^i dx^j,$$

where $\gamma_{\mu\nu}$ is the metric of m-dimensional constant curvature space, and i and j run from 1 to n. Under the assumption of the above metric form, we solve the vacuum Einstein equations,

$$e^{2\alpha}R_\eta^{\ \eta} = n\left[\beta'^2 - \alpha'\beta' + \beta''\right] + m\alpha'' = 0, \tag{8.28}$$

$$e^{2\alpha}R_\mu^{\ \nu} = \delta_\mu^{\ \nu}\left[K(m-1) + n\alpha'\beta' + (m-1)\alpha'^2 + \alpha''\right] = 0, \tag{8.29}$$

$$e^{2\alpha}R_i^{\ j} = \delta_i^{\ j}\left[n\beta'^2 + (m-1)\alpha'\beta' + \beta''\right] = 0, \tag{8.30}$$

where the prime denotes differentiation with respect to η. Eliminating α'' and α' from the above three equations, we have

$$\beta'' = \mp\sqrt{\frac{n(m+n-1)}{m}\beta'^4 - K(m-1)^2\beta'^2}.$$

We can easily integrate this equation. For example, when $K = 1$ (compactified on the m-sphere S^m), the solution of this equation becomes

$$n\beta' = \frac{\pm(m-1)q}{\sin[(m-1)\eta]},$$

where

$$q := \sqrt{\frac{mn}{m+n-1}}, \tag{8.31}$$

and the integration constant was used to shift the origin of time. Thus we find

$$e^{\beta} = \left[\tan\left(\frac{m-1}{2}\eta\right)\right]^{\pm q/n}.$$

Substituting this result into Eq. (8.30), we have

$$\alpha' = \cot[(m-1)\eta] \mp \frac{q}{\sin[(m-1)\eta]}.$$

This can be integrated as

$$e^{(m-1)\alpha} = \sin[(m-1)\eta]\left[\cot\left(\frac{m-1}{2}\eta\right)\right]^{\pm q}.$$

The solution for $K = -1$ is easily obtained by replacing sin, tan, and cot in the above expressions by sinh, tanh, and coth, respectively. The solution for $K = 0$ behaves as $e^{\beta} \propto \eta^{\pm q/n}$ and $e^{(m-1)\alpha} \propto \eta^{1\mp q}$.

Let us further investigate cosmology of the above example. Setting $n = 3$ and $K = 1$, the induced four-dimensional metric becomes

$$q_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = e^{2m\alpha/3}\left[-e^{2\alpha}d\eta^2 + e^{2\beta}\delta_{ij}dx^i dx^j\right], \tag{8.32}$$

where

$$\begin{aligned} e^{\alpha} &= \{\sin[(m-1)\eta]\}^{\frac{1}{m-1}}\{\cot[(m-1)\eta/2]\}^{\frac{q}{m-1}}, \\ e^{\beta} &= \{\tan[(m-1)\eta/2]\}^{q/3}, \end{aligned} \tag{8.33}$$

and $q = \sqrt{3m/(m+2)}$ [1]. We set q to be positive without any loss of generality since the signature of q is flipped by a shift of the origin of time, $\eta \to \eta+\pi$. Here one remark is in order. In the original $(D+1)$-dimensional model m represents the number of compactified dimensions and therefore is supposed to be an integer. However, m is just a number parameterizing the form of the scalar field potential when we start with the action (8.14) obtained after dimensional reduction. We therefore find that m can be any real positive number in this

[1] We should remark that the dynamical solutions in Ref. [29] can be obtained if we compactify the n-dimensional section $e^{2\beta}\delta_{ij}dx^i dx^j$ and regard the m-dimensional section $e^{2\alpha}\gamma_{\mu\nu}dy^{\mu}dy^{\nu}$ as our three-space instead. The flat case ($K = 0$) corresponds to the solution in Ref. [118]. We should also mention that recently there has been a lot of discussion about the solutions with $K = -1$ in an attempt to explain accelerated expansion of the universe in the context of M or string theory [117, 140, 119, 120, 128, 28, 20, 40, 21]. Note, however, that these arguments are not in the braneworld context.

context. The positivity needs to be assumed to keep the appropriate signature of the kinetic term for the scalar field ϕ, or equivalently to keep the relation between ϕ and φ real. (Strictly speaking, the case with $m < -n = -3$ is also allowed.) Then q can be regarded as a continuous parameter with its range $0 < q < \sqrt{3}$.

The cosmological time τ is related to η via

$$d\tau = e^{(m+3)\alpha/3} d\eta. \tag{8.34}$$

Recall that $q_{\hat{a}\hat{b}}$, the metric induced on the brane in the five[$= n+2$]-dimensional model (8.14), is related to $g_{\hat{a}\hat{b}}$ by Eq. (8.23). Hence the scale factor associated with the metric $q_{\hat{a}\hat{b}}$ is given by $a = e^{m\alpha/3+\beta}$, and therefore the Hubble parameter on the brane $H := a^{-1}da/d\tau$ is given by $H = (m\alpha'/3 + \beta')e^{-m\alpha/3-\alpha}$. Substituting the above solution (8.33) into these expressions, we obtain

$$\begin{aligned} a &= \{\sin[(m-1)\eta]\}^{\frac{2q^2}{9(q^2-1)}} \{\tan[(m-1)\eta/2]\}^{\frac{q(q^2-3)}{9(q^2-1)}}, \\ H &= \frac{q\left(q^2-3+2q\cos[(m-1)\eta]\right)}{3(q^2-3)} \{\sin[(m-1)\eta]\}^{-\frac{8q^2}{9(q^2-1)}} \{\tan[(m-1)\eta/2]\}^{-\frac{q(q^2-9)}{9(q^2-1)}}. \end{aligned}$$

The relation between the coordinate time η and the cosmological time τ (8.34) is not so obvious, but the asymptotic behavior can be easily studied. When $\eta \to 0$, we have $\tau \propto \eta^{\frac{q(q+9)}{9(q+1)}} \to 0$ and

$$\begin{aligned} a &\propto \eta^{\frac{q(q+3)}{9(q+1)}} \propto \tau^{p_-}, \\ H &\propto \eta^{-\frac{q(q+9)}{9(q+1)}} \propto a^{-\frac{q+9}{q+3}}, \\ e^{\alpha} &\propto \eta^{-\frac{3-q^2}{3(q+1)}} \to +\infty, \end{aligned}$$

where the exponent p_- is defined below in Eq. (8.35). For $\bar{\eta} := \eta - \pi/(m-1) \to 0$, the cosmological time is (locally) expressed as $\tau \propto \bar{\eta}^{-\frac{q(q-9)}{9(q-1)}}$. Therefore the range of the proper time τ is infinite for the parameter region $0 < q < 1$, while it is finite for $1 < q < \sqrt{3}$. In this limit $\bar{\eta} \to 0$, the scale factor, the Hubble parameter, and the scalar field behave as

$$\begin{aligned} a &\propto \bar{\eta}^{-\frac{q(q-3)}{9(q-1)}} \propto \begin{cases} \tau^{p_+}, & (0 < q < 1), \\ (\tau_{end} - \tau)^{p_+}, & (1 < q < \sqrt{3}), \end{cases} \\ H &\propto \bar{\eta}^{\frac{q(q-9)}{9(q-1)}} \propto a^{-\frac{q-9}{q-3}}, \\ e^{\alpha} &\propto \bar{\eta}^{-\frac{3-q^2}{3(1-q)}} \to \begin{cases} +\infty, & (0 < q < 1), \\ 0, & (1 < q < \sqrt{3}). \end{cases} \end{aligned}$$

In the above expressions, we have used

$$p_{\pm} := \frac{m+3}{4m+9 \pm \sqrt{3m(m+2)}}. \tag{8.35}$$

The ranges of p_+ and p_- are $1/(4+\sqrt{3}) < p_+ < 1/3$ and $1/3 < p_- < 1/(4-\sqrt{3})$.

The behavior of this solution is easily understood from the viewpoint of the four-dimensional effective theory described by the action (8.26), as was discussed in Ref. [28]. The potential (8.27) with $\Lambda_b = 0$ and $K = 1$ is shown in Fig. 8.2. For $0 < q < 1$ $(0 < m < 1)$ the

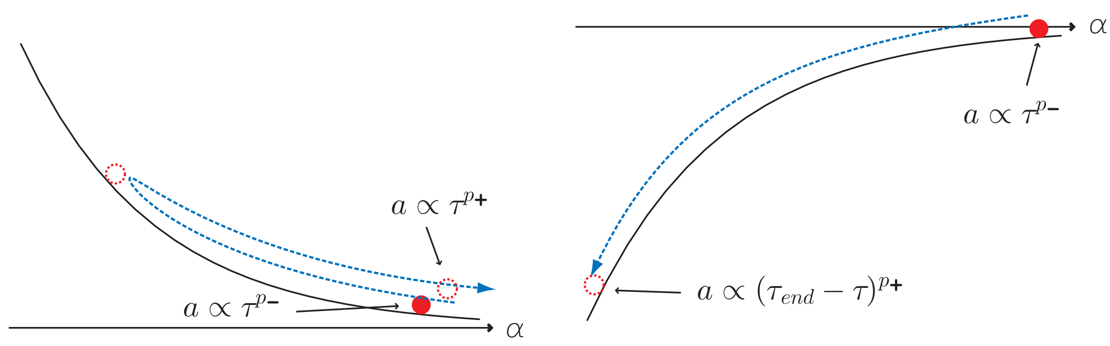

Figure 8.2: The motion of the scalar field in the potential. The behavior of the scale factor in the Jordan frame is also presented. The potential is positive for $0 < q < 1$ (left figure), whereas it is negative for $1 < q < \sqrt{3}$ (right figure).

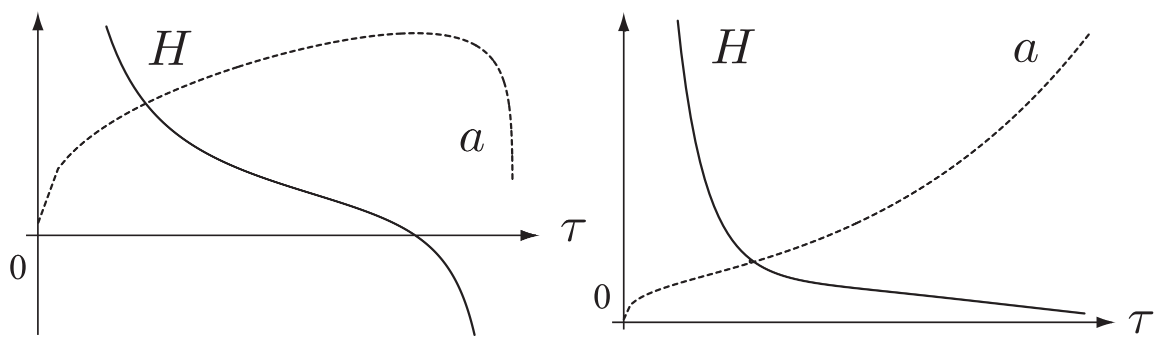

Figure 8.3: Sketch of the behavior of the scale factor a (dashed lines) and the Hubble parameter H (solid lines) in the Jordan frame as functions of the proper time on the brane τ. The left figure shows the solution of (8.33) with $q > 1$. The right figure describes the solution of (8.40) with the plus sign in the exponent.

potential is positive, while for $q > 1$ ($m > 1$) the potential is negative. In the former case, the scalar field α starts at $\alpha = \infty$, climbs up the slope of the potential, turns around somewhere, and finally goes back to $\alpha = \infty$. In the latter case, α starts to roll down from $\alpha = \infty$. The universe expands for a period of time and eventually it starts to contract. Finally α falls into the bottomless pit within a finite time, where the universe ends up with a singularity.

Here we note that the above picture based on the four-dimensional effective theory describes the dynamics in the conformally transformed frame in which the metric is given by $\tilde{q}_{\hat{a}\hat{b}}$, whereas we suppose that the "physical" metric is given by the induced metric on the brane $q_{\hat{a}\hat{b}}$. In principle, the cosmic expansion law can look very different depending on the frame we choose. The dynamics in the "physical" frame therefore can be apparently very different. However, the above discussion is still useful since the conformal rescaling does not change the causal structure of the spacetime.

8.3.2 Kasner-type solutions with a cosmological constant

The next example is a generalization of the Kasner-type spacetime including a cosmological constant Λ_b. Let us assume that the metric is in the form of

$$g_{\hat{A}\hat{B}}dx^{\hat{A}}dx^{\hat{B}} = -dt^2 + e^{2\alpha(t)}\delta_{\mu\nu}dy^\mu dy^\nu + e^{2\beta(t)}\delta_{ij}dx^i dx^j, \tag{8.36}$$

where i and j again run from 1 to n, but here the metric of m-dimensional space is chosen to be flat ($K = 0$) because otherwise the solution with $\Lambda_b \neq 0$ is not obtained analytically. The $D[= n+1+m]$-dimensional vacuum Einstein equations with a cosmological constant reduce to

$$m\left(\ddot{\alpha} + \dot{\alpha}^2\right) + n\left(\ddot{\beta} + \dot{\beta}^2\right) = (m+n)H_0^2, \tag{8.37}$$

$$\ddot{\alpha} + \dot{\alpha}\left(m\dot{\alpha} + n\dot{\beta}\right) = (m+n)H_0^2, \tag{8.38}$$

$$\ddot{\beta} + \dot{\beta}\left(m\dot{\alpha} + n\dot{\beta}\right) = (m+n)H_0^2. \tag{8.39}$$

From this we obtain two types of solutions (see Sec. 8.7). One is a trivial solution, namely, D-dimensional de Sitter spacetime,

$$\alpha = \beta = H_0 t.$$

Of the other type are the following two solutions,

$$e^{m\alpha+n\beta} = \sinh[(m+n)H_0 t],$$
$$e^{\alpha-\beta} = \left[\tanh\left(\frac{m+n}{2}H_0 t\right)\right]^{\pm 1/q}, \tag{8.40}$$

where q is the one that has been introduced in Eq. (8.31). The range of the time coordinate t is $(-\infty, \infty)$ for the former de Sitter solution and $[0, \infty)$ for the latter non-trivial solutions. Applying the method discussed in Sec. 8.1.1 to these solutions, one can construct background solutions for a $(D+1)$-dimensional braneworld model.

First we briefly mention the relation to the bulk inflaton model proposed by Koyama and Takahashi [75, 77]. Identifying their model parameters Δ and δ as

$$\Delta = 4b^2 - 8/3 = -\frac{2(m+4)}{m+3}, \qquad \delta = \frac{m+4}{4(m+3)}\frac{\ell^2 H_0^2}{1+\ell^2 H_0^2}, \tag{8.41}$$

we will find that our model is equivalent to theirs. The parameter m is supposed to take any positive number. Thus it follows that Δ varies in the same region $-8/3 \leq \Delta \leq -2$ considered in [75, 77]. The background metric obtained by substituting the simplest solution $\alpha = \beta = H_0 t$ is indeed the case discussed in their paper.

Next we consider the cosmic expansion law. We start with the simplest case $\alpha = \beta = H_0 t$. The dimensionally reduced metric (on the brane) is

$$q_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = e^{2mH_0 t/3}\left(-dt^2 + e^{2H_0 t}\delta_{ij}dx^i dx^j\right).$$

Introducing the cosmological time τ and the conformal time η defined by

$$d\tau = a d\eta = e^{mH_0 t/3}dt,$$

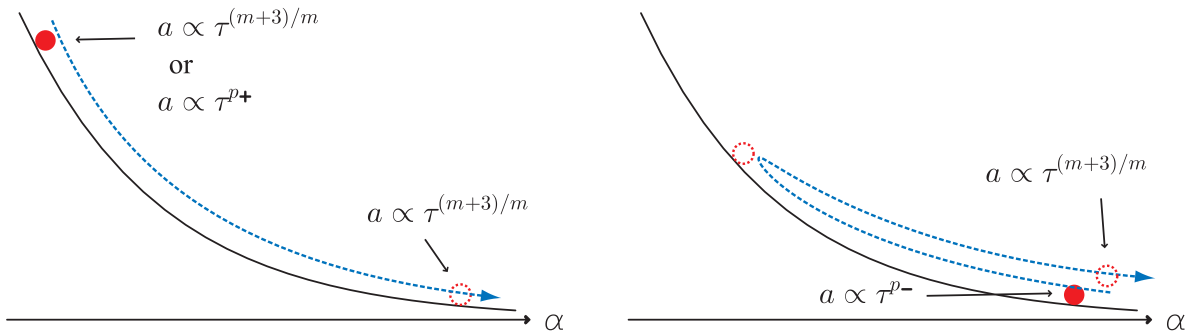

Figure 8.4: The motion of the scalar field in the potential and the behavior of the scale factor in the Jordan frame. The solution found by Koyama and Takahashi [75, 77] is described in the left figure. One of the non-trivial solutions with its behavior at the starting point $a \propto \tau^{p_+}$ is also shown in the left figure, while the other one with the exponent p_- behaves as shown in the right figure.

the scale factor on the brane $a = e^{(m+3)H_0 t/3}$ is written in terms of τ or η as

$$a \propto \tau^{(m+3)/m} \propto \eta^{-(m+3)/3}. \tag{8.42}$$

Since $1 \leq (m+3)/m < \infty$, power-law inflation with any exponent can be realized.

Furthermore, we have non-trivial solutions (8.40). The behavior of the solutions is as follows. At early times ($t \sim 0$), the scale factor behaves like $a = e^{m\alpha/3+\beta} \sim t^{1/3}$, and the cosmological time is given by $d\tau \sim t^{\left(m\pm\sqrt{3m(m+2)}\right)/3(m+3)}dt$ (and so $\tau \to 0$ as $t \to 0$). Therefore, we have

$$a \sim \tau^{p_\pm}, \tag{8.43}$$

with $p_\pm$ introduced previously, which implies that the universe is not accelerated at early times. At late times ($t \to \infty$), we see that $\alpha \to H_0 t$ and $\beta \to H_0 t$, and the solution shows power-law expansion as is given in Eq. (8.42).

A rather intuitive interpretation of the behavior of these three solutions can be made from the four-dimensional point of view again. The situation is summarized in Fig. 8.4. This time the potential is always positive. For one of the non-trivial solutions with the exponent p_+ in (8.43), the scalar field starts to roll down from $\alpha = -\infty$. For the other non-trivial solution with the exponent p_-, the field starts to climb up the slope of the potential from $\alpha = +\infty$, turns around somewhere, and rolls down back to $\alpha = +\infty$. Suppose that the field α is increasing. Let us trace the evolution of α backward in time. If the kinetic energy is larger than a certain critical value, α will not have a turning point in the past. In this case α continues to decrease, reaching $-\infty$. This corresponds to the case with the exponent p_+. If the kinetic energy is lower, α will have a turning point. Then, we will have $\alpha \to +\infty$ at $t \to -\infty$. This corresponds to the case with the exponent p_-. The case of the power-law inflation (8.42) is, in fact, the marginal case between these two. In this case, α does not turn around. Therefore the evolution of α is similar to the case with p_+. In any case, information about the initial velocity is lost as the universe expands. Therefore the late time behavior of the solutions is unique, and is given by Eq. (8.42).

8.4 Cosmological perturbations

We consider cosmological perturbations in the $(n+2)$-dimensional bulk inflaton models defined by Eqs. (8.12) and (8.13). The analysis of perturbations is complicated if we work in the original $(n+2)$-dimensional models with a bulk scalar field. Our $(n+2)$-dimensional system, however, is equivalent to the $(D+1)[= n+2+m]$-dimensional one defined by Eqs. (8.2) and (8.3). We will show that the perturbation analysis becomes very simple and transparent in the $(D+1)$-dimensional picture, in which we just need to consider pure gravity without any matter fields.

We begin with the following form of background metric:

$$G_{AB}dX^A dX^B = e^{2\omega(z)}\left(dz^2 - dt^2 + e^{2\alpha(t)}\gamma_{\mu\nu}dy^\mu dy^\nu + e^{2\beta(t)}\delta_{ij}dx^i dx^j\right), \tag{8.44}$$

where the latin and roman indices in the lower case, respectively, run m- and n-dimensional subspaces, and the warp factor is given by $e^{\omega(z)} = \ell H_0/\sinh(H_0 z)$. We assume that $\alpha(t)$ and $\beta(t)$ are chosen so that $g_{\hat{A}\hat{B}}$ is a solution of the $(n+1+m)$-dimensional vacuum Einstein equations with a cosmological constant,

$$m\left(\ddot{\alpha} + \dot{\alpha}^2\right) + n\left(\ddot{\beta} + \dot{\beta}^2\right) = (m+n)H_0^2, \tag{8.45}$$

$$\ddot{\alpha} + \dot{\alpha}\left(m\dot{\alpha} + n\dot{\beta}\right) + K(m-1)e^{-2\alpha} = (m+n)H_0^2, \tag{8.46}$$

$$\ddot{\beta} + \dot{\beta}\left(m\dot{\alpha} + n\dot{\beta}\right) = (m+n)H_0^2. \tag{8.47}$$

The background solutions with $\gamma_{\mu\nu} = \delta_{\mu\nu}$ $(K=0)$, $H_0 \neq 0$ and with $K = \pm 1$, $H_0 = 0$ were discussed in the preceding section. In the following discussions, we include more general cases with $K = \pm 1$ and $H_0 \neq 0$. Although the background solution cannot be obtained in an explicit form for such non-flat compactifications with a cosmological constant, we will find that general properties of perturbations can be explored to a great extent.

We write the perturbed metric as

$$\begin{aligned}&(G_{AB} + \delta G_{AB})dX^A dX^B \\ &\quad = e^{2\omega}\big\{(1+2N)dz^2 + 2A dt dz - (1+2\Phi)dt^2 + e^{2\alpha}(1+2S)\gamma_{\mu\nu}dy^\mu dy^\nu \\ &\qquad + e^{2\beta}\left[(1+2\Psi)\delta_{ij}dx^i dx^j + 2E_{ij}dx^i dx^j + 2B_i dx^i dt + 2C_i dx^i dz\right]\big\}.\end{aligned} \tag{8.48}$$

These perturbations are assumed to be homogeneous and isotropic in the directions of the m-dimensional compactified space spanned by the coordinates y^μ. From the assumption of isotropy, mixed components such as $\delta G_{\mu t}dy^\mu dt$ and $\delta G_{\mu i}dy^\mu dx^i$ are set to zero. Concerning the metric perturbations of the compactified space, therefore, only the overall volume perturbation S is considered. After reduction to $(n+2)$ dimensions, S is to be interpreted as the scalar field perturbation. Here, we also assume that the dependence on the n-dimensional coordinates x^i is given by $e^{ik_i x^i}$.

Metric perturbations are decomposed into scalar, vector, and tensor components based on the behavior under the transformation of the n-dimensional spatial coordinates x^i in the following manner:

$$\begin{aligned}&B_i = \frac{k_i}{ik}B^S + B_i^V, \quad k^i B_i^V = 0, \\ &E_{ij} = \left(-\frac{k_i k_j}{k^2} + \frac{\delta_{ij}}{n}\right)E^S + \frac{1}{ik}k_{(i}E_{j)}^V + \frac{1}{2}E_{ij}^T, \\ &k^i E_i^V = k^i E_{ij}^T = 0, \quad \delta^{ij}E_{ij}^T = 0.\end{aligned} \tag{8.49}$$

The quantities with a superscript S, V, and T represent scalar, vector and tensor perturbations, respectively. The perturbations δG_{AB} obey the linearized Einstein equations supplemented by boundary conditions at the position of the brane, $\delta\mathcal{K}_A{}^B|_{z=z_b}=0$ where $\mathcal{K}_{AB}$ is the extrinsic curvature of the brane. From the $(D+1)$-dimensional point of view matter sources are absent on the D-dimensional brane, and this makes boundary conditions considerably simple. Each component of the Einstein equations is presented in Sec. 8.6.

Here we would like to discuss the number of physical degrees of freedom in scalar, vector, and tensor perturbations. The transverse traceless tensor E^T_{ij} has $(n+1)(n-2)/2$ independent components, each of which obeys a second order differential equation. For vector perturbations there are three variables B^V_i, E^V_j, and C^V_i. The coordinate transformation has one vector mode, and correspondingly there is one vector constraint equation. Therefore we have only $1[=3-1\times 2]$ vector remaining as a physical mode. Since a transverse vector has $(n-1)$ independent components, we find that there are $1\times(n-1)$ degrees of freedom in vector perturbations, corresponding to the "graviphoton." For scalar perturbations there are 8 variables, and the coordinate transformation has 3 independent modes. Since there are the same number of constraint equations, the number of physical modes is $2[=8-3\times 2]$. One of them corresponds to the bulk scalar field and the other corresponds to the "graviscalar." In total, there are $1+(n+2)(n-1)/2=2+(n-1)+(n+1)(n-2)/2$ physical degrees of freedom. The first "1" on the left hand side corresponds to the bulk scalar and the other $(n+2)(n-1)/2$ degrees of freedom to those of $(n+2)$-dimensional gravitational waves.

8.4.1 Tensor perturbations

Since tensor perturbations are gauge invariant from the beginning, they are in general easy to analyze. The equations for tensor perturbations are read from the $\{i,j\}$-component of the Einstein equations (8.114) as

$$\mathcal{L}E^T_{ij}=0, \tag{8.50}$$

where we have defined a differential operator

$$\mathcal{L}:=\partial_t^2+\left(m\dot{\alpha}+n\dot{\beta}\right)\partial_t+e^{-2\beta}k^2-\partial_z^2-(m+n)(\partial_z\omega)\partial_z. \tag{8.51}$$

The perturbed junction condition implies that boundary conditions are Neumann on the brane,

$$\partial_z E^T_{ij}\big|_{z=z_b}=0. \tag{8.52}$$

Since the perturbation equations are manifestly separable, we write $E^T_{ij}=\chi(t)\psi(z)Y^T_{ij}(x^\ell)$ where Y^T_{ij} is a transverse, traceless tensor harmonics. Then $\chi(t)$ and $\psi(z)$ obey

$$\ddot{\chi}+\left(m\dot{\alpha}+n\dot{\beta}\right)\dot{\chi}+\left(e^{-2\beta}k^2+M^2\right)\chi=0, \tag{8.53}$$

$$\partial_z^2\psi+(m+n)(\partial_z\omega)\partial_z\psi+M^2\psi=0. \tag{8.54}$$

Here M^2 is a separation constant and represents the squared Kaluza-Klein mass for observers on the D-dimensional brane.

Now we discuss the mode function in the z-direction, $\psi(z)$. Using a canonical variable $\hat{\psi}:=e^{\mu\omega}\psi$, with

$$\mu:=\frac{m+n}{2}\geq\frac{3}{2}, \tag{8.55}$$

Eq. (8.54) is rewritten into a Schrödinger-type equation

$$-\partial_z^2\hat{\psi} + V(z)\hat{\psi} = M^2\hat{\psi}, \tag{8.56}$$

where the potential is

$$V(z) = \mu(\mu+1)\frac{H_0^2}{\sinh^2(H_0 z)} + \mu^2 H_0^2 - \frac{\kappa_{D+1}^2\sigma}{2}\delta(z - z_b). \tag{8.57}$$

The delta-function term is introduced so that ψ automatically satisfies the boundary condition $\psi'(z_b) = 0$. The presence of the zero mode, for which ψ is constant in z, is obvious from Eq. (8.54). From the asymptotic value of the potential $V(\infty) = \mu^2 H_0^2$, we can say that there is a mass gap $\delta M = \mu H_0$ between the zero mode and the KK continuum.

The z-dependence of the massive modes is given in terms of the associated Legendre functions by

$$\begin{aligned}\psi_M(z) \quad = \quad & c^T(\sinh(H_0 z))^{1/2+\mu}\Big[Q^{-1/2-\mu}_{1/2+i\nu}(\cosh(H_0 z_b))P^{-1/2-\mu}_{-1/2+i\nu}(\cosh(H_0 z)) \\ & -P^{-1/2-\mu}_{1/2+i\nu}(\cosh(H_0 z_b))Q^{-1/2-\mu}_{-1/2+i\nu}(\cosh(H_0 z))\Big],\end{aligned} \tag{8.58}$$

where c^T is a normalization constant and

$$\nu := \sqrt{\frac{M^2}{H_0^2} - \mu^2}. \tag{8.59}$$

These general properties of the mass spectrum and the mode functions in the z-direction hold irrespective of the specific form of the background solution α and β.

Let us move on to the time dependence of tensor perturbations. Using the cosmological time on the brane defined by $\tau = \int e^{m\alpha/n}dt$, Eq. (8.53) is rewritten as

$$\left[\frac{d^2}{d\tau^2} + \left(nH + \frac{m}{n}\frac{d\alpha}{d\tau}\right)\frac{d}{d\tau} + \frac{k^2}{a^2} + e^{-2m\alpha/n}M^2\right]\chi = 0, \tag{8.60}$$

where one must recall that $a = e^{m\alpha/n+\beta}$ and $H = a^{-1}da/d\tau$. For the zero mode ($M^2 = 0$), this reduces to the equation for the tensor perturbations in the scalar-tensor theory defined by the action (8.21). An apparent difference from Einstein gravity is the presence of the term $(m/n)(d\alpha/d\tau)$. The Kaluza-Klein mass with respect to observers on the brane is expressed as

$$m_{\rm KK}^2(t) = e^{-2m\alpha(t)/n}M^2, \tag{8.61}$$

and so $m_{\rm KK} = e^{-m\alpha/n}\mu H_0$ for the lightest one, whereas the Hubble parameter at that time is given by

$$H = e^{-m\alpha/n}\left(\frac{m}{n}\dot{\alpha} + \dot{\beta}\right).$$

For $\alpha = \beta = H_0 t$, this implies $H = 2n^{-1}e^{-m\alpha/n}\mu H_0$ and therefore the mass gap and H are of the same order. On the other hand, when the background is given by Eq. (8.40), we have

$H = 2n^{-1}e^{-m\alpha/n}\mu H_0 \coth[(m+n)H_0 t]$ and the mass gap can be very small compared to H, but only for a short period near $t = 0$.

Despite its rather simple form, Eq. (8.53) cannot be solved analytically in general. One exception is the case $\alpha = \beta = H_0 t$ discussed in [75]. In this case, Eq. (8.53) reads

$$\ddot{\chi} + (m+n)H_0\dot{\chi} + \left(e^{-2H_0 t}k^2 + M^2\right)\chi = 0,$$

which, using the conformal time $\eta = -e^{-H_0 t}/H_0$, can be rewritten as

$$\left[\frac{d^2}{d\eta^2} + \frac{1-m-n}{\eta}\frac{d}{d\eta} + k^2 + \frac{M^2}{H_0^2\eta^2}\right]\chi = 0. \tag{8.62}$$

This indeed has analytic solutions,

$$\chi_0 \propto (-\eta)^{\mu} H_{\mu}^{(1)}(-k\eta), \tag{8.63}$$

$$\chi_M \propto (-\eta)^{\mu} H_{i\nu}^{(1)}(-k\eta). \tag{8.64}$$

This is not a surprise because the background of the current model is just an AdS_{m+2+n} bulk with a de Sitter brane.

8.4.2 Vector perturbations

Next we consider vector perturbations. From the perturbed junction conditions $\delta\mathcal{K}_i{}^j|_{z=z_b} = 0$ and $\delta\mathcal{K}_i{}^t|_{z=z_b} = 0$, we have

$$\partial_z E_i^V\big|_{z=z_b} = 0, \quad C_i^V\big|_{z=z_b} = 0, \quad \partial_z B_i^V\big|_{z=z_b} = 0. \tag{8.65}$$

Under a vector gauge transformation $x^i \to \bar{x}^i = x^i + \xi^{iV}$, the metric variables transform as

$$\bar{E}_i^V = E_i^V + k\xi_i^V, \qquad \bar{B}_i^V = B_i^V - \dot{\xi}_i^V, \qquad \bar{C}_i^V = C_i^V - \partial_z\xi_i^V. \tag{8.66}$$

Thus we are allowed to set $E_i^V = 0$ by choosing an appropriate gauge. We expand the remaining variables by using the transverse vector harmonics Y_i^V as

$$B_i^V = \mathcal{B}Y_i^V, \qquad C_i^V = \mathcal{C}Y_i^V. \tag{8.67}$$

For convenience, we introduce

$$\Omega := k^{-2}e^{m\alpha+(n+2)\beta}e^{(m+n)\omega}\left(\partial_z\mathcal{B} - \dot{\mathcal{C}}\right). \tag{8.68}$$

Then Eqs. (8.117) and (8.120) are written as

$$\mathcal{B} = e^{-m\alpha-n\beta}e^{-(m+n)\omega}\partial_z\Omega, \tag{8.69}$$

$$\mathcal{C} = e^{-m\alpha+n\beta}e^{-(m+n)\omega}\dot{\Omega}. \tag{8.70}$$

It is easy to see that the remaining third equation is automatically satisfied if the above two equations hold. Substituting these two into Eq. (8.68), we obtain a master equation

$$\left[\ddot{\Omega} - \left(m\dot{\alpha} + n\dot{\beta}\right)\dot{\Omega} + e^{-2\beta}k^2\Omega\right] - \left[\partial_z^2\Omega - (m+n)(\partial_z\omega)\partial_z\Omega\right] = 0. \tag{8.71}$$

This equation looks similar to the equation for tensor perturbations (8.50). The difference is that the signatures of the terms containing first derivatives such as $\dot{\Omega}$ and Ω' are reversed. From Eqs. (8.65), the boundary condition for Ω on the brane turns out to be Dirichlet,

$$\Omega|_{z=z_b} = 0. \tag{8.72}$$

Since the master equation (8.71) is separable, we write $\Omega(t,z) = \chi(t)\Omega(z)$. The canonical variable $\hat{\Omega}(z) := e^{-\mu\omega}\Omega$ again obeys a Schrödinger-type equation

$$-\partial_z^2\hat{\Omega} + V(z)\hat{\Omega} = M^2\hat{\Omega}, \tag{8.73}$$

with the potential

$$V(z) = \mu(\mu-1)\frac{H_0^2}{\sinh^2(H_0 z)} + \mu^2 H_0^2. \tag{8.74}$$

The crucial difference from tensor perturbations is the absence of the delta-function potential well. Because of this, there is no zero mode and only the massive modes with $M^2 \geq V(\infty) = \mu^2 H_0^2$ exist. The z-dependence of the mode functions is given by

$$\begin{aligned}\Omega_M(z) = \quad & c^V(\sinh(H_0 z))^{1/2-\mu} \quad \Big[Q^{-1/2+\mu}_{-1/2+i\nu}(\cosh(H_0 z_b))P^{-1/2+\mu}_{-1/2+i\nu}(\cosh(H_0 z)) \\ & \qquad - P^{-1/2+\mu}_{-1/2+i\nu}(\cosh(H_0 z_b))Q^{-1/2+\mu}_{-1/2+i\nu}(\cosh(H_0 z))\Big]\end{aligned} \tag{8.75}$$

where c^V is a normalization constant. When $\alpha = \beta = H_0 t$, we can find an analytic solution for the time-dependence of the mode functions, which, using the conformal time, is given by

$$\chi_M(\eta) \propto (-\eta)^{-\mu} H^{(1)}_{i\nu}(-k\eta).$$

8.4.3 Scalar perturbations

Gauge choice, the boundary condition, and the mode decomposition

Since scalar perturbations are more complicated, we begin with fixing the gauge appropriately in order to simplify the perturbed Einstein equations. We impose the Gaussian-normal gauge conditions

$$N = A = C = 0. \tag{8.76}$$

Different from the case of vector perturbations, these conditions do not fix the gauge completely. In the case of scalar perturbations we need to take care of perturbations of the brane location. Here we make use of the remaining gauge degrees of freedom to keep the brane location unperturbed at $z = z_b$. In the Gaussian-normal gauge, boundary conditions on the brane for all remaining variables become Neumann:

$$\partial_z\Psi|_{z=z_b} = \partial_z E|_{z=z_b} = \partial_z S|_{z=z_b} = \partial_z\Phi|_{z=z_b} = \partial_z B|_{z=z_b} = 0. \tag{8.77}$$

Three of eight scalar perturbation equations are the constraint equations, and the other five are the evolution equations. First, let us examine the constraint equations (8.116), (8.122), and (8.123). Eq. (8.122) reduces to $\partial_z^2(\Phi + n\Psi + mS) + \partial_z \omega \partial_z(\Phi + n\Psi + mS) = 0$. Taking into account the boundary conditions, this equation is once integrated to give

$$\partial_z(\Phi + n\Psi + mS) = 0. \tag{8.78}$$

Eqs. (8.116) and (8.123) reduce to

$$\partial_z \left[kB + 2\dot{\Phi} + 2\left(m\dot{\alpha} + n\dot{\beta}\right)\Phi - 2m\dot{\alpha}S - 2n\dot{\beta}\Psi \right] = 0, \tag{8.79}$$

$$\partial_z \left\{ \dot{B} + \left(m\dot{\alpha} + n\dot{\beta} + 2\dot{\beta}\right) B + 2e^{-2\beta}k \left[\Psi + \left(\frac{1}{n} - 1\right) E \right] \right\} = 0. \tag{8.80}$$

With the aid of Eq. (8.78), we find that all the perturbation equations are separable. Furthermore, the z-dependent parts of these equations are the same as those of tensor perturbations with the same type of boundary conditions. Therefore we can expand all variables by using the same mode functions in the z-direction as those for tensor perturbations:

$$E = E_0(t)\psi_0(z) + \sum E_M(t)\psi_M(z), \qquad \Phi = \Phi_0(t)\psi_0(z) + \sum \Phi_M(t)\psi_M(z), \tag{8.81}$$

where ψ_0 is constant, ψ_M is given by Eq. (8.58), and $M^2 \geq \mu^2 H_0^2$ [2]. Consequently, Eqs. (8.78), (8.79), and (8.80) are automatically satisfied for the zero mode. For the massive modes these constraint equations give

$$\Phi + n\Psi + mS = 0, \tag{8.82}$$

$$kB + 2\dot{\Phi} + 2\left(m\dot{\alpha} + n\dot{\beta}\right)\Phi - 2m\dot{\alpha}S - 2n\dot{\beta}\Psi = 0, \tag{8.83}$$

$$\dot{B} + \left(m\dot{\alpha} + n\dot{\beta} + 2\dot{\beta}\right) B + 2e^{-2\beta}k \left[\Psi + \left(\frac{1}{n} - 1\right) E \right] = 0, \tag{8.84}$$

where the subscript M was abbreviated. Note that these three equations are nothing but the components of the divergence of the metric perturbations,

$$\nabla^A \delta G_{zA} = \nabla^A \delta G_{tA} = \nabla^A \delta G_{iA} = 0.$$

In other words, the transverse traceless conditions are automatically satisfied if one imposes the Gaussian-normal gauge conditions except for the contribution coming from the zero mode. ($\nabla^A \delta G_{zA}$ gives the traceless condition.) Below we discuss the KK modes and the zero mode separately.

KK modes

By using the constraint equations (8.82)-(8.84), the Einstein equations (8.111), (8.112), (8.119), (8.121), and (8.124) are simplified to give

$$\mathcal{L}\Psi = -\left(\frac{2}{n} - 1\right) k\dot{\beta}B + 2\dot{\beta}\dot{\Phi} + 2(m+n)H_0^2\Phi, \tag{8.85}$$

$$\mathcal{L}E = 2k\dot{\beta}B, \tag{8.86}$$

$$\mathcal{L}S = k\dot{\alpha}B + 2\dot{\alpha}\dot{\Phi} + 2e^{-2\alpha}K(m-1)(S - \Phi) + 2(m+n)H_0^2\Phi, \tag{8.87}$$

$$\mathcal{L}\Phi = -2\left(m\ddot{\alpha} + n\ddot{\beta}\right)\Phi + 2k\dot{\beta}B + 2m\ddot{\alpha}S + 2n\ddot{\beta}\Psi + 2(m+n)H_0^2\Phi, \tag{8.88}$$

$$\mathcal{L}B = -\left(m\ddot{\alpha} + n\ddot{\beta}\right) B - 4\dot{\beta}\dot{B} - 4\dot{\beta}^2 B - 2(m+n)H_0^2 B - 4e^{-2\beta}k\dot{\beta}\Phi, \tag{8.89}$$

[2]In Ref. [145], it was shown that the non-normalizable graviscalar mode with KK mass $M^2 = (2\mu - 1)H_0^2$ excites a small black hole in the bulk in the long wavelength limit, which corresponds to the dark radiation.

where $\mathcal{L}$ is defined in Eq. (8.51). Two of them give independent master equations for the massive modes, and the remaining three equations do not give any new conditions. With the aid of the constraint equations (8.82)-(8.84), Eqs. (8.85) and (8.88) can be rewritten as

$$\begin{aligned}\mathcal{L}\Psi &= -2(n-2)\dot{\beta}\left(\dot{\alpha}-\dot{\beta}\right)\Psi+\frac{4}{n}\dot{\beta}\dot{\Phi}\\ &\quad -2\left\{\frac{n-2}{n}\dot{\beta}\left[(m+1)\dot{\alpha}+n\dot{\beta}\right]-(m+n)H_0^2\right\}\Phi, \qquad (8.90)\\ \mathcal{L}\Phi &= -4\dot{\beta}\dot{\Phi}-2\left\{(m+1)\ddot{\alpha}+n\ddot{\beta}+2\dot{\beta}\left[(m+1)\dot{\alpha}+n\dot{\beta}\right]-(m+n)H_0^2\right\}\Phi\\ &\quad -2n\left[\ddot{\alpha}-\ddot{\beta}+2\dot{\beta}\left(\dot{\alpha}-\dot{\beta}\right)\right]\Psi. \qquad (8.91)\end{aligned}$$

Unfortunately, except for the simplest case (to be discussed later) we do not know how to disentangle these two equations, although there is no problem in solving these equations numerically. Once we solve these coupled equations, the other variables S, B, E are easily determined just by using the constraint equations.

Zero mode

To discuss the zero mode, it is useful to look at the cosmological perturbations in the corresponding $(n+1)$-dimensional theory defined by the action (8.26). In the case of the $(n+1)$-dimensional Friedmann universe with a single scalar field, there is only one physical degree of freedom in scalar perturbations. One can derive a second order differential equation for one master variable [110]. Back in the braneworld context, the background metric and its zero-mode perturbations are also described by the same effective action (8.26). Therefore, the analysis of the zero mode is no different from the conventional $(n+1)$-dimensional cosmological perturbation theory. Below we will explain this fact more explicitly.

To begin with, we consider $(n+1+m)$-dimensional spacetime whose metric is given by

$$\begin{aligned}ds^2 &= -(1+2\Phi)dt^2+e^{2\alpha}(1+2S)\gamma_{\mu\nu}dy^\mu dy^\nu\\ &\quad +e^{2\beta}\left[(1+2\Psi)\delta_{ij}dx^i dx^j+2E_{ij}dx^i dx^j+2B_i dx^i dt\right], \qquad (8.92)\end{aligned}$$

where only scalar perturbations are imposed and they are again assumed to be homogeneous and isotropic with respect to the m-dimensional compactified space spanned by y^μ. Then, the perturbed Einstein equations $\delta R_{\hat{A}\hat{B}}=(m+n)H_0^2\delta g_{\hat{A}\hat{B}}$ become identical to Eqs. (8.111), (8.112), (8.119), (8.121), and (8.124) with N, A, C, and the terms differentiated by z dropped. Hence, it is manifest that the analysis of zero-mode perturbations in our $(n+2+m)$-dimensional spacetime is equivalent to that of the above system.

As for perturbations in $(n+2+m)$-dimensional spacetime, we have already fixed the gauge by imposing three gauge conditions (8.76). However, these gauge conditions do not fix the gauge completely. As is manifest from Eqs. (8.126), gauge transformations satisfying $\xi^z=0$ and $\xi^{S'}=\xi^{t'}=0$ do not disturb the conditions (8.76). On the other hand, on the $(n+1)$-dimensional side there are two scalar gauge transformations

$$\begin{aligned}t &\to \bar{t}=t+\xi^t,\\ x^i &\to \bar{x}^i=x^i+k^i\xi^S/ik.\end{aligned}$$

The transformation of metric variables under these gauge transformations is the same as that obtained by setting $\xi^z=0$ and $\xi^{S'}=\xi^{t'}=0$ in the last five equations in (8.126).

If we think of the size of the compactified dimension S as a scalar field in $(n+1)$-dimensional spacetime, the system reduces to a conventional $(n+1)$-dimensional model with a scalar field. In the conventional cosmological perturbation theory, Φ and Ψ in the longitudinal gauge ($B=E=0$) are known to be convenient variables. Here one remark is that we need to take account of a conformal transformation to map the theory to the conventional $(n+1)$-dimensional one,

$$d\tilde{s}^2 = e^{\frac{2m}{n(n-1)}(\alpha+S)} \cdot e^{\frac{2m}{n}(\alpha+S)} ds^2 = e^{\frac{2m}{n-1}(\alpha+S)} ds^2, \tag{8.93}$$

which follows from the discussion in Sec. 8.2. Then the variables corresponding to the so-called Sasaki-Mukhanov variables are

$$\hat{\Phi} := \Phi + \frac{m}{n-1} S, \tag{8.94}$$

$$\hat{\Psi} := \Psi + \frac{m}{n-1} S, \tag{8.95}$$

in the longitudinal gauge.

Eliminating N, A, B, C, E and the terms differentiated by z, Eq. (8.112) becomes

$$\hat{\Phi} + (n-2)\hat{\Psi} = 0. \tag{8.96}$$

Similarly, from Eqs. (8.119), (8.111), and (8.124), we have

$$\begin{aligned}
&(n-1)\dot{\hat{\Psi}} + (n-2)\left[m\dot{\alpha} + (n-1)\dot{\beta}\right]\hat{\Psi} = -\frac{m+n-1}{n-1} m\dot{\alpha} S, \\
&\mathcal{L}_0 \Psi = -2(n-1)\dot{\beta}\dot{\hat{\Psi}} + 2(m+n)H_0^2 \Phi, \\
&\mathcal{L}_0 S = -2(n-1)\dot{\alpha}\dot{\hat{\Psi}} + 2(m+n)H_0^2 \Phi - 2K(m-1)e^{-2\alpha}(\Phi - S),
\end{aligned} \tag{8.97}$$

where $\mathcal{L}_0 f := \ddot{f} + \left(m\dot{\alpha} + n\dot{\beta}\right)\dot{f} + e^{-2\beta}k^2 f$. Combining all these, we obtain the equation of motion for $\hat{\Psi}$:

$$\mathcal{L}_0 \hat{\Psi} - 2\left(\dot{\beta} + \frac{\ddot{\alpha}}{\dot{\alpha}}\right)\dot{\hat{\Psi}} + 2(n-2)\left(\ddot{\beta} - \dot{\beta}\frac{\ddot{\alpha}}{\dot{\alpha}}\right)\hat{\Psi} = 0. \tag{8.98}$$

Using the conformal time $\eta = \int e^{-\beta} dt$, we can rewrite this into a more familiar form as

$$\hat{\Psi}'' + \left[(n-1)\mathcal{H} - 2\frac{\alpha''}{\alpha'}\right]\hat{\Psi}' + k^2 \hat{\Psi} + 2(n-2)\left(\mathcal{H}' - \mathcal{H}\frac{\alpha''}{\alpha'}\right)\hat{\Psi} = 0, \tag{8.99}$$

where we have defined

$$\mathcal{H} := (\ln \tilde{a})' = \frac{1}{n-1}\left[m\alpha + (n-1)\beta\right]', \tag{8.100}$$

and $\tilde{a} = e^{m\alpha/(n-1)+\beta}$ is the scale factor in the Einstein frame.

Since there is a mass gap between the zero mode and the massive modes in general in our models except for a short period in the cases of $K \neq 0$, the massive modes would not be excited easily. Hence, the behavior of the zero mode is especially important. Since we found that the zero mode can be described by the corresponding $(n+1)$-dimensional conventional cosmology, it can be easily analyzed in general.

Exactly solvable case

Let us consider the simplest background given by $\alpha = \beta = H_0 t$ with $K = 0$. In this special case, scalar perturbations including the KK modes are solved exactly. The most remarkable advantage of our approach may be that z-dependence of the modes can be derived for a general background as we did in the earlier part of this section. The time-dependent part, which is usually non-trivial especially for the KK modes, is also solved easily as shown below when $\alpha = \beta = H_0 t$.

Substituting $\alpha = \beta = H_0 t$ into Eq. (8.91), the equation for Φ is decoupled first,

$$\mathcal{L}\Phi = -4H_0\dot{\Phi} - 2(m+n+2)H_0^2\Phi. \tag{8.101}$$

By assuming the z-dependence given in Eq. (8.58), we expand as $\Phi = \Phi_M \psi_M$. Then, using the conformal time, the above equation is rewritten as

$$\left[\frac{d^2}{d\eta^2} - \frac{m+n+3}{\eta}\frac{d}{d\eta} + k^2 + \frac{1}{\eta^2}\left(\frac{M^2}{H_0^2} + 2(m+n+2)\right)\right]\Phi_M = 0. \tag{8.102}$$

The solution is given in terms of the Hankel function by

$$\Phi_M = c_1^S(-\eta)^2\rho(\eta) \tag{8.103}$$

with

$$\rho = (-\eta)^\mu H_{i\nu}^{(1)}(-k\eta),$$

where c_1^S is a constant and μ and ν were defined in Eqs. (8.55) and (8.59). Then, substituting this into Eq. (8.84) with the aid of Eq. (8.83), B_M is immediately obtained as

$$B_M = -2c_1^S k^{-1} H_0\left[(-\eta)^3\rho' + (2\mu - 1)(-\eta)^2\rho\right]. \tag{8.104}$$

The result is consistent with the evolution equation for B [Eq. (8.89)].

Eqs. (8.86), (8.87) and (8.89) are combined to give a simple equation,

$$\mathcal{L}[\Psi_M + E_M/n - S_M] = 0.$$

The operator $\mathcal{L}$ is the one that appeared in tensor perturbations, and so the mode solutions are already known:

$$\Psi_M + E_M/n - S_M = c_2^S\rho, \tag{8.105}$$

where c_2^S is another constant. Substituting Φ_M and B_M, the constraints (8.82) and (8.84) reduce to two algebraic equations for Ψ_M, E_M, and S_M as

$$n\Psi_M + mS_M = -c_1^S(-\eta)^2\rho, \tag{8.106}$$

$$\Psi_M + (1/n - 1)E_M = -c_1^S(-\eta)^2\rho - c_1^S k^{-2} \times\left[(2\mu - 1)\eta\rho' + (\nu^2 - 3\mu^2 + 2\mu)\rho\right]. \tag{8.107}$$

Solving Eqs. (8.105), (8.106), and (8.107), we obtain the expressions for Ψ_M, E_M, and S_M. Thus, all the metric variables can be analytically solved. Note that one of the above two independent solutions was already obtained in Ref. [75, 77].

The zero-mode solution is also easily obtained. In this background, the master equation becomes

$$\hat{\Psi}'' - \frac{m+n-3}{\eta}\hat{\Psi}' + k^2\hat{\Psi} = 0, \tag{8.108}$$

and the solution is

$$\hat{\Psi} = c^S(-\eta)^{\mu-1}H^{(1)}_{\mu-1}(-k\eta). \tag{8.109}$$

8.5 Summary

We have shown that a wide class of braneworld models with bulk scalar fields can be constructed by dimensional reduction from a higher dimensional extension of the Randall-Sundrum model with an empty bulk. The sizes of compactified dimensions translate into scalar fields with exponential potentials both in the bulk and on the brane. We have mainly concentrated on models with a single scalar field, which include the power-law inflation solution of Ref. [75, 77].

First we have investigated the evolution of five[$= n+2$]-dimensional background cosmologies, giving an intuitive interpretation based on the four[$= n+1$]-dimensional effective description. Then we have studied cosmological perturbations in such braneworld models. Lifting the models to $(5+m)[= n+2+m]$-dimensions is a powerful technique for this purpose. The degrees of freedom of a bulk scalar field in $(n+2)$-dimensions are deduced from a purely gravitational theory in the $(n+2+m)$-dimensional Randall-Sundrum braneworld, which consists of a vacuum brane and an empty bulk. We would like to emphasize that the analysis is greatly simplified thanks to the absence of matter fields. From the $(n+2+m)$-dimensional perspective, we have derived master equations for all types of perturbations. We have shown that mode decomposition is possible for all models which are constructed by using this dimensional reduction technique. Moreover, the dependence in the direction of the extra dimension perpendicular to the brane can always be solved analytically.

As for scalar perturbations, there are two physical degrees of freedom for the massive modes and the equations are not decoupled in general. For the zero mode, however, the situation is equivalent to the standard four-dimensional inflation driven by a single scalar field. Hence, only one degree of freedom is physical. Therefore we end up with a single master equation. To sum up, our "embedding and reduction" approach enables a systematic study of cosmological perturbations in a class of braneworld models with bulk scalar fields.

In this chapter, we have not discussed quantum mechanical aspects. In order to evaluate the amplitude of the quantum fluctuations, the overall normalization factor of the perturbations must be determined. For this purpose, one needs to write down the perturbed action up to the second order written solely in terms of physical degrees of freedom as is done in the standard cosmological perturbation theory.

In this chapter our investigation is restricted to the parameter region $m > 0$, where m is the number of compactified dimensions. If $m > 0$, a singularity could exist at $z = \infty$, but it is null. For $m < -3$, we have the right sign for the kinetic term of the scalar field and so it is possible to consider such models. In this parameter region, however, there is a timelike singularity at $z=0$ and therefore we need a regulator brane to hide it. This case includes the cosmological solution of heterotic M theory [97] (which corresponds to $m=-18/5$), and the analysis of cosmological perturbations in such two-brane models [133] would also be meaningful. This issue is left for future work.

8.6 Appendix: Perturbed Einstein equations and gauge transformations

In this section, we write down the components of the perturbed Einstein equations

$$\delta R_{AB} = \nabla_C\nabla_{(A}\delta G_{B)}{}^C - \frac{1}{2}\Box\delta G_{AB} - \frac{1}{2}\nabla_A\nabla_B\delta G = -\frac{1+m+n}{\ell^2}\delta G_{AB}.$$

The perturbed quantities are decomposed into scalar, vector, and tensor components whose basic definitions are given by Eq. (8.49). Note that in the following expressions no gauge conditions have been imposed yet.

- $\{i,j\}$-component

$$\begin{aligned}
&\frac{1}{2}\left[\ddot{h}_{ij} + (m\dot\alpha + n\dot\beta)\dot{h}_{ij} - h''_{ij} - (m+n)\omega' h'_{ij} - e^{-2\beta}\partial^k\partial_k h_{ij} + 2e^{-2\beta}\partial^k\partial_{(i}h_{j)k}\right]\\
&-\left[\partial_{(i}\dot{B}_{j)} + (m\dot\alpha + n\dot\beta)\partial_{(i}B_{j)} - \partial_{(i}C'_{j)} - (m+n)\omega'\partial_{(i}C_{j)}\right]\\
&-\delta_{ij}\left(\dot\beta\,\partial^k B_k - \omega'\partial^k C_k\right) + \delta_{ij}\Big\{2(m+n)H_0^2(N-\Phi) + 2(1+m+n)\omega'' N + \omega' N'\\
&-\left[m\dot\alpha + (m+2n)\dot\beta\right]\omega' A - \dot\beta A' - \omega'\dot{A} - \omega'\left(\Phi + mS + n\Psi\right)'\\
&+\dot\beta\left(N - \Phi + mS + n\Psi\right)^{\cdot}\Big\} - e^{-2\beta}\partial_i\partial_j\left(N + \Phi + mS + n\Psi\right) = 0,
\end{aligned} \tag{8.110}$$

where $h_{ij} = 2\Psi\delta_{ij} + 2E_{ij}$ and the dot (prime) denotes $\partial/\partial t$ $(\partial/\partial z)$. (We use the prime to denote differentiation with respect to z only in Sec. 8.6.)

Trace Part:

$$\begin{aligned}
&\ddot\Psi + (m\dot\alpha + n\dot\beta)\dot\Psi + e^{-2\beta}k^2\Psi - \Psi'' - (m+n)\omega'\Psi'\\
&-e^{-2\beta}k^2\frac{2}{n}\left[\Psi + \left(\frac{1}{n} - 1\right)E\right] - k\left[\dot{B} + \left(m\dot\alpha + 2n\dot\beta\right)B - C' - (m+2n)\omega' C\right]\frac{1}{n}\\
&+\Big\{2(m+n)H_0^2(N-\Phi) + 2(1+m+n)\omega'' N + \omega' N'\\
&-\left[m\dot\alpha + (m+2n)\dot\beta\right]\omega' A - \dot\beta A' - \omega'\dot{A} - \omega'\left(\Phi + mS + n\Psi\right)'\\
&+\dot\beta\left(N - \Phi + mS + n\Psi\right)^{\cdot}\Big\}\\
&+e^{-2\beta}k^2\left(N + \Phi + mS + n\Psi\right)\frac{1}{n} = 0.
\end{aligned} \tag{8.111}$$

Trace-free Part:

$$\begin{aligned}
&\ddot{E} + (m\dot\alpha + n\dot\beta)\dot{E} + e^{-2\beta}k^2 E - E'' - (m+n)\omega' E' + 2e^{-2\beta}k^2\left[\Psi + \left(\frac{1}{n} - 1\right)E\right]\\
&+k\left[\dot{B} + (m\dot\alpha + n\dot\beta)B\right] - k\left[C' + (m+n)\omega' C\right]\\
&-e^{-2\beta}k^2\left(N + \Phi + mS + n\Psi\right) = 0.
\end{aligned} \tag{8.112}$$

Vector:

$$\ddot{E}_i^V + (m\dot{\alpha} + n\dot{\beta})\dot{E}_i^V - E_i^{V\prime\prime} - (m+n)\omega' E_i^{V\prime}$$
$$+k\left[\dot{B}_i^V + (m\dot{\alpha} + n\dot{\beta})B_i^V\right] - k\left[C_i^{V\prime} + (m+n)\omega' C_i^V\right] = 0. \tag{8.113}$$

Tensor:

$$\left[\partial_t^2 + (m\dot{\alpha} + n\dot{\beta})\partial_t + e^{-2\beta}k^2 - \partial_z^2 - (m+n)\omega'\partial_z\right] E_{ij}^T = 0. \tag{8.114}$$

- $\{z, i\}$-component

$$\left(\partial^j h_{ij}\right)' - (\partial_i A)^{\cdot} - \left(m\dot{\alpha} + n\dot{\beta} - 2\dot{\beta}\right)\partial_i A + \partial^j\partial_i C_j - \partial^j\partial_j C_i$$
$$-2\partial_i\left(\Phi' + mS' + n\Psi'\right) + 2(m+n)\omega'\partial_i N$$
$$+e^{2\beta}\left[\ddot{C}_i - \dot{B}_i{}' + \left(m\dot{\alpha} + n\dot{\beta} + 2\dot{\beta}\right)\left(\dot{C}_i - B_i{}'\right)\right] = 0. \tag{8.115}$$

Scalar:

$$k\left[2\Psi' + 2\left(\frac{1}{n} - 1\right)E' - \dot{A} - \left(m\dot{\alpha} + n\dot{\beta} - 2\dot{\beta}\right)A - 2\left(\Phi' + mS' + n\Psi'\right)\right.$$
$$\left.+2(m+n)\omega' N\right] - e^{2\beta}\left[\ddot{C} - \dot{B}' + \left(m\dot{\alpha} + n\dot{\beta} + 2\dot{\beta}\right)\left(\dot{C} - B'\right)\right] = 0. \tag{8.116}$$

Vector:

$$kE_i^{V\prime} + k^2 C_i^V + e^{2\beta}\left[\ddot{C}_i^V - \dot{B}_i^{V\prime} + \left(m\dot{\alpha} + n\dot{\beta} + 2\dot{\beta}\right)\left(\dot{C}_i^V - B_i^{V\prime}\right)\right] = 0. \tag{8.117}$$

- $\{t, i\}$-component

$$\left(\partial^j h_{ij}\right)^{\cdot} + (\partial_i A)' + (m+n)\omega'\partial_i A + \partial^j\partial_i B_j - \partial^j\partial_j B_i$$
$$+2\left(m\dot{\alpha} + n\dot{\beta} - \dot{\beta}\right)\partial_i\Phi - 2\partial_i\left(\dot{N} + m\dot{S} + n\dot{\Psi}\right) + 2\partial_i\left[\dot{\beta}N + m\left(\dot{\beta} - \dot{\alpha}\right)S\right]$$
$$+e^{2\beta}\left[-B_i{}'' + \dot{C}_i{}' + (m+n)\omega'\left(\dot{C}_i - B_i{}'\right)\right] = 0. \tag{8.118}$$

Scalar:

$$k\left[2\dot{\Psi} + 2\left(\frac{1}{n} - 1\right)\dot{E} + A' + (m+n)\omega' A + 2\left(m\dot{\alpha} + n\dot{\beta} - \dot{\beta}\right)\Phi\right.$$
$$\left.-2\left(\dot{N} + m\dot{S} + n\dot{\Psi}\right) + 2\dot{\beta}N + 2m\left(\dot{\beta} - \dot{\alpha}\right)S\right]$$
$$-e^{2\beta}\left[-B'' + \dot{C}' + (m+n)\omega'\left(\dot{C} - B'\right)\right] = 0. \tag{8.119}$$

Vector:

$$k\dot{E}_i^V + k^2 B_i + e^{2\beta}\left[-B_i^{V\prime\prime} + \dot{C}_i^{V\prime} + (m+n)\omega'\left(\dot{C}_i^V - B_i^{V\prime}\right)\right] = 0. \tag{8.120}$$

- $\{t,t\}$-component

$$k\left(\dot{B}+2\dot{\beta}B-\omega' C\right)-(n\Psi+mS)^{\cdot\cdot}+\omega'\left(\Phi+n\Psi+mS\right)'+\dot{A}'+(1+m+n)\omega'\dot{A}$$
$$+\Phi''+(m+n)\omega'\Phi'-e^{-2\beta}k^2\Phi+\left(m\dot{\alpha}+n\dot{\beta}\right)\dot{\Phi}-2n\dot{\beta}\dot{\Psi}-2m\dot{\alpha}\dot{S}$$
$$-\ddot{N}-\omega' N'-2(1+m+n)\omega'' N-2(m+n)H_0^2(N-\Phi)=0. \tag{8.121}$$

- $\{z,z\}$-component

$$\ddot{N}+e^{-2\beta}k^2N+k\left(C'+\omega' C\right)-\left(\Phi+n\Psi+mS\right)''-\omega'\left(\Phi+n\Psi+mS\right)'$$
$$-\left(\dot{A}'+\omega'\dot{A}\right)-\left(m\dot{\alpha}+n\dot{\beta}\right)\left(A'+\omega' A-\dot{N}\right)$$
$$+(1+m+n)\left(\omega' N'+2\omega'' N\right)=0. \tag{8.122}$$

- $\{t,z\}$-component

$$k\left(\dot{C}+2\dot{\beta}C+B'\right)+e^{-2\beta}k^2A+2\left(m\dot{\alpha}+n\dot{\beta}\right)\Phi'+2\omega'(m+n)\dot{N}$$
$$-2\left(n\dot{\Psi}'+m\dot{S}'\right)-2n\dot{\beta}\Psi'-2m\dot{\alpha}S'-2(m+n)H_0^2A=0. \tag{8.123}$$

- $\{\mu,\nu\}$-component

$$\ddot{S}+\left(2m\dot{\alpha}+n\dot{\beta}\right)\dot{S}-S''-(2m+n)\omega' S'+\left[e^{-2\beta}k^2-2e^{-2\alpha}K(m-1)\right]S$$
$$+n\dot{\alpha}\dot{\Psi}-\dot{\alpha}\dot{\Phi}-\omega'\left(\Phi'+n\Psi'\right)+2e^{-2\alpha}K(m-1)\Phi+k\left(\omega' C-\dot{\alpha}B\right)$$
$$+\dot{\alpha}\dot{N}+\omega' N'+2(1+m+n)\omega'' N-\omega'\dot{A}-\dot{\alpha}A'-\left[(2m+n)\dot{\alpha}+n\beta\right]\omega' A$$
$$+2(m+n)H_0^2(N-\Phi)=0. \tag{8.124}$$

Lastly we summarize the gauge transformations of the metric variables. Under a scalar gauge transformation,

$$\begin{aligned} t &\rightarrow \quad \bar{t}=t+\xi^t, \\ z &\rightarrow \quad \bar{z}=z+\xi^z, \\ x^i &\rightarrow \quad \bar{x}^i=x^i+\frac{k^i}{ik}\xi^S, \end{aligned} \tag{8.125}$$

the metric variables transform as

$$\begin{aligned} N &\rightarrow \quad \bar{N}=N-\xi^{z\prime}-\omega'\xi^z, \\ A &\rightarrow \quad \bar{A}=A+\xi^{t\prime}-\dot{\xi}^z, \\ C &\rightarrow \quad \bar{C}=C+e^{-2\beta}k\xi^z-\xi^{S\prime} \\ B &\rightarrow \quad \bar{B}=B-e^{-2\beta}k\xi^t-\dot{\xi}^S, \\ \Phi &\rightarrow \quad \bar{\Phi}=\Phi-\dot{\xi}^t-\omega'\xi^z, \\ \Psi &\rightarrow \quad \bar{\Psi}=\Psi-\frac{1}{n}k\xi^S-\omega'\xi^z-\dot{\beta}\xi^t, \\ E &\rightarrow \quad \bar{E}=E+k\xi^S, \\ S &\rightarrow \quad \bar{S}=S-\omega'\xi^z-\dot{\alpha}\xi^t. \end{aligned} \tag{8.126}$$

8.7 Appendix: Multiple scalar field generalization

Let us give the generalization of the Kasner-type metric discussed in Sec. 8.3. First we generalize the case without Λ_b but including the curvature for one of the spatial sections:

$$g_{MN}dx^M dx^N = e^{2\alpha(\eta)}\left[-d\eta^2 + \gamma_{\mu\nu}dy^\mu dy^\nu\right] + \sum_{i=1}^{\mathcal{N}} e^{2\gamma_i(\eta)}\delta^{(i)}_{MN}dx^M dx^N,$$

where $\delta^{(i)}_{MN}$ is the metric of a j_i-dimensional flat space and $\gamma_{\mu\nu}$ is the metric of a $\bar{m}$-dimensional maximally symmetric space. As before we identified j_1 with n. If we compactify $(\bar{m}+\sum_{i=2}^{\mathcal{N}} j_i)$ dimensions leaving $(n+1)$ dimensions, the compactified space is divided into $\mathcal{N}$ sectors having different scale factors. Then the resulting cosmology after dimensional reduction will possess $\mathcal{N}$ scalar fields.

The set of vacuum Einstein equations becomes

$$e^{2\alpha}R_\eta{}^\eta = \sum j_i\gamma_i'^2 - \alpha' u + u' + \bar{m}\alpha'' = 0, \tag{8.127}$$

$$e^{2\alpha}R_\mu{}^\nu = \delta_\mu{}^\nu\left[K(\bar{m}-1) + \alpha' u + (\bar{m}-1)\alpha'^2 + \alpha''\right] = 0, \tag{8.128}$$

$$e^{2\alpha}R_M^N\delta^{(i)}_{NL} = \delta^{(i)}_{ML}\left[\gamma_i' u + (\bar{m}-1)\alpha'\gamma_i' + \gamma_i''\right] = 0, \tag{8.129}$$

where we have introduced

$$u := \sum j_i\gamma_i'. \tag{8.130}$$

From Eq. (8.129), we obtain

$$u^2 + (\bar{m}-1)\alpha' u + u' = 0. \tag{8.131}$$

Then it is easy to see that Eqs. (8.128) and (8.130) are equivalent to Eqs. (8.29) and (8.30) in the example of Sec. 8.3.1 by identifying u with $n\beta'$. Therefore the solution of Eqs. (8.128) and (8.130) for $K = 1$ is written as

$$u = \frac{\pm(\bar{m}-1)\bar{q}}{\sin[(\bar{m}-1)\eta]}, \quad \alpha' = \cot[(\bar{m}-1)\eta] \mp \frac{\bar{q}}{\sin[(\bar{m}-1)\eta]}. \tag{8.132}$$

Since these two equations (8.128) and (8.130) do not have dependence on the number of dimensions, $\bar{q}$ has not been fixed yet. Substituting this solution into Eq. (8.129), we obtain

$$(\bar{m}-1)\gamma_i'\cot[(\bar{m}-1)\eta] + \gamma_i'' = 0,$$

which is integrated to give

$$\gamma_i' = \frac{(\bar{m}-1)c_i}{\sin[(\bar{m}-1)\eta]},$$

where c_i is an integration constant. Substituting this into the remaining equation (8.127) and the definition of u (8.130), we find that the solution is given by

$$e^{(\bar{m}-1)\alpha} = \sin[(\bar{m}-1)\eta]\left[\cot\left(\frac{\bar{m}-1}{2}\eta\right)\right]^{\pm\bar{q}},$$

$$e^{\gamma_i} = \left[\tan\left(\frac{\bar{m}-1}{2}\eta\right)\right]^{c_i},$$

with

$$\sum j_i c_i^2 = \frac{\bar{m} - \bar{q}^2}{\bar{m} - 1}, \quad \sum j_i c_i = \pm \bar{q}.$$

The next is a generalization of the Kasner-type spacetime including a cosmological constant Λ_b. Let us assume that the metric is in the form of

$$g_{MN} dx^M dx^N = -dt^2 + \sum_{i=0}^{\mathcal{N}} e^{2\gamma_i(t)} \delta_{MN}^{(i)} dx^a dx^b. \tag{8.133}$$

Here all the spatial sections are taken to be flat ($K_i = 0$), because otherwise an analytic solution with $\Lambda_b \neq 0$ cannot be found. The Einstein equations $R_{MN}[g] = NH_0^2 g_{MN}$, with $N := D - 1 = \sum n_i$ reduce to

$$\sum j_i \left(\ddot{\gamma}_i + \dot{\gamma}_i^2 \right) = NH_0^2, \tag{8.134}$$

$$\ddot{\gamma}_i + \dot{\gamma}_i \sum j_i \dot{\gamma}_i = NH_0^2, \tag{8.135}$$

where the overdot denotes differentiation with respect to t. These equations admit a trivial solution of D-dimensional de Sitter spacetime,

$$\gamma_i = H_0 t. \tag{8.136}$$

There is another type of non-trivial solution. From Eq. (8.135) we find that $u := \sum j_i \dot{\gamma}_i$ obeys

$$\dot{u} + u^2 = N^2 H_0^2.$$

The solution for this equation is

$$u = NH_0 \coth(NH_0 t).$$

Substituting this into Eq. (8.135), we obtain

$$[\sinh(NH_0 t)\dot{\gamma}_i]^{\cdot} = NH_0^2 \sinh(NH_0 t).$$

This can be easily integrated and the integration constants are determined from Eq. (8.134). Then we have

$$\dot{\gamma}_i = \frac{H_0 \left[\cosh(NH_0 t) + c_i\right]}{\sinh(NH_0 t)}, \tag{8.137}$$

$$\sum j_i c_i = 0, \quad \sum j_i c_i^2 = N(N-1). \tag{8.138}$$

Finally, integrating Eq. (8.137), we obtain

$$e^{N\gamma_i} = \sinh(NH_0 t) \left[\tanh\left(\frac{NH_0}{2} t\right)\right]^{c_i}. \tag{8.139}$$

Integration constants were removed by rescaling the spatial coordinates. There are only these two types of solutions (8.136) and (8.139) for Eqs. (8.134) and (8.135).

Chapter 9

Conclusions

In this thesis we have discussed aspects of cosmological perturbations in the Randall-Sundrum-type braneworlds. In Chapter 4 we have calculated leading order corrections to the evolution of tensor perturbations in the RS2 braneworld cosmology at low energies by using the perturbative expansion scheme of Ref. [139], and analytically determined the precise numerical factors of the correction terms of $\mathcal{O}(\ell^2)$ and $\mathcal{O}(\ell^2 \ln \ell)$. Thus we have confirmed that leading order corrections are indeed small and conventional four-dimensional cosmology can be reproduced at low energies on the brane.

In Chapter 5 we have introduced the so called "junction model," in which two maximally symmetric branes are joined in order to investigate the effect of nontrivial motion of the brane, and calculated the primordial spectrum of gravitational waves generated quantum-mechanically from inflation. Through the analysis of this toy model, we have conjectured that the primordial tensor spectrum in the RS braneworld is quite well approximated by the "rescaled" spectrum obtained by using a simple map speculated by the exact result of pure de Sitter inflation. To check this, in Chapter 6 we have developed a new numerical method based on the Wronskian formulation. Our numerical results strongly indicate that the above speculation is correct in general inflationary models. The vacuum fluctuations of the initial KK gravitons contribute to the final amplitude of the zero mode at a significant level for high-energy inflation, and thus we have obtained an interesting picture: when the expansion rate changes during inflation, zero mode gravitons escape into the bulk as KK gravitons, but at the same time bulk gravitons come onto the brane to compensate for the loss, and these two effects almost cancel each other. In Chapter 7, we have followed the same line as the previous chapter and numerically studied the late time power spectrum of tensor perturbations, focusing on the high-energy effects in the radiation-dominated stage. In our analysis, initial conditions of perturbations are imposed quantum-mechanically. The issue of initial conditions has been less considered in previous studies. We have found that the effect of initial KK vacuum fluctuations are subdominant, contributing not more than 10% of the total power spectrum so far as the present calculation is concerned, and thus the damping due to the generation of KK modes and the enhancement due to the modification of the background Friedmann equation mainly work. The combination of these two effects leads to the same spectral tilt as the standard four-dimensional result. We have also estimated the energy density of the "dark radiation," which is generated via KK mode excitation, and showed that only a tiny amount is generated due to that process.

The topic of Chapter 8 was on more complicated braneworld models with scalar fields in the bulk. However, concerning the bulk inflaton models in Chapter 8 the analysis of perturbations is essentially the same as that of the simplest setup: tensor perturbations in the Randall-Sundrum-type de Sitter braneworld. The key idea is that the five-dimensional bulk inflaton models can be described by $(5+m)$-dimensional vacuum gravity via dimensional reduction. We have shown that various background solutions including the power-law inflation model [75, 77] are generated from known $(5+m)$-dimensional vacuum solutions of pure gravity, and derived master equations supplemented by simple boundary conditions for all types of perturbations.

As a final remark, a number of issues still remain to be solved. In this thesis we have elaborated mainly on tensor perturbations in the Randall-Sundrum-type braneworld, but of course it is important to study scalar cosmological perturbations, aiming to give a clear prediction about the CMB temperature anisotropy on the brane. The generation and evolution of curvature perturbations are much more complicated than that of tensor perturbations, because the problem reduces to solving the system of coupled brane-bulk fields [80]. We believe that the approaches reviewed in this thesis will be helpful as well for understanding the behavior of scalar perturbations in the braneworld. From the point of view of general relativistic cosmology, the analysis of the Randall-Sundrum-type models is accessible thanks to their simple construction. It would be interesting to explore general relativistic and cosmological consequences of more "stringy" models than those relying on just five-dimensional AdS geometry, going in the direction of, e.g., Refs. [59, 116] inspired by the recent seminal proposal of KKLT [55] and KKLMMT [56].

Acknowledgments

First of all I would like to thank Takahiro Tanaka for valuable discussions and collaborations. He has guided me through the fields of general relativity and cosmology. I have learned a lot from him during these five years. It is my pleasure to thank Takashi Nakamura, head of the theoretical astrophysics group, who provides an encouraging atmosphere around me. If I have succeeded in making my work scientifically interesting and important, I owe it to his critical comments. I am grateful to all the colleagues in Department of Physics at Kyoto University and Yukawa Institute for Theoretical Physics, especially to Hiroyuki Tashiro, Teruaki Suyama, Shuichiro Yokoyama, and Makoto Yoshikawa, for everyday discussions and conversations (and for an occasional drink). Many thanks to my parents, my friends, and my favorite rock music! T.K. is supported by the JSPS.

Appendix

Appendix A

Review of cosmological perturbation theory in four dimensions

Here we give a brief review of cosmological perturbation theory in the conventional four-dimensional Universe. For comprehensive reviews, see Refs. [67, 110, 4, 94].

A.1 Governing equations

A.1.1 The unperturbed Universe

The unperturbed Universe is described by the Friedmann-Robertson-Walker spacetime, whose metric is given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{A.1}$$

Just for simplicity we assume that the geometry of a homogeneous and isotropic 3-space is flat. The energy-momentum tensor which is compatible with homogeneity and isotropy of 3-space is that of a perfect fluid:

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}. \tag{A.2}$$

The components of the Einstein equations are

$$G^0{}_0 = -3H^2 = -8\pi G\rho, \tag{A.3}$$

$$G^i{}_j = -\left(3H^2 + 2\dot{H}\right)\delta^i{}_j = 8\pi G p \delta^i{}_j, \tag{A.4}$$

where $H := \dot{a}/a = (da/dt)/a$ is the Hubble parameter. The first equation is called the Friedmann equation. The energy-momentum conservation equation $\nabla_\mu T^\mu{}_\nu = 0$ reads

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{A.5}$$

A.1.2 Perturbed metric and energy-momentum tensor

Perturbed quantities can be decomposed into scalar, vector, and tensor parts according to their transformation properties with respect to the three-dimensional space, where scalar parts are related to a scalar potential, vector parts to transverse vectors and tensor parts to transverse, traceless tensors. Such a splitting is found to be quite useful to study cosmological perturbation theory at least in linear order, because the different modes are decoupled from each other in the field equations.

Scalar perturbations

We write the perturbed metric as

$$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dt dx^i + a^2\left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j. \tag{A.6}$$

The components of the Christoffel symbol and the perturbed Einstein tensor are, respectively,

$$\begin{aligned}
&\Gamma^0_{00} = \dot{A}, \quad \Gamma^i_{00} = \partial^i\left[A/a^2 + \left(\dot{B} + HB\right)/a\right], \quad \Gamma^0_{0i} = \partial_i\left(A + aHB\right),\\
&\Gamma^0_{ij} = a^2 H\delta_{ij} - 2a^2 HA\delta_{ij} - a\partial_i\partial_j B + \frac{d}{dt}\left[a^2\left(-\psi\delta_{ij} + \partial_i\partial_j E\right)\right],\\
&\Gamma^i_{0j} = H\delta^i{}_j - \dot{\psi}\delta^i{}_j + \partial^i\partial_j\dot{E},\\
&\Gamma^k_{ij} = -aH\partial^k B\delta_{ij} - \partial_i\psi\delta^k{}_j - \partial_j\psi\delta^k{}_i - \partial^k\psi\delta_{ij} + \partial^k\partial_i\partial_j E,
\end{aligned}$$

and

$$\delta G^0{}_0 = 6H\left(\dot{\psi} + HA\right) - 2a^{-2}\partial^k\partial_k\left[\psi + H\left(a^2\dot{E} - aB\right)\right], \tag{A.7}$$

$$\delta G^0{}_i = -2\partial_i\left(\dot{\psi} + HA\right), \tag{A.8}$$

$$\begin{aligned}
\delta G^i{}_j = {} & \left(\partial^i\partial_j - \delta^i{}_j\partial^k\partial_k\right)\left[a^{-2}(\psi - A) + \left(\dot{E} - B/a\right)^{\cdot} + 3H\left(\dot{E} - B/a\right)\right]\\
& +2\left[\ddot{\psi} + 3H\dot{\psi} + H\dot{A} + \left(3H^2 + 2\dot{H}\right)A\right]\delta^i{}_j.
\end{aligned} \tag{A.9}$$

The fluid four-velocity is given by

$$u^\mu = \left(1 - A,\ a^{-1}\partial^i v\right), \quad u_\mu := g_{\mu\nu}u^\nu = \left(-1 - A,\ a\partial_i(v + B)\right), \tag{A.10}$$

and then the perturbed energy-momentum tensor is found to be

$$\delta T^0{}_0 = -\delta\rho, \tag{A.11}$$

$$\delta T^0{}_i = \partial_i\left[a(\rho + p)(v + B)\right] =: \partial_i\delta q, \tag{A.12}$$

$$\delta T^i{}_j = \delta p\delta^i{}_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\partial^k\partial_k\right)\delta\Pi. \tag{A.13}$$

The perturbed quantities are generally not invariant under a scalar gauge transformation

$$t \to t + \delta t, \quad x^i \to x^i + \partial^i\delta x, \tag{A.14}$$

but they transform as

$$
\begin{aligned}
A &\rightarrow A - \dot{\delta t}, &\text{(A.15)}\\
B &\rightarrow B + a^{-1}\delta t - a\dot{\delta x}, &\text{(A.16)}\\
\psi &\rightarrow \psi + H\delta t, &\text{(A.17)}\\
E &\rightarrow E - \delta x, &\text{(A.18)}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta\rho &\rightarrow \delta\rho - \dot{\rho}\delta t, &\text{(A.19)}\\
\delta q &\rightarrow \delta q + (\rho + p)\delta t, &\text{(A.20)}\\
\delta\Pi &\rightarrow \delta\Pi. &\text{(A.21)}
\end{aligned}
$$

The following gauge-invariant combinations have been widely used in the literature:

$$
\begin{aligned}
\Phi &:= A - \frac{d}{dt}\left(a^2\dot{E} - aB\right), &\text{(A.22)}\\
\Psi &:= \psi + H\left(a^2\dot{E} - aB\right). &\text{(A.23)}
\end{aligned}
$$

In terms of these variables we can write the gauge-invariant Poisson equation

$$
\frac{k^2}{a^2}\Psi = -4\pi G\delta\epsilon, \tag{A.24}
$$

where we have defined the gauge-invariant comoving density perturbation

$$
\delta\epsilon := \delta\rho - 3H\delta q, \tag{A.25}
$$

and the spatial Laplacian has been replaced by their eigenvalue: $\partial^i\partial_i \rightarrow -k^2$. The traceless part of the Einstein equation $\delta G^i{}_j = 8\pi G T^i{}_j$ gives the constraint equation

$$
\Psi - \Phi = 8\pi G a^2 \delta\Pi. \tag{A.26}
$$

Thus we have $\Psi = \Phi$ when anisotropic stresses vanish, which is the case for a cosmic fluid in which there is negligible diffusion or freestreaming. Assuming that $\delta\Pi$ is negligible, we obtain the evolution equation for the potential

$$
\ddot{\Phi} + (4 + 3c_s^2)H\dot{\Phi} + \left[2\dot{H} + 3H^2(1 + c_s^2)\right]\Phi + c_s^2\frac{k^2}{a^2}\Phi = 4\pi G\delta p_{\rm nad}, \tag{A.27}
$$

where $c_s^2 := \dot{p}/\dot{\rho}$ is the sound velocity and the source term in the right hand side,

$$
\delta p_{\rm nad} := \delta p - c_s^2\delta\rho, \tag{A.28}
$$

is the non-adiabatic pressure perturbation or entropy perturbation.

The perturbation of energy-momentum conservation $\delta\left(\nabla_\mu T^\mu_\nu\right) = 0$ gives

$$
\begin{aligned}
\dot{\delta\rho} + 3H(\delta\rho + \delta p) - 3\dot{\psi}(\rho + p) &= \frac{k^2}{a^2}\left[\delta q + (\rho + p)\left(a^2\dot{E} - aB\right)\right], &\text{(A.29)}\\
\dot{\delta q} + 3H\delta q + (\rho + p)A + \delta p &= \frac{2}{3}k^2\delta\Pi. &\text{(A.30)}
\end{aligned}
$$

Using the curvature perturbation on uniform density hypersurfaces,

$$\zeta := -\psi - \frac{H}{\dot{\rho}}\delta\rho, \tag{A.31}$$

the first equation can be rewritten into a quite important expression:

$$\dot{\zeta} = -\frac{H}{\rho + p}\delta p_{\rm nad} + \mathcal{O}(k^2). \tag{A.32}$$

This implies that in the absence of entropy perturbations ζ is conserved on super-horizon scales ($k^2 \ll a^2H^2$). What should be stressed here is that the above result is obtained without invoking the Einstein equations; so long as energy-momentum conservation holds, we have Eq. (A.32) in a braneworld or in alternative theories of gravity [143].

Another gauge-invariant variable which is commonly used is the comoving curvature perturbation

$$\mathcal{R} := \psi - \frac{H}{\rho + p}\delta q. \tag{A.33}$$

Since with the aid of the Poisson equation (A.24) we have

$$\mathcal{R} + \zeta = -\frac{H}{\dot{\rho}}\delta\epsilon = \frac{H}{4\pi G\dot{\rho}}\frac{k^2}{a^2}\Psi, \tag{A.34}$$

$\mathcal{R}$ and $-\zeta$ coincide and thus $\mathcal{R}$ is constant for adiabatic perturbations on super-horizon scales.

Now let us consider a universe filled with a scalar field φ, for which the perturbed energy-momentum tensor is given by

$$\begin{aligned}
\delta T^0_{\ 0} &= -\delta\rho = -\left(\dot{\varphi}\dot{\delta\varphi} + V_{,\varphi}\delta\varphi - \dot{\varphi}^2 A\right), & \text{(A.35)}\\
\delta T^0_{\ i} &= \partial_i\delta q = \partial_i\left(-\dot{\varphi}\delta\varphi\right), & \text{(A.36)}\\
\delta T^i_{\ j} &= \delta p\delta^i_{\ j} = \left(\dot{\varphi}\dot{\delta\varphi} - V_{,\varphi}\delta\varphi - \dot{\varphi}^2 A\right)\delta^i_{\ j}. & \text{(A.37)}
\end{aligned}$$

Note that anisotropic stress $\delta\Pi$ vanishes for a scalar field (and for multiple scalar fields). The non-adiabatic pressure can be calculated as

$$\delta p_{\rm nad} = -\frac{2V_{,\varphi}}{3H\dot{\varphi}}\delta\epsilon, \tag{A.38}$$

from which we find that in the case of a single scalar field, perturbations become adiabatic on large scales.

The evolution equation for the gauge-invariant potential is obtained by using the Einstein equations as

$$\ddot{\Phi} + \left(H - 2\frac{\ddot{\varphi}}{\dot{\varphi}}\right)\dot{\Phi} + 2\left(\dot{H} - H\frac{\ddot{\varphi}}{\dot{\varphi}}\right)\Phi + \frac{k^2}{a^2}\Phi = 0. \tag{A.39}$$

Vector perturbations

The perturbed metric is

$$ds^2 = -dt^2 - 2aB_i dt dx^i + a^2 \left[\delta_{ij} + 2\partial_{(i}E_{j)}\right] dx^i dx^j, \tag{A.40}$$

where B_i and E_i are transverse vectors: $\partial^i B_i = 0$ and $\partial^i E_i = 0$.

The components of the perturbed Einstein tensor are

$$\begin{aligned} \delta G^0{}_i &= -\frac{1}{2}\partial^k \partial_k \sigma_i, & \text{(A.41)} \\ a^{-2}\delta G_{ij} &= \partial_{(i}\dot{\sigma}_{j)} + 3H\partial_{(i}\sigma_{j)} - 2\left(3H^2 + 2\dot{H}\right)\partial_{(i}E_{j)}, & \text{(A.42)} \end{aligned}$$

where

$$\sigma_i := \dot{E}_i + B_i/a, \tag{A.43}$$

while the perturbed energy-momentum tensor is given by

$$\begin{aligned} \delta T^0{}_i &= \delta q_i, & \text{(A.44)} \\ \delta T_{ij} &= 2p\partial_{(i}E_{j)} + \partial_{(i}\delta\Pi_{j)}. & \text{(A.45)} \end{aligned}$$

Under a vector gauge transformation $x^i \to x^i + \delta x^i$, the metric perturbations transform as

$$B_i \to B_i + a\dot{\delta x}_i, \quad E_i \to E_i - \delta x_i, \tag{A.46}$$

and so σ_i is a gauge-invariant quantity. Both δq_i and $\delta\Pi_i$ are gauge-invariant. Thus, we obtain the following equations that govern vector perturbations:

$$\begin{aligned} \dot{\sigma}_i + 3H\sigma_i &= 8\pi G\delta\Pi_i & \text{(A.47)} \\ k^2\sigma_i &= 16\pi G\delta q_i. & \text{(A.48)} \end{aligned}$$

Equation (A.47) shows that in the absence of anisotropic stress sources, vector metric perturbations decay away due to Hubble friction.

Tensor perturbations

Tensor perturbations correspond to gravitational waves. We write the perturbed metric as

$$ds^2 = -dt^2 + a^2\left(\delta_{ij} + h_{ij}\right) dx^i dx^j, \tag{A.49}$$

where h_{ij} is transverse and traceless: $\partial^i h_{ij} = 0$ and $\delta^{ij}h_{ij} = 0$. The equation of motion is given by

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 16\pi G\delta\Pi_{ij}, \tag{A.50}$$

where $\delta\Pi_{ij}$ stands for the tensor component of the matter anisotropic stress, and two possible polarization states, "+" and "×", are suppressed.

A.2 Quantum theory of perturbations

Quantization of fields in a nontrivial background generally gives rise to particle production, which is responsible for structure formation in the inflationary universe and thus provides the *initial conditions* for the classical evolution of perturbations described in the previous section. We now explain how we quantize cosmological perturbations.

Scalar perturbations

Quantization of scalar perturbations in single-field inflation is based on the comoving curvature perturbation,

$$\mathcal{R} = \psi + \frac{H}{\dot{\varphi}}\delta\varphi, \tag{A.51}$$

and in terms of this we define the new variable [110]

$$u := z\mathcal{R} = a\left(\delta\varphi + \frac{\dot{\varphi}}{H}\psi\right), \tag{A.52}$$

where

$$z := \frac{a\dot{\varphi}}{H}. \tag{A.53}$$

Then, the Einstein-Hilbert action truncated to quadratic order in a small fluctuation u is given by

$$S = \frac{1}{2}\int d\eta d^3x\mathcal{L} = \frac{1}{2}\int d\eta d^3x\left[(\partial_\eta u)^2 - \delta^{ij}\partial_i u\partial_j u + \frac{z''}{z}u^2\right], \tag{A.54}$$

where η is the conformal time and a prime denotes partial differentiation with respect to η. This is equivalent to the action for a scalar field in flat spacetime with a time-dependent effective mass $m^2 = -z''/z$, and here its origin is attributed to the variation of the background spacetime.

The theory is now quantized by promoting u and its conjugate momentum,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_\eta u)} = \partial_\eta u, \tag{A.55}$$

to operators satisfying the following equal time commutation relations:

$$[\hat{u}(\eta,\mathbf{x}),\hat{\pi}(\eta,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{A.56}$$

$$[\hat{u}(\eta,\mathbf{x}),\hat{u}(\eta,\mathbf{y})] = [\hat{\pi}(\eta,\mathbf{x}),\hat{\pi}(\eta,\mathbf{y})] = 0. \tag{A.57}$$

The operator $\hat{u}$ can be expanded as

$$\hat{u}(\eta,\mathbf{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\left[u_k(\eta)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{A.58}$$

where the mode u_k and its complex conjugate u_k^* form a complete orthonormal basis with respect to the Wronskian:

$$(u_k\cdot u_k) = -\left(u_k^*\cdot u_k^*\right) = 1, \quad (u_k\cdot u_k^*) = 0, \tag{A.59}$$

where

$$(\phi \cdot \varphi) := -i\left(\phi \partial_\eta \varphi^* - \varphi^* \partial_\eta \phi\right). \tag{A.60}$$

The equal time commutation relations for $\hat{u}$ and $\hat{\pi}$ imply the usual commutation relations for the annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$:

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^{(3)}\left(\mathbf{k} - \mathbf{k}'\right), \tag{A.61}$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0. \tag{A.62}$$

Then the vacuum is defined as the state annihilated by $\hat{a}_{\mathbf{k}}$, so that $\hat{a}_{\mathbf{k}}|0\rangle = 0$.

The field equation for u_k is given by

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0, \tag{A.63}$$

and the correct form of the mode is determined so that ordinary quantum field theory in flat spacetime is reproduced at short distances. Namely, we impose, in the limit of $k/aH \to \infty$,

$$u_k \to \frac{1}{\sqrt{2k}} e^{-ik\eta}. \tag{A.64}$$

The spectrum of the comoving curvature perturbation is defined by

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = \frac{(2\pi)^3}{4\pi k^3} \delta^{(3)}(\mathbf{k} + \mathbf{k}') \mathcal{P}_{\mathcal{R}}(k), \tag{A.65}$$

where the statistical average $\langle \cdots \rangle$ can now be replaced by the expectation value $\langle 0| \cdots |0\rangle$, leading to the formula

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \left| \frac{u_k}{z} \right|^2. \tag{A.66}$$

Tensor perturbations

For tensor perturbations, the Einstein-Hilbert action truncated to quadratic order reduces to

$$S = \frac{M_{\mathrm{Pl}}^2}{8} \int d\eta d^3x \; a^2(\eta) \partial_\mu h_{ij} \partial^\mu h^{ij}, \tag{A.67}$$

and defining $v^{(A)}$ $(A = +, \times)$ by

$$\frac{M_{\mathrm{Pl}}}{\sqrt{2}} a(\eta) h_{ij} = v^{(+)} e^{(+)}_{ij} + v^{(\times)} e^{(\times)}_{ij}, \tag{A.68}$$

we obtain the following effective action

$$S = \sum_{A=+,\times} \frac{1}{2} \int d\eta d^3x \left[\left(\partial_\eta v^{(A)}\right)^2 - \delta^{ij} \partial_i v^{(A)} \partial_j v^{(A)} + \frac{a''}{a} \left(v^{(A)}\right)^2 \right]. \tag{A.69}$$

For each polarization state, this gives the action equivalent to that for a scalar field in flat spacetime with a time-dependent mass $m^2 = -a''/a$. Thus, along the same path as scalar perturbations, we can quantize tensor perturbations and compute the power spectrum just by replacing z by a. Since a depends only on the background geometry, the situation is much easier now.

Unlike the case of scalar perturbations, where metric and matter perturbations are simultaneously quantized and hence a nonvanishing matter component is required, it is possible to quantize gravitational waves in a de Sitter background. For de Sitter inflation, we have $a''/a = 1/\eta^2$. The mode solution v_k that satisfies the condition $v_k \to e^{-ik\eta}/\sqrt{2k}$ as $k/aH = -k\eta \to \infty$ is given by

$$v_k = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{A.70}$$

Thus, the spectrum well after horizon exit is

$$\mathcal{P}_{\rm GW} = \frac{2}{M_{\rm Pl}^2} \cdot \frac{k^3}{2\pi^2} \left|\frac{v_k}{a}\right|^2 = \frac{2}{M_{\rm Pl}^2} \left(\frac{H}{2\pi}\right)^2. \tag{A.71}$$

The growing mode of tensor perturbations h_{ij} stays constant well outside the horizon and hence the amplitude is basically determined by the Hubble parameter at horizon exit. As a result, slow-roll (i.e., non de Sitter) inflation predicts a slightly tilted tensor spectrum:

$$\mathcal{P}_{\rm GW}(k) = \frac{2}{M_{\rm Pl}^2} \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH}. \tag{A.72}$$

For a detailed calculation including scalar power spectra, see Ref. [137].

Appendix B

Derivation of the effective Einstein equations on the brane

In this appendix we replicate the derivation of the well-known projected Einstein equations on the brane [134] (see also [101]). The extension to more complicated braneworld models is found in Refs. [103, 104, 105].

Our starting point is the five-dimensional Einstein equations that determine the bulk geometry:

$$^{(5)}G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 {}^{(5)}T_{AB}. \tag{B.1}$$

The Gauss and Codazzi equations relate the four-dimensional Riemann tensor $R_{\mu\nu\lambda\sigma}$, the extrinsic curvature $K_{\mu\nu}$ on the brane, and the five-dimensional curvature tensor as follows:

$$R_{\mu\nu\lambda\sigma} = {}^{(5)}R_{ABCD} q_\mu{}^A q_\nu{}^B q_\lambda{}^C q_\sigma{}^D + K_{\mu\lambda}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\lambda}, \tag{B.2}$$

$$\nabla_\nu K^\nu{}_\mu - \nabla_\mu K = {}^{(5)}R_{AB} q_\mu{}^A n^B, \tag{B.3}$$

where n^A is the unit normal to the brane, $q_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu$ is the induced metric on the brane, and ∇_μ is the covariant derivative associated with $q_{\mu\nu}$. Contracting μ and λ in the Gauss equation (B.2) we obtain the four-dimensional Ricci tensor:

$$R_{\mu\nu} = {}^{(5)}R_{AB} q_\mu{}^A q_\nu{}^B - {}^{(5)}R_{ABCD} n^A n^C q_\mu{}^B q_\nu{}^D + KK_{\mu\nu} - K_\mu{}^\lambda K_{\lambda\nu}. \tag{B.4}$$

It follows from Eqs. (B.1) and (B.4) that

$$\begin{aligned} G_{\mu\nu} \quad = \quad & -\frac{1}{2}\Lambda_5 + \frac{2}{3}\kappa_5^2 \left[{}^{(5)}T_{AB} q_\mu{}^A q_\nu{}^B + \left({}^{(5)}T_{AB} n^A n^B - \frac{1}{4} {}^{(5)}T \right) q_{\mu\nu} \right] \\ & + KK_{\mu\nu} - K_\mu{}^\lambda K_{\lambda\nu} + \frac{1}{2}\left(K^{\lambda\sigma}K_{\lambda\sigma} - K^2 \right) q_{\mu\nu} - E_{\mu\nu}, \end{aligned} \tag{B.5}$$

where $E_{\mu\nu}$ is the projected Weyl tensor[1]:

$$E_{\mu\nu} = {}^{(5)}C_{ABCD} n^A n^C q_\mu{}^B q_\nu{}^D. \tag{B.7}$$

[1]The d-dimensional Riemann tensor can be decomposed into Ricci and traceless parts:

$$R_{\mu\alpha\nu\beta} = \frac{2}{d-2}\left(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu} \right) - \frac{2}{(d-1)(d-2)} R g_{\mu[\nu}g_{\beta]\alpha} + C_{\mu\alpha\nu\beta}. \tag{B.6}$$

This is the definition of the Weyl tensor $C_{\mu\nu\lambda\sigma}$.

Let y be a Gaussian normal coordinate orthogonal to the brane so that $n_A dx^A = dy$. Let the energy-momentum tensor be of the form

$$
\begin{aligned}
{}^{(5)}T_{AB} &= \mathcal{T}_{AB} + S_{AB}\delta(y), &\text{(B.8)}\\
S_{AB} &= -\sigma q_{AB} + T_{AB}, &\text{(B.9)}
\end{aligned}
$$

where σ is the tension of the brane and T_{AB} is the contribution from brane matter. (The brane is located at $y = 0$ without loss of generality.) To make the final result as general as possible, we allow for the energy-momentum tensor $\mathcal{T}_{AB}$ of any bulk matter (e.g., dilaton and moduli fields). The junction conditions[2] imply that

$$
K_{\mu\nu}|_{y=0^+} = -\frac{\kappa_5^2}{2}\left(S_{\mu\nu} - \frac{1}{3}Sq_{\mu\nu}\right), \tag{B.10}
$$

where we assumed the Z_2-symmetry across the brane. Substituting this into Eq. (B.5), we obtain

$$
G_{\mu\nu} = -\Lambda q_{\mu\nu} + \kappa^2 T_{\mu\nu} + 6\frac{\kappa^2}{\sigma}\pi_{\mu\nu} - E_{\mu\nu} + 4\frac{\kappa^2}{\sigma}F_{\mu\nu}, \tag{B.11}
$$

where

$$
\begin{aligned}
\kappa^2 &= \frac{1}{6}\kappa_5^4\sigma, &\text{(B.12)}\\
\Lambda &= \frac{1}{2}\left(\Lambda_5 + \kappa^2\sigma\right), &\text{(B.13)}
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_{\mu\nu} &:= -\frac{1}{4}T_{\mu\alpha}T_\nu{}^\alpha + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}q_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}q_{\mu\nu}T^2, &\text{(B.14)}\\
\kappa_5^2 F_{\mu\nu} &:= \mathcal{T}_{AB}q_\mu{}^A q_\nu{}^B + \left(\mathcal{T}_{AB}n^A n^B - \frac{1}{4}\mathcal{T}\right)q_{\mu\nu}. &\text{(B.15)}
\end{aligned}
$$

This is the field equations projected on the brane. The effect of Kaluza-Klein modes is encoded into $E_{\mu\nu}$. If the bulk is empty except for the cosmological constant, $F_{\mu\nu}$ vanishes, as is the case in the Randall-Sundrum model. These two terms carry bulk information and therefore in general the geometry on the brane cannot be determined solely from the above

[2]The junction conditions will be most easily derived as follows. Let us consider a $(d-1)$-dimensional discontinuous (timelike) hypersurface in d-dimensional spacetime, and let the energy-momentum tensor be of the form ${}^{(d)}T_{AB} = \cdots + S_{\mu\nu}\delta(y)$, as in the main text. In the Gaussian normal coordinates we have

$$
K_{\mu\nu} = -\Gamma^y_{\mu\nu} = \frac{1}{2}\partial_y g_{\mu\nu},
$$

which is discontinuous across the hypersurface. It is straightforward to show that the Ricci tensor is expressed in terms of the extrinsic curvature as $R_{\mu\nu} = -\partial_y K_{\mu\nu} + \cdots$. Then we have

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} R_{\mu\nu}dy &= \kappa_d^2\left(S_{\mu\nu} - \frac{1}{d-2}Sq_{\mu\nu}\right)\\
&= -K_{\mu\nu}|_{y=0^+} + K_{\mu\nu}|_{y=0^-},
\end{aligned}
$$

where we used the d-dimensional Einstein equations in the first line. Here κ_d^2 is the d-dimensional gravitational constant. This together with the Z_2-symmetry $K_{\mu\nu}|_{y=0^+} = -K_{\mu\nu}|_{y=0^-}$ yields Eq. (B.10).

effective equations; one needs information on the bulk geometry which is determined from the full five-dimensional field equations. Nevertheless, the effective Einstein equations are useful in that they provide us some insight into what is the modification to the standard four-dimensional general relativity.

Using the five-dimensional Einstein equations (B.1), the Codazzi equation (B.3), and the junction condition (B.10), we can derive the (non-)conservation equation:

$$\nabla_\nu T^{\nu}_{\ \mu} = -2\mathcal{T}_{AB} n^A g^B_{\ \mu}, \tag{B.16}$$

which means that generally there is exchange of energy-momentum between the brane and the bulk. The standard conservation equation $\nabla_\nu T^{\nu}_{\ \mu} = 0$ holds in the absence of the bulk matter.

Appendix C

Units

Natural units

- $M_{\rm Pl} = (\hbar c/G)^{1/2} = 2.177 \times 10^{-5}$ g
- $l_{\rm Pl} = 1.616 \times 10^{-33}$ cm
- $t_{\rm Pl} = 5.391 \times 10^{-44}$ sec
- $T_{\rm Pl} = 1.416 \times 10^{32}$ K $= 1.221 \times 10^{19}$ GeV

Conversion from natural units

- 1 cm $= 5.068 \times 10^{13}$ GeV$^{-1}\hbar$
- 1 sec $= 1.519 \times 10^{24}$ GeV$^{-1}\hbar/c$
- 1 g $= 5.608 \times 10^{25}$ GeV$/c^2$
- 1 erg $= 6.242 \times 10^{2}$ GeV
- 1 K $= 8.618 \times 10^{-14}$ GeV$/k_B$

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